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ADVANCED TOPICS IN SCATTERING AND BIOMEDICAL ENGINEERING

Proceedings of the Eighth International Workshop on Mathematical Methods in Scattering Theory and Biomedical Engineering

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Preface

This volume includes papers which were presented in the 8th International Workshop on Applied Mathematics on Scattering Theory and Biomedical Engineering held in the island of Lefkas, Greece, from 27-29 September, 2007.

The workshop is organized every two years by the University of Ioannina, the National Technical University of Athens and the University of Patras. For almost two decades, the workshop has been making positive impacts on the use of mathematics and computing in scattering and biomedical engineering by fostering collaboration among scientists both young and senior from a variety of disciplines. The vision of the organizers has certainly been realised by the more than 600 papers that have been presented at the workshop over the years.

Scattering theory is a framework for studying the scattering of waves and particles. Biomedical Engineering is the application of engineering principles to the medical field. Both fields share methodological approaches such as applied mathematics, numerical analysis, scientific computing and we try to merge all those in a single workshop. This is one of the continuing strengths of our workshop since it offers a wonderful opportunity for exchange of ideas and matching of needs between scattering and biomedical engineering. In a troubled world the workshop serves as an ideal place for scientific cooperation and development of collegiality and friendship.

We were all enthusiastic for the organization on this workshop and we are grateful to Ms Maria Pikou and Mrs Vicky Papageorgiou for the organization of such a successful meeting. We are also indebted to our Universities, The Open University of Greece and others for their financial support. We all are looking forward to the next workshop, which will be held in Patras in 2009, and we expect to see there our friends and colleagues.

Ioannina, December 2007

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Contents

Preface

Scattering Theory

ON THE RECONSTRUCTION OF A SMALL ELASTIC SPHERE IN THE NEAR FIELD BY POINT–SOURCES *

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A near-field reconstruction method which locates the radius and the position of a small elastic rigid sphere in the low–frequency sense is considered. In particular, the direct scattering problem for a rigid sphere by a point generated dyadic field is presented in a dyadic form, and the exact Green's function as well as the elastic far–field patterns of the radiating solution in form of infinite series are obtained. Finally, the inversion scheme is based on a closed form approximation of the scattered field at the source for various point-source locations.

1. Introduction

This paper is concerned with scattering of elastic point–sources by a bounded obstacle, as well as with a related near–field inverse problem for small scatterers. One main type of boundary value problem, which char-

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acterize the scattering region will be examined. In particular, we consider the rigid problem, where the displacement field is vanishing on the surface of the scatterer. A dyadic formulation for the aforementioned scattering problem is considered, in order to gain the symmetry–compactness of the dyadic analysis [10].

For acoustic and electromagnetic scattering, results on incident waves generated by a point–source appear in [2]; see also references therein, and in particular the book by Dassios–Kleinmann [7], and therein related references. All of the aforementioned studies deal with scattering relations by point–sources, and related simple inversion algorithms for small scatterers. For elasticity now, related problems such as the location and identification of a small three–dimensional elastic inclusion, using arrays of elastic source transmitters and receivers, is considered in [1].

The present paper provides results on the direct scattering problem by point–generated elastic waves for the two and the three–dimensional elastic case. Further, a related near–field inversion algorithm for a small rigid sphere, in the low–frequency sense is established, where the key idea is to measure the scattered field for various point–source locations.

2. Formulation of the problem in R^N, $N = 2, 3$

We assume that $\mathbb{R}^N$, $N = 2$ or 3, is filled by an isotropic and homogeneous elastic medium with positive Lamé constants λ, μ and density ϱ. The propagation of time-harmonic elastic waves in such a medium is described by the reduced Navier equation

$$\mu\,\Delta\widetilde{\mathbf{u}}(\mathbf{r}) + (\lambda+\mu)\ \mathrm{grad\,div}\,\widetilde{\mathbf{u}}(\mathbf{r}) + \varrho\,\omega^2\widetilde{\mathbf{u}}(\mathbf{r}) = \widetilde{\mathbf{0}}, \tag{1}$$

where $\omega > 0$ is the angular frequency, and the overtilde (" $\sim$ ") is used to denote dyadic fields. Using the standard abbreviation

$$\Delta^* := \mu\,\Delta + (\lambda+\mu)\ \mathrm{grad\,div}, \tag{2}$$

an alternative form of equation (1) (which will be considered from now on), is given by

$$(\Delta^* + \varrho\,\omega^2)\,\widetilde{\mathbf{u}}(\mathbf{r}) = \widetilde{\mathbf{0}}. \tag{3}$$

Let now V be a open, bounded and simply connected subset of $\mathbb{R}^N$ with C^2- boundary S. The set V will be referred to as the scatterer. The physical parameters of the elastic background medium lead to the mathematical formulation of the problem through a main type of boundary condition that is described on the surface of the scatterer.

From the mathematical point of view, the scattering problem is described by the following exterior boundary-value problem: For a given point source incident field $\widetilde{\mathbf{u}}_a^{inc}$ at $\mathbf{a}$, and zero body forces, find a solution $\widetilde{\mathbf{u}}_a \in [C^2(\mathbb{R}^N \setminus \overline{V}) \cap C^1(\mathbb{R}^N \setminus V)]^N$ such that

$$(\Delta^* + \varrho\,\omega^2)\,\widetilde{\mathbf{u}}_a(\mathbf{r}) = \widetilde{\mathbf{0}}, \qquad \mathbf{r} \in \mathbb{R}^N \setminus V, \tag{4}$$

$\overline{V} = V \cup S$ which for the rigid body problem, satisfies the Dirichlet boundary condition

$$\widetilde{\mathbf{u}}_a(\mathbf{r}) = \widetilde{\mathbf{0}}, \qquad \mathbf{r} \in S. \tag{5}$$

Due to the point source incident field at $\mathbf{a}$, the corresponding component of the scattered field is denoted by $\widetilde{\mathbf{u}}_a^{sct}$. Then the total field $\widetilde{\mathbf{u}}_a^{tot}$ in the exterior $R^N \setminus V$, $\mathrm{N} = 2, 3$, of the scatterer, is given by

$$\widetilde{\mathbf{u}}_a^{tot}(\mathbf{r}) = \widetilde{\mathbf{u}}_a^{inc}(\mathbf{r}) + \widetilde{\mathbf{u}}_a^{sct}(\mathbf{r}), \tag{6}$$

where the incident, the scattered and the total field satisfy (4). In addition, for the well-posedness of the problem, the well known radiation conditions due to Kupradze should also be satisfied by the scattered field [9].

3. The 2D case: elastic point–sources

We irradiate our object by an incident elastic wave due to a source located at a point with position vector $\mathbf{a}$, i.e.,

$$\widetilde{\mathbf{u}}_a^{\mathrm{inc}}(\mathbf{r}) = \frac{i}{4\omega^2}\left(\nabla_{\mathbf{r}} \otimes \nabla_{\mathbf{r}} + (k_s)^2\,\widetilde{\mathbf{I}}\right) H_0^{(1)}(k_s\,|\mathbf{r} - \mathbf{a}|)$$
$$- \frac{i}{4\omega^2}\,\nabla_{\mathbf{r}} \otimes \nabla_{\mathbf{r}} H_0^{(1)}(k_p\,|\mathbf{r} - \mathbf{a}|), \quad \mathbf{r} \in \mathbb{R}^2 \quad \mathbf{r} \neq \mathbf{a}. \tag{7}$$

In (7) $\widetilde{\mathbf{I}}$ is the identity dyadic, $H_0^{(1)}(z)$, is the Hankel function of first kind and zero order and "$\otimes$" is the juxtaposition between two vectors (this gives a dyadic). We note that when $a = |\mathbf{a}| \to \infty$, we recover the plane–wave incidence case in the direction $-\hat{\mathbf{a}}$, i.e.,

$$\widetilde{\mathbf{u}}^{\mathrm{inc}}(\mathbf{r}; -\hat{\mathbf{a}}) = A_p\,(\hat{\mathbf{a}} \otimes \hat{\mathbf{a}})\,e^{-ik_p\,\mathbf{r}\cdot\hat{\mathbf{a}}} + A_s(\widetilde{\mathbf{I}} - \hat{\mathbf{a}} \otimes \hat{\mathbf{a}})\,e^{-ik_s\,\mathbf{r}\cdot\hat{\mathbf{a}}}, \tag{8}$$

where A_p, A_s are constant amplitudes, given as

$$A_p := \frac{1}{\lambda + 2\mu}\,\frac{(1+i)\,e^{ik_p\,a}}{4\sqrt{\pi k_p\,a}} \quad \text{and} \quad A_s := \frac{1}{\mu}\,\frac{(1+i)\,e^{ik_s\,a}}{4\sqrt{\pi k_s\,a}}. \tag{9}$$

In what follows, we consider the scatterer to be a circular disk of radius R. We take polar coordinates and using cylinder Navier eigenvectors $\mathbf{\Phi}^{e,i}_{m,\sigma}, \mathbf{\Psi}^{e,i}_{m,\sigma}$ [6], $\sigma = 1, 2$, we obtain for (7) the following expansion

$$\widetilde{\mathbf{u}}^{\mathrm{inc}}_a(\mathbf{r}) = -\frac{i}{4\mu(k_s)^2} \sum_{m=0}^{+\infty} \sum_{\sigma=1}^{2} \left[\mathbf{\Phi}^i_{m,\sigma}(\mathbf{r}) \otimes \mathbf{\Phi}^e_{m,\sigma}(\mathbf{a}) + \mathbf{\Psi}^i_{m,\sigma}(\mathbf{r}) \otimes \mathbf{\Psi}^e_{m,\sigma}(\mathbf{a})\right] \tag{10}$$

for $r < a$. The scattered field has a similar expression and takes the form

$$\begin{aligned}
\widetilde{\mathbf{u}}^{\mathrm{sct}}_a(\mathbf{r}) = -\frac{i}{4\mu k_s^2} \times \\
\{ \sum_{m=0}^{+\infty} a_m \sqrt{\varepsilon_m}\, k_p\, H^{(1)'}_m(k_p r)\, \hat{\mathbf{r}} \otimes [\cos(m\varphi) \mathbf{\Phi}^e_{m,1}(\mathbf{a}) \\
+ \sin(m\varphi)\, \mathbf{\Phi}^e_{m,2}(\mathbf{a})\,] \\
+ \sum_{m=0}^{+\infty} \beta_m \sqrt{\varepsilon_m}\, \frac{m}{r}\, H^{(1)}_m(k_s r)\, \hat{\mathbf{r}} \otimes [\cos(m\varphi)\, \mathbf{\Psi}^e_{m,2}(\mathbf{a}) \\
- \sin(m\varphi)\, \mathbf{\Psi}^e_{m,1}(\mathbf{a})\,] \\
+ \sum_{m=0}^{+\infty} \gamma_m \sqrt{\varepsilon_m}\, \frac{m}{r}\, H^{(1)}_m(k_p r)\, \hat{\varphi} \otimes [\cos(m\varphi)\, \mathbf{\Phi}^e_{m,2}(\mathbf{a}) \\
- \sin(m\varphi)\, \mathbf{\Phi}^e_{m,1}(\mathbf{a})\,] \\
- \sum_{m=0}^{+\infty} \delta_m \sqrt{\varepsilon_m}\, k_s\, H^{(1)'}_m(k_s r)\, \hat{\varphi} \otimes [\cos(m\varphi)\, \mathbf{\Psi}^e_{m,1}(\mathbf{a}) \\
+ \sin(m\varphi)\, \mathbf{\Psi}^e_{m,2}(\mathbf{a})\,]\, \},
\end{aligned} \tag{11}$$

where " $\times$ " denotes the standard multiplication and the coefficients a_m, β_m, γ_m and δ_m are to be determined. We use the Dirichlet boundary condition (5), on $r = R$, (circular disk of radius R) and using orthogonality arguments, lengthy calculations yield

$$a_m = -\frac{J'_m(k_pR)}{H_m^{(1)'}(k_pR)}, \qquad \beta_m = -\frac{J_m(k_sR)}{H_m^{(1)}(k_sR)} \tag{12}$$

$$\gamma_m = -\frac{J_m(k_pR)}{H_m^{(1)}(k_pR)}, \qquad \delta_m = -\frac{J'_m(k_sR)}{H_m^{(1)'}(k_sR)}. \tag{13}$$

We now calculate the elastic far-field patterns in the form of infinite series. In order to find the longitudinal and transverse far–field pattern of the radiating solution $\widetilde{\mathbf{u}}_a^{\mathrm{sct}}(\mathbf{r})$ of the exterior boundary–value problem (4)–(5), we take into account the asymptotic forms of the Hankel functions $H_m(k_cr)$, $H'_m(k_cr)$, for $c = p, s$ and $r \to \infty$. The scattered field (11) takes the following form

$$\widetilde{\mathbf{u}}_a^{\mathrm{sct}}(\mathbf{r}) = \widetilde{\mathbf{u}}_a^{\infty,p}(\hat{\mathbf{r}})\,\frac{e^{ik_pr}}{\sqrt{r}} + \widetilde{\mathbf{u}}_a^{\infty,s}(\hat{\mathbf{r}})\,\frac{e^{ik_sr}}{\sqrt{r}} + O(r^{-3/2}), \qquad r \to \infty, \tag{14}$$

where the longitudinal and transverse far–field pattern of the scattered field is given by

$$\widetilde{\mathbf{u}}_a^{\infty,p}(\hat{\mathbf{r}}) = -\frac{1-i}{4(\lambda+2\mu)\sqrt{\pi k_p}} \times$$
$$\sum_{m=0}^{+\infty} \alpha_m\sqrt{\varepsilon_m}\, e^{-\frac{i\pi m}{2}}\, \hat{\mathbf{r}} \otimes \left[\cos(m\varphi)\,\mathbf{\Phi}_{m,1}^e(\mathbf{a}) + \sin(m\varphi)\,\mathbf{\Phi}_{m,2}^e(\mathbf{a})\right], \tag{15}$$

and

$$\widetilde{\mathbf{u}}_a^{\infty,s}(\hat{\mathbf{r}}) = \frac{1-i}{4\,\mu\sqrt{\pi k_s}} \times$$
$$\sum_{m=0}^{+\infty} \delta_m\sqrt{\varepsilon_m}\, e^{-\frac{i\pi m}{2}}\, \hat{\varphi} \otimes \left[\cos(m\varphi)\,\mathbf{\Psi}_{m,1}^e(\mathbf{a}) + \sin(m\varphi)\,\mathbf{\Psi}_{m,2}^e(\mathbf{a})\right], \tag{16}$$

respectively. As one can easily see in (15) and (16), the coefficients β_m and γ_m do not appear in the series, (although in (11) exist). This is justified from the fact that these coefficients are contained in the terms of $O(r^{-3/2})$ of the scattered field (14) at the radiation zone.

4. The scattered field at the point–source

We measure the scattered field at the source for various point–source locations. Hence, if we compute the scattered field (11) at point–source **a**, i.e., **r**=**a**, due to the orthogonal base $\{\hat{\mathbf{r}},\ \hat{\boldsymbol{\varphi}}\}$ of the polar coordinate system, and taking into account the cylinder Navier eigenvectors, then with some computational effort we arrive at

$$\widetilde{\mathbf{u}}_a^{sct}(\mathbf{a}) = -\frac{i}{4\,\mu\,k_s^2}\ \times$$

$$\{\sum_{m=0}^{\infty} \varepsilon_m \times [\alpha_m\, k_p^2\ \left(H_m^{1'}(k_p\, a)\right)^2$$

$$+\ \beta_m\ \frac{m^2}{\alpha^2}\ \left(H_m^1(k_s\, a)\right)^2\ \ (\hat{\mathbf{r}} \otimes \hat{\mathbf{r}})\,]$$

$$+\sum_{m=0}^{\infty} \varepsilon_m \times [\gamma_m\ \frac{m^2}{\alpha^2}\ \left(H_m^1(k_p\, a)\right)^2$$

$$+\ \delta_m\, k_s^2\ \left(H_m^{1'}(k_s\, a)\right)^2\ ]\ (\hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\varphi}})\ \}.$$

This formula is exact. Let us now consider the low–frequency assumption $k_c R \to 0,\ \ c = p,\ s$. Hence, the coefficients α_m, β_m, γ_m and δ_m (see (12)–(13), are computed and we arrive at

$$\alpha_m \simeq \frac{i\,\pi m}{2^{2m}\,(m!)^2}\ (k_p\, R)^{2m},$$

$$\delta_m \simeq \frac{i\,\pi m}{2^{2m}\,(m!)^2}\,(k_s R)^{2m},$$

$$\beta_m \simeq \frac{i\,\pi}{(m-1)!m!\,2^{2m}}\ (k_s\, R)^{2m},$$

$$\gamma_m \simeq -\frac{i\,\pi}{(m-1)!m!\,2^{2m}}\,(k_p R)^{2m}, \quad \text{for} \quad m \geq 1,$$

while for $m = 0$ we obtain

$$\alpha_0 \simeq \frac{i\,\pi}{4}\ (k_p\, R)^2, \quad \delta_0 \simeq \frac{i\,\pi}{4}\,(k_s R)^2,$$

$$\beta_0 \simeq \frac{i\,\pi}{2\,ln\frac{k_s R}{2}}, \qquad \gamma_0 \simeq \frac{i\,\pi}{2\,ln\frac{k_p R}{2}},$$

as $k_c R \to 0,\ c = p,\ s \to 0.$