Fourier Analysis

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Fourier Analysis

To My Wife
Dawn Manire

Preface

The deep study of nature is the most fruitful source of knowledge.
Joseph Fourier

Fourier's Theorie analytique de la chaleur is the bible of the mathematical physicist.
Arnold Sommerfeld

My purpose in writing this book is to explain the basic mathematical theory as well as some of the principal applications of Fourier analysis. Vibrations and sound, heat conduction, optics, and CAT scanning are just some of the many areas to which Fourier analysis contributes deep insights. Although there are many fine books that cover any one of these applications in detail, each such book must necessarily be rather brief in its coverage of mathematical techniques. Therefore, I feel that there is a real need for a text that covers most of the basic mathematics of Fourier analysis and gives concise discussions of how that mathematics is applied.

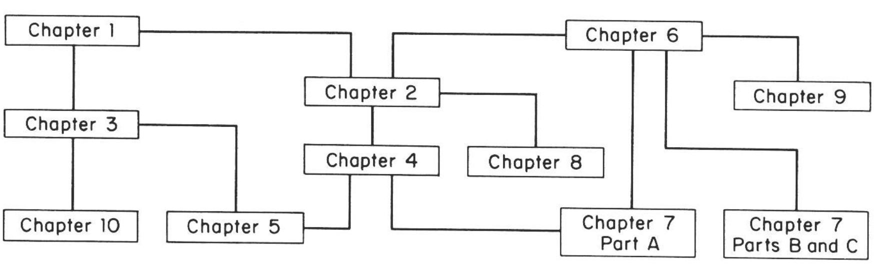
This book arose from the lecture notes that I have given to my students in courses and seminars at the University of Wisconsin at Eau Claire. These classes consisted mostly of seniors majoring in mathematics and physics. My intention has been to provide them with a firm foundation for the modern ideas and applications of Fourier analysis and not just stick to the simplest concepts. Hence the book should also be suitable for first-year graduate students. A previous course in advanced calculus would make the book most easily comprehensible; all the necessary background can be found in any of the following books: Bartle (1964), Buck (1978), Kaplan (1984), or Rudin (1964). I have found, however, that talented physics majors have been able to read the text with a fair degree of comprehension.

To make this book accessible to as wide an audience as possible I have not utilized the Lebesgue theory of integration. Those readers who are familiar with this sophisticated theory, however, should have no trouble making minor, obvious modifications in theorems and proofs to obtain more general results. Also, for reasons of space, I have not included a complete discussion of computer methods. I hope that the section on Fast Fourier Transforms in Chapter 7 will serve as a brief introduction to this aspect of Fourier analysis. Perhaps in a second edition I might expand this discussion. In any case, I have provided ample references to the literature of computerized Fourier analysis in §13 of Chapter 7.

When I first planned to write this book I wanted to include more material on the history of Fourier analysis, since it has united much of mathematical analysis and physics for nearly two hundred years. But such material would have made the book even longer than it already is. Readers who are interested in the rich history of Fourier analysis might begin by consulting the References for Chapter 1.

The exercises in the text range from straightforward applications of formulas to collections of problems aiming toward extensions of results in the text or describing additional concepts. It is very important that the reader try as many exercises as possible; the text cannot be comprehended without them. *Those exercises that are absolutely essential for understanding the text are marked by a star* (★). Some exercises that might be found difficult are marked by an asterisk (*).

This book was written for a one-year course, in which case there should be time to cover the entire book. If the book is used for one semester, then one possible syllabus, which I have found effective, consists of Chapters 1, 2, §§1–4, 3, 2, §§6–8, 6, 7, and (optionally), 8, §§1–4. In classes with a high proportion of physics majors I have found it expedient to cover Chapter 3 after §§1–4 of Chapter 2, and then return to §§6–8 of Chapter 2. There is some flexibility for presentation of material, as indicated by the following diagram of the dependency of chapters.



Dependency of Chapters.

Acknowledgments

I would like to thank Professors Gabriel Kojoian, Andrew Balas, and Kevin Gough for their participation in my classes in Fourier analysis and for their encouragement when I was writing this book. Thanks also go to Professor Walter "Doc" Reid for many enjoyable conversations about the book.

Professor S. G. Lipson was kind enough to send me the photographs in Chapter 7 and I thank him for that. Thanks also to Roland T. Katz for the moral support he has given me.

My students also deserve acknowledgment. I especially want to thank Sue Kelly, Doug Pearson, and Ken Dykema for their many helpful questions and suggestions. Ken Dykema must be thanked again for his help with the computer graphs.

I owe a debt of gratitude to my wife for all of her patience and kind support during the long hours, days, and months that it took to write this book.

May 1987 J.S.W.
Eau Claire, Wisconsin

Notation

The notation in the text is fairly standard—for example, a, b, c, d are usually real constants, j, k, m, n are usually integers, and t, u, v, x, y, z are usually real variables. We will sometimes write t_0, u_0, v_0, x_0, y_0, z_0 to denote fixed values of those variables. The letters f, g, h are reserved for functions; f', g', h' will denote derivatives of those functions when they depend on a single variable. The open interval $a < x < b$ will be denoted by (a, b) while the closed interval $a \le x \le b$ will be denoted by $[a, b]$.

The following is a list of some special notations that are employed.

$\hat{f}$ = Fourier transform of the function f

$f_x, f_y, (f_z)$ = partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $(\partial f/\partial z)$ of a function f that depends on x, y, (z)

$f(x_0+)$ = one-sided limit from the right $\lim\limits_{\substack{u\to 0\\(u>0)}} f(x_0+u) = \lim\limits_{u\to 0+} f(x_0+u)$

$f(x_0-)$ = one-sided limit from the left $\lim\limits_{\substack{u\to 0\\(u<0)}} f(x_0+u) = \lim\limits_{u\to 0-} f(x_0+u)$

$f^{>}$ = Laplace transform of a function f

$f^{\#}$ = Radon transform of a function f (*This notation for Radon transforms will be used only in Chapter* 9; it will occasionally be used in earlier chapters to denote auxiliary functions with no fixed meaning.)

Π = function $\Pi(x) = \begin{cases} 1 & \text{if} \quad |x| < \frac{1}{2} \\ 0 & \text{if} \quad |x| > \frac{1}{2} \end{cases}$

Λ = function $\Lambda(x) = \begin{cases} 1 - |x| & \text{if} \quad |x| \le 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$

sinc = function $\operatorname{sinc} x = \sin \pi x/\pi x$

Ψ = harmonic function

$\mathbb{R}$ = set of real numbers

$\mathbb{R}^2$, $(\mathbb{R}^3, \mathbb{R}^n)$ = set of ordered pairs (triples, n-tuples) of real numbers

Brief Review

We will now briefly discuss, through exercises, some concepts and techniques that will ease the burden of some of our later work.

1. Integrate the following integrals by parts.

   (a) $\int_0^1 x \sin 2\pi x \, dx$

   (b) $\int_{-\pi}^{\pi} e^x \sin 3x \, dx$

2. *Even and Odd Functions.* The definition of an *even function* f over an interval $(-a, a)$ symmetric about the origin is that

$$f(-x) = f(x) \qquad \text{for each } x \text{ in } (-a, a)$$

   An *odd function* f over $(-a, a)$ is a function that satisfies

$$f(-x) = -f(x) \qquad \text{for each } x \text{ in } (-a, a)$$

   Given these definitions, do the following exercises.

   (a) State which of the following functions are odd, even, or neither, on the given interval.

   $f(x) = x^2$ on $(-1, 1)$ $\qquad$ $g(x) = x^3 \cos 4x$ on $(-\pi, \pi)$

   $h(x) = x^2 \cos 4x$ on $(-\pi, \pi)$ $\qquad$ $k(x) = x + \frac{1}{4}x^2$ on $(-2, 2)$

   (b) Show that for all (continuous) even functions

$$\int_{-a}^{a} f(x) \, dx = 2\int_0^a f(x) \, dx,$$

   and for all (continuous) odd functions

$$\int_{-a}^{a} f(x) \, dx = 0.$$

(c) Fill in the following table by writing odd, even, or neither, in the appropriate blanks.

f	g	fg	f/g	$f+g$	$f-g$
odd	odd				
even	even				
even	odd				
odd	even				

3. Using the results of Exercise 2, evaluate the following integrals.

(a) $\int_{-\pi}^{\pi} 3x \cos x \, dx$

(b) $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (x^3 + x + 3) \cos x \, dx$

[*Note*: For (b), split the integral into even and odd terms.]

4. *Kronecker's Rule.* With the aid of *Kronecker's rule,* many of the integrals needed in Fourier series are easier to evaluate. The following exercises are intended to explain Kronecker's rule.

(a) Let $p(x)$ be a polynomial in x of degree m, and $f(x)$ a continuous function. Let $F_1(x) = \int f(x)\,dx$, $F_2(x) = \int F_1(x)\,dx, \ldots, F_{m+1}(x) = \int F_m(x)\,dx$. And, let $p^{(j)}(x)$ be the jth derivative of $p(x)$. Prove that

$$\int p(x)f(x)\,dx = p(x)F_1(x) - p^{(1)}(x)F_2(x) + p^{(2)}(x)F_3(x) - \cdots + (-1)^m p^{(m)}(x)F_{m+1}(x) + C$$

$$= \sum_{j=0}^{m} (-1)^j p^{(j)}(x)F_{j+1}(x) + C$$

[*Hint*: Integrate repeatedly by parts until $p^{(m+1)}(x) = 0$.] It follows immediately from this result that

$$\int_a^b p(x)f(x)\,dx = \left[\sum_{j=0}^{m} (-1)^j p^{(j)}(x)F_{j+1}(x)\right]\Bigg|_a^b$$

For example,

$$\int_{-\pi}^{\pi} x^3 \sin x\,dx = 2\int_0^{\pi} x^3 \sin x\,dx \qquad (x^3 \sin x \text{ is even})$$

$$= 2[x^3(-\cos x) - (3x^2)(-\sin x) + (6x)(\cos x) - (6)(\sin x)]|_0^{\pi}$$

$$= 2\pi^3 - 12\pi$$

(b) Using Kronecker's rule, evaluate the following integrals.

$$\int_{-\pi}^{\pi} x^2 \cos nx\, dx \qquad \int_{-1}^{1} (x^3 + 2x^2 + 1)\sin n\pi x\, dx$$

[*Note:* For the second integral, do not forget about even and odd functions.]

5. *Complex Numbers.* A complex number z is a quantity $a + ib$ where a and b are real numbers and $i^2 = -1$. The arithmetic of complex numbers is the usual one subject to the relations

$$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$$
$$(a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

It is often convenient to think of z as the point (a, b) in the Cartesian plane. Therefore, the distance from z to $0 = (0, 0)$ is $(a^2 + b^2)^{1/2}$. We define $|z| = |a + ib|$ to equal that distance $(a^2 + b^2)^{1/2}$ and call $|z|$ the *modulus* of z. We call a the *real part* and b the *imaginary part* of z and write $a = \text{Re}\, z$ and $b = \text{Im}\, z$. Finally, we define $\bar{z}$ to be the complex number $a - ib$ and call it the *complex conjugate* of z. Using these definitions do the following exercises.

(a) Compute $3 - 2i - (4 + 3i) \qquad (3 + 2i)(6 - 5i) \qquad 3 + i - (4 + i)i$

(b) Prove that $\overline{zw} = \bar{z}\,\bar{w}$ and that $|zw| = |z| \cdot |w|$ for all complex numbers z and w. Conclude that $|z^n| = |z|^n$ for all positive integers n. [*Hint:* $|z| = (z\bar{z})^{1/2}$.]

(c) Prove that the following inequalities hold for all complex numbers z and w

$$|z \pm w| \le |z| + |w| \qquad |\text{Re}\, z| \le |z| \qquad |\text{Im}\, z| \le |z|$$

(d) Suppose that $a + ib \ne 0 + i0$. We then define division as follows

$$\frac{c + id}{a + ib} = \frac{(c + id)(a - ib)}{a^2 + b^2} = \left(\frac{ac + bd}{a^2 + b^2}\right) + i\left(\frac{ad - bc}{a^2 + b^2}\right)$$

Using that definition, compute

$$\frac{3 - 2i}{4 + i} \quad \text{and} \quad \frac{2 - 4i}{2 - i}$$

(e) Prove that for all complex numbers z and w we have

$$z + w = w + z \qquad zw = wz$$

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