DEVELOPMENTS IN MATHEMATICS

SYMBOLIC COMPUTATION, NUMBER THEORY, SPECIAL FUNCTIONS, PHYSICS AND COMBINATORICS

Edited by
Frank G. Garvan
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Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics

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Developments in Mathematics

VOLUME 4

Series Editor:

Krishnaswami Alladi, *University of Florida, U.S.A.*

Aims and Scope

Developments in Mathematics is a book series publishing

(i) Proceedings of Conferences dealing with the latest research advances,

(ii) Research Monographs, and

(iii) Contributed Volumes focussing on certain areas of special interest.

Editors of conference proceedings are urged to include a few survey papers for wider appeal. Research monographs which could be used as texts or references for graduate level courses would also be suitable for the series. Contributed volumes are those where various authors either write papers or chapters in an organized volume devoted to a topic of special/current interest or importance. A contributed volume could deal with a classical topic which is once again in the limelight owing to new developments.

Preface

These are the proceedings of the conference "Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics" held at the Department of Mathematics, University of Florida, Gainesville, from November 11 to 13, 1999. The main emphasis of the conference was Computer Algebra (i.e. symbolic computation) and how it related to the fields of Number Theory, Special Functions, Physics and Combinatorics. A subject that is common to all of these fields is q-series. We brought together those who do symbolic computation with q-series and those who need q-series including workers in Physics and Combinatorics. The goal of the conference was to inform mathematicians and physicists who use q-series of the latest developments in the field of q-series and especially how symbolic computation has aided these developments.

Over 60 people were invited to participate in the conference. We ended up having 45 participants at the conference, including six one hour plenary speakers and 28 half hour speakers. There were talks in all the areas we were hoping for. There were three software demonstrations.

Plenary Lectures:

George Andrews (Pennsylvania State University)
"Search algorithms in the study of q-series"

Ken Ono (Pennsylvania State University and the University of Wisconsin at Madison)
"Congruences for $p(n)$ and some questions of Serre on the Fourier coefficients of modular forms"

Barry McCoy (Institute for Theoretical Physics, Stony Brook)
"Rogers-Ramanujan identities in statistical mechanics and conformal field theory"

Doron Zeilberger (Temple University)
"A tutorial on Mint: Akalu Tefera's brilliant fully-automated implementation of the continuous multi-WZ method"

Sergei Suslov (Arizona State University)
"Basic Fourier series: Introduction, analytic and numerical investigation"

Dennis Stanton (University of Minnesota)
"Open problems in q-series"

The papers in this volume represent many of the topics covered at the conference. Although Bill Gosper and Mike Hirschhorn were unable to attend the conference, they were able to contribute papers to these proceedings. The order of articles is alphabetical by author.

We would like the thank the sponsors of our conference: the Institute for Fundamental Theory (University of Florida), the National Science Foundation, the National Security Agency, the UF Department of Mathematics and The Number Theory Foundation. We would also like to thank Denise Marks (University of South Florida) for typing some of the papers.

Frank G. Garvan
University of Florida, Gainesville
March 8, 2001.

Mourad E. H. Ismail
University of South Florida, Tampa

March 8, 2001.

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Bruce Berndt*† (University of Illinois)
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Youn-Seo Choi* (Korean Advanced Institute of Science and Technology, Seoul)
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* Speaker. Links to abstracts of all talks are available at
`http://www.math.ufl.edu/~frank/qsconf.html`

† Contributed paper to these proceedings.

‡ David and Gregory Chudnovsky were unable to make it to the conference. Their talk *Orthogonal Polynomials and the Solution of the Pulse Width Modulation Problem*, was delivered by Mourad Ismail.

Photograph of partipants, Symbolic Computation,
Number Theory, Special Functions, Physics and Combinatorics Conference

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GAUSSIAN HYPERGEOMETRIC SERIES AND COMBINATORIAL CONGRUENCES

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Abstract We study the Gaussian hypergeometric series of type ${}_3F_2$ over finite fields $\mathbb{F}_p$. For each prime p and each $\lambda \in \mathbb{F}_p$, we explicitly determine $p^2{}_3F_2(\lambda)_p \pmod{p^2}$. Using this perspective, we are able to give a direct proof of one of Beukers' conjectured "supercongruences" between certain Apéry numbers and the coefficients of a weight three modular form of CM type. Finally, we record many new supercongruences of this form.

Keywords: Gaussian hypergeometric series, Apéry numbers

1. INTRODUCTION

In a recent paper [1], the author and K. Ono study the "Gaussian" hypergeometric series ${}_4F_3(1)_p$ over the finite field $\mathbb{F}_p$. They describe relationships between values of these series, Fourier coefficients of modular forms, and the arithmetic of a certain algebraic variety. These relationships, together with tools from p-adic analysis and some unexpected combinatorial identities, lead to the proof of one of Beukers "supercongruence" conjectures for the Apéry numbers $A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$.

Our purpose in this paper is to investigate similar phenomena for the hypergeometric series ${}_3F_2(\lambda)_p$. We begin by recalling some definitions. If p is an odd prime, then let $\mathbb{F}_p$ be the field with p elements. We extend each multiplicative character χ of $\mathbb{F}_p^\times$ to $\mathbb{F}_p$ by defining $\chi(0) := 0$. If A and B are two such characters, then we define the normalized Jacobi sum $\binom{A}{B}$ by

$$\binom{A}{B} := \frac{B(-1)}{p} J(A, \bar{B}) = \frac{B(-1)}{p} \sum_{x \in \mathbb{F}_p} A(x)\bar{B}(1-x).$$

F.G. Garvan and M.E.H. Ismail (eds.),
Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics, 1–12.

Let $A_0, A_1, \ldots A_n$, and $B_1, B_2, \ldots B_n$ be characters of $\mathbb{F}_p$. Following Greene [5], we define the Gaussian hypergeometric series over $\mathbb{F}_p$ by

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \mid x\right)_p := \frac{p}{p-1}\sum_{\chi}\binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x) \tag{1.1}$$

(here the sum runs over all characters χ of $\mathbb{F}_p$). Let ϕ_p and ϵ_p denote the quadratic and trivial characters of $\mathbb{F}_p$, respectively, and define ${}_{n+1}F_n(x)_p$ by

$$ {}_{n+1}F_n(x)_p := {}_{n+1}F_n\left(\begin{matrix} \phi_p, & \phi_p, & \ldots, & \phi_p \\ & \epsilon_p, & \ldots, & \epsilon_p \end{matrix} \mid x\right)_p .$$

In what follows, the prime p will be clear from context. Therefore we will sometimes suppress the subscript p in our notation.

For odd primes p, define the quantities

$$\begin{aligned} A(p,\lambda) &:= \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{\frac{p-1}{2}+j}{j} \lambda^{pj}, \\ B(p,\lambda) &:= \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{\frac{p-1}{2}+j}{j} \lambda^{j} \left\{ 1 + \frac{1}{2}\sum_{i=\frac{p+1}{2}}^{\frac{p-1}{2}+j} \frac{1}{i} + 3j \sum_{i=1+j}^{\frac{p-1}{2}+j} \frac{1}{i} \right\}. \end{aligned} \tag{1.2}$$

All of the results in this paper are consequences of the following

Theorem 1. *If p is an odd prime and $\lambda \in \mathbb{Q}\setminus\{0\}$ has $\mathrm{ord}_p(\lambda) \geq 0$, then*

$$p^2 {}_3F_2(\lambda)_p \equiv A(p,\lambda) + pB(p,\lambda) \pmod{p^2}.$$

Consider the family of elliptic curves

$$ {}_3E_2(\lambda) \ : \ y^2 = (x-1)(x^2+\lambda), \qquad \lambda \in \mathbb{Q}\setminus\{0,-1\}, \tag{1.3}$$

and let $L({}_3E_2(\lambda), s) = \sum_{n=1}^{\infty} \frac{{}_3a_2(n,\lambda)}{n^s}$ be the usual Hasse–Weil L-function for ${}_3E_2(\lambda)$. Ono [11, Thm. 5] proved that if p is an odd prime and $\lambda \in \mathbb{Q}\setminus\{0,1\}$ has $\mathrm{ord}_p(\lambda(\lambda-1)) = 0$, then

$$ {}_3a_2(p, \tfrac{1}{\lambda-1})^2 = p + \phi_p(1-\lambda)\cdot p^2 {}_3F_2(\lambda)_p$$

(we have made a change of variables in the curves which Ono calls ${}_3E_2(\lambda)$ in order to simplify notation). Together with Theorem 1, this yields

Corollary 1. *Suppose that p is an odd prime and that $\lambda \in \mathbb{Q} \setminus \{0,1\}$ has* $\mathrm{ord}_p(\lambda(\lambda-1)) = 0$. *Then*

$$_3a_2(p, \tfrac{1}{\lambda-1})^2 \equiv \phi_p(1-\lambda)A(p,\lambda) + p + p \cdot \phi_p(1-\lambda)B(p,\lambda) \pmod{p^2}.$$

By a theorem of Hasse, we know that $|_3a_2(p, \frac{1}{\lambda-1})| < 2\sqrt{p}$. This yields the following curious corollary.

Corollary 2. *If p is an odd prime and $\lambda \in \mathbb{Q}\setminus\{0,1\}$ has* $\mathrm{ord}_p(\lambda(\lambda-1)) = 0$, *then the quantity*

$$\phi_p(1-\lambda)A(p,\lambda) + p + p \cdot \phi_p(1-\lambda)B(p,\lambda)$$

is congruent modulo p^2 to one of the numbers $0, 1, 2, 3, \ldots, 4p-1$.

We remark that a similar phenomenon occurs for another family of elliptic curves. In particular, define the curves

$$_2E_1(\lambda) \ : \ y^2 = x(x-1)(x-\lambda), \qquad \lambda \in \mathbb{Q} \setminus \{0,1\},$$

and let $L(_2E_1(\lambda), s) = \sum_{n=1}^{\infty} \frac{_2a_1(n,\lambda)}{n^s}$ be the associated L-function. Then combining Proposition 5 and Theorem 1 of [11] (see also [10, Prop. 1]) yields the following result.

Theorem 2. *If p is an odd prime and $\lambda \in \mathbb{Q}\setminus\{0,1\}$ has* $\mathrm{ord}_p(\lambda(\lambda-1)) = 0$, *then*

$$_2a_1(p,\lambda) \equiv \phi_p(-1) \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \binom{\frac{p-1}{2}+j}{j} (-\lambda)^j \pmod{p}.$$

In the latter part of the paper, we consider the topic of "supercongruences". For $n \geq 0$, define the Apéry number

$$b(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}. \tag{1.4}$$

Beukers made the following

Conjecture. (Beukers, [2]) *Suppose that $p \geq 5$ is a prime. Then we have*

$$b\left(\tfrac{p-1}{2}\right) \equiv \begin{cases} 0 \pmod{p^2} & \textit{if } p \equiv 3 \pmod 4, \\ 4a^2 - 2p \pmod{p^2} & \textit{if } p = a^2 + b^2 \textit{ and } a \textit{ is odd.} \end{cases} \tag{1.5}$$

We will give a proof of the following result.

Theorem 3. *The conjecture is true.*

This theorem had already been proved in the case $p \equiv 3 \pmod 4$ by Van Hamme [13], and in the general case by Ishikawa [7]. Our proof is direct, and is of interest since it sheds some light on the relationships between the supercongruence, special values of Gaussian hypergeometric series, and certain unexpected combinatorial identities which arise in its proof (see Theorem 4 below).

In the last section we will attempt to insert Beukers' supercongruence (1.5) into a larger framework by giving eight new examples of supercongruences of the same form. The combinatorial sums which arise in the new congruences are somewhat more complicated than the quantity $b(\frac{p-1}{2})$ of Beukers' original conjecture; this difference is explained by the combinatorial identities (Theorem 4 below) which intervene in the latter case. It seems that the common thread in these supercongruences is the presence of a weight three modular form with complex multiplication. The quantity on the right side of (1.5), for example, defines the pth Fourier coefficient of the weight three CM form $\eta^6(4z)$ (here $\eta(z)$ denotes Dedekind's eta-function). Such a modular form lies in the background of each of the new examples which we give.

Acknowledgements

The author is indebted to Peter Paule and Carsten Schneider at RISC-Linz for sharing their expertise, and for performing the computations necessary to prove Theorem 4.

2. PRELIMINARIES

In order to prove Theorem 1, we will use the Gross-Koblitz formula [6] in order to develop the first two terms in the p-adic expansion of ${}_3F_2(\lambda)_p$. In this section we collect some preliminaries on Gauss sums and the p-adic gamma function.

The gamma function is defined on the ring $\mathbb{Z}_p$ of p-adic integers by

$$\Gamma_p(n) := (-1)^n \prod_{j<n,\, p\nmid j} j, \quad \text{for } n \in \mathbb{N},$$
$$\Gamma_p(x) := \lim_{n \to x} \Gamma_p(n), \text{ for } x \in \mathbb{Z}_p.$$

We have the fundamental facts, which may be found, for example, in [8]:

$$n! = (-1)^{n+1}\Gamma_p(n+1), \qquad 0 \le n \le p-1, \tag{2.1}$$
$$|\Gamma_p(x)| = 1, \qquad x \in \mathbb{Z}_p. \tag{2.2}$$

Further, if $x \in \mathbb{Z}_p$, and $R(x)$ denotes the representative of $x \pmod p$ in the set $\{1, \ldots, p\}$, then we have

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{R(x)}. \tag{2.3}$$

The following are known for $p \geq 5$ (see [3], or [1, section 6] for (2.5)):

$$x \equiv y \pmod{p^n} \implies \Gamma_p(x) \equiv \Gamma_p(y) \pmod{p^n} \qquad (x, y \in \mathbb{Z}_p,\ n \geq 1), \tag{2.4}$$

$$\Gamma'_p(x_0 + z) \equiv \Gamma'_p(x_0) \pmod p \qquad (x_0 \in \mathbb{Z}_p,\ |z| \leq |p|), \tag{2.5}$$

$$\Gamma_p(x_0 + z) \equiv \Gamma_p(x_0) + z\Gamma'_p(x_0) \pmod{p^2} \qquad (x_0 \in \mathbb{Z}_p,\ |z| \leq |p|). \tag{2.6}$$

Finally, define $G(x) := \frac{\Gamma'_p(x)}{\Gamma_p(x)}$. Then if $x \in \mathbb{Z}_p$ we have $G(x) \in \mathbb{Z}_p$. Further,

$$G(x+1) - G(x) = \frac{1}{x}, \qquad \text{if } x \in \mathbb{Z}_p, \quad |x| = 1. \tag{2.7}$$

We also require some background on Gauss sums. Let $\pi \in \mathbb{C}_p$ be a fixed root of $x^{p-1} + p = 0$, and let ζ_p be the unique p-th root of unity in $\mathbb{C}_p$ such that $\zeta_p \equiv 1 + \pi \pmod{\pi^2}$. Then for a character $\chi : \mathbb{F}_p \mapsto \mathbb{C}_p$, we define the Gauss sum $g(\chi) = \sum_{x=0}^{p-1} \chi(x)\zeta_p^x$. We have the following well-known properties:

(1) $g(\chi)g(\bar{\chi}) = \chi(-1)p$.

(2) If χ_1 and χ_2 are not both trivial, but $\chi_1\chi_2 = \epsilon$, then $J(\chi_1, \chi_2) = -\chi_1(-1)$.

(3) If $\chi_1\chi_2 \neq \epsilon$, then $J(\chi_1, \chi_2) = \dfrac{g(\chi_1)g(\chi_2)}{g(\chi_1\chi_2)}$.

Let ω denote the Teichmüller character; ω is a primitive character which is defined uniquely by the property that $\omega(x) \equiv x \pmod p$ for $x = 0, \ldots, p-1$. Then the Gross-Koblitz formula [6] states that

$$g(\bar{\omega}^j) = -\pi^j \Gamma_p\left(\frac{j}{p-1}\right), \qquad 0 \leq j \leq p-2. \tag{2.8}$$

3. PROOF OF THEOREM 1

For simplicity, we break the proof into a number of lemmas. Recall that $G(x)$ is the logarithmic derivative of $\Gamma_p(x)$.

Lemma 3.1. *If p is an odd prime and $\lambda \in \mathbb{Q} \setminus \{0\}$ has $\mathrm{ord}_p(\lambda) \geq 0$, then*