ELEMENTARY DIFFERENTIAL EQUATIONS

with Applications

Edwards and Penney

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Elementary Differential Equations with Applications

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To Alice and Carol

Preface

We wrote this book to provide a concrete and readable text for the traditional course in elementary differential equations that science, engineering, and mathematics students take following calculus. It includes enough material appropriately arranged for different courses varying in length from one quarter to two semesters. Our treatment is shaped throughout by the goal of an exposition that students will find accessible, attractive, and interesting. We hope that we have anticipated and addressed most of the questions and difficulties that students typically encounter when they study differential equations for the first time.

The book begins (in Section 1.1) and ends (in Section 9.4) with discussions of the mathematical modeling of real-world situations. The fact that differential equations have diverse and important applications is too familiar for extensive comment here. But these applications have played a singular role in the historical development of this subject. Whole areas of the subject exist mainly because of their applications. So in teaching it, we want our students to learn first to solve those differential equations that enjoy the most frequent application.

We therefore make consistent use of appealing applications for both motivation and illustration of the standard elementary techniques of solution of differential equations. A number of the more substantial applications are placed in optional sections, each marked with an asterisk (in the table of contents and in the text). These sections can be omitted without loss of con-

tinuity, but their availability can provide instructors with flexibility for variations in emphasis.

While according real-world applications their due, we also think the first course in differential equations should be a window on the world of mathematics. Matters of definition, classification, and logical structure deserve (and receive here) careful attention—for the first time in the mathematical experience of many of the students (and perhaps for the last time in some cases). While it is neither feasible nor desirable to include proofs of the fundamental existence and uniqueness theorems along the way in an elementary course, students need to see precise and clearcut statements of these theorems, and to understand their role in the subject. We do include some existence and uniqueness proofs in Section 9.4, and occasionally refer to them in the main body of the text.

The list of introductory topics in differential equations is quite standard, and a glance at our chapter titles will reveal no major surprises, though in the fine structure we have attempted to add a bit of zest here and there. A number of different permutations in the order of topics are possible, and the table that follows this preface exhibits the logical dependence between chapters. In most chapters the principal ideas of the topic are introduced in the first few sections of the chapter, and the remaining sections are devoted to extensions and applications. Hence the instructor has a wide range of choice regarding breadth and depth of coverage.

Chapter 1 naturally treats first order equations, with separable equations (Section 1.4), linear equations (Section 1.5), substitution methods (Section 1.6), and exact equations (Section 1.7) comprising the core of the chapter. Chapter 2 is devoted to linear equations of higher order. In order to make the concepts of linear independence and general solutions more concrete and tangible, we discuss only second order equations in Section 2.1, and follow with the *n*th order case in Section 2.2.

Chapter 3 begins with a review of the basic facts about power series that will be needed. The first three sections of the chapter treat the standard power series techniques for the solution of linear equations with variable coefficients. We devote more attention than usual to certain matters—such as shifting indices of summation—that are mathematically routine but nevertheless troublesome for many students. In Section 3.4 (optional) we include for reference more detail on the method of Frobenius than ordinarily will be covered in the classroom. Similarly, we go slightly further than is customary in Section 3.6 (optional) with applications of Bessel functions. Chapter 4 on Laplace transforms is rather standard, though our discussion in Section 4.6 (optional) of impulses and Dirac delta functions may have some merit.

There is much variation in the treatment of linear systems in introductory courses, depending on the background in linear algebra that is assumed. The first two sections of Chapter 5 can stand alone as an introduction to linear systems without the use of linear algebra and matrices. The last four sections of Chapter 5 employ the notation and terminology (though not so much

theory) of elementary linear algebra. For ready reference, we have included in Section 5.3 a complete and self-contained account of the needed notation and terminology of determinants, matrices, and vectors.

Chapter 6 on numerical methods requires some special comment. Personal computers are now with us and here to stay. Pocket computers are relatively inexpensive and already in the hands of some students. The great difference (in the perception of students) between personal computing and mainframe computing may not yet be universally appreciated. Students can now envision the numerical approximation of solutions as a routine and commonplace matter. Our viewpoint in Chapter 6 is that understanding and appreciation of numerical algorithms is enhanced (and rendered more concrete to students) by discussion of their computer implementations. We decided to include illustrative programs because no flowchart has the convincing tangibility of a program that actually runs. Our choice of programming language was motivated by the recent adoption of BASIC as the lingua franca of personal computers. Moreover, only in BASIC could we include programs that without extensive discussion would be intelligible and informative to students with little or no programming experience. In another vein, it is pointed out in the Chapter 1 summary that the first four sections of Chapter 6 can be covered at any point in the course subsequent to Chapter 1. The increasingly widespread use of computers may provide a motive for covering numerical methods earlier than has been the custom in the past.

Chapters 1 through 6 are devoted to ordinary differential equations. Chapters 7 and 8 treat the applications of Fourier series, separation of variables, and Sturm-Liouville theory to partial differential equations and boundary value problems. After the introduction of Fourier series, the three classical equations—the wave and heat equations and Laplace's equation—are discussed in the last three sections of Chapter 7. The Sturm-Liouville methods of Chapter 8 are developed sufficiently to include some rather significant and realistic applications.

Apart from its final section on existence and uniqueness, Chapter 9 is a brief introduction to qualitative properties and stability of solutions, with numerous applications to competition, survival, and extinction of species.

Probably in no other mathematics course beyond calculus are the exercises and problem sets so crucial to student learning as in the introductory differential equations course. We therefore devoted great effort to the development and selection of the approximately 1750 problems in this book. Each section contains more computational problems ("solve the following equations," and so on) than any class will ordinarily use, plus an ample number of applied problems. We were, however, very sparing in our inclusion of purely theoretical problems. The answer section includes the answers to all odd-numbered problems and to some of the even-numbered ones.

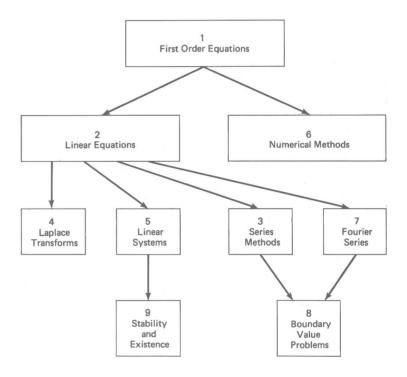
All experienced textbook authors know the value of critical reviewing during the preparation and revision of a manuscript. In writing this book we profited greatly from the advice of the following exceptionally able reviewers:

W. Dan Curtis, Kansas State University; Bruce Conrad, Temple University; James W. Cushing, University of Arizona; James L. Heitsch, University of Illinois at Chicago; Erich Zauderer, Polytechnic Institute of New York; Anthony Peressini, University of Illinois; and William Rundell, Texas A & M University.

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> C. H. E., Jr. D. E. P.

Dependence of Chapters



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Introduction and First Order Differential Equations

1

1.1 Introduction

The laws of the universe are written largely in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are best described by equations that relate changing quantities.

Because the derivative dy/dt = f'(t) of the function f may be regarded as the rate at which the quantity y = f(t) changes with respect to the independent variable t, it is natural that equations involving derivatives are those that describe the changing universe. An equation involving an unknown function and one or more of its derivatives is called a **differential equation**, and the study of differential equations has two principal goals:

- 1. To discover the differential equation that describes a physical situation;
- 2. To find the appropriate solution of that equation.

Unlike algebra, in which we seek the unknown numbers that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$, in solving a differential equation we are challenged to find the unknown functions y = g(x) for which an identity such as g'(x) - 2xg(x) = 0—in Leibniz notation,

$$\frac{dy}{dx} - 2xy = 0$$

—holds on some interval of real numbers. Ordinarily we will want to find *all* solutions of the differential equation if possible.

The following three examples illustrate the process of translating scientific laws and principles into differential equations, by interpreting rates of change as derivatives. In each of these examples the independent variable is time t, but we will see numerous applications in which some quantity other than time is the independent variable.

EXAMPLE 1 Newton's law of cooling may be stated in the following form: The *time rate of change* (the rate of change with respect to time t) of the temperature T(t) of a body is proportional to the difference between T and the temperature A of the surrounding medium. That is,

$$\frac{dT}{dt} = k(A - T) \tag{1}$$

where k is a positive constant.

Thus the physical law is translated into a differential equation. If we are given the values of k and A, we hope to find an explicit formula for T(t), and then—with the aid of this formula—we can predict the future temperature of the body.

EXAMPLE 2 The *time rate of change* of a population P(t) with constant birth and death rates is, in many simple cases, proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP \tag{2}$$

where k is the constant of proportionality.

EXAMPLE 3 Torricelli's law implies that the *time rate of change* of the volume V of water in a draining tank is proportional to the square root of the depth y of the water in the tank:

$$\frac{dV}{dt} = -ky^{1/2} \tag{3}$$

where k is constant. If the tank is a cylinder with cross-sectional area A, then V = Ay, and so dV/dt = A(dy/dt). In this case Eq. (3) takes the form

$$\frac{dy}{dt} = -hy^{1/2} \tag{4}$$

where h = k/A.

Let us discuss Example 2 further. Note first that each function of the form

$$P(t) = Ce^{kt} (5)$$

is a solution of the differential equation, Eq. (2),

$$\frac{dP}{dt} = kP$$
.

We verify this assertion as follows:

$$P'(t) = Cke^{kt} = k(Ce^{kt}) = kP(t)$$

for all real numbers t. Because substitution of each function of the form given in (5) into Eq. (2) produces an identity, all these functions are solutions of Eq. (2).

Thus, even if the value of the constant k is known, the differential equation dP/dt = kP has infinitely many different solutions of the form $P(t) = Ce^{kt}$ —one for each choice of the "arbitrary" constant C. This is typical of differential equations in general. It is also fortunate, because it allows us to use additional information to select from all the solutions a particular one that fits the situation under study.

EXAMPLE 4 Suppose that P(t) is the population of a bacterial colony at time t, that the population at time t = 0 (hours, h) was 1000, and that the population doubled after 1 h. This additional information about the function P(t) yields the following equations:

$$1000 = P(0) = Ce^0 = C,$$

 $2000 = P(1) = Ce^k.$

It follows that C = 1000 and that $k = \ln 2$. Thus the function P(t) describing the population of this particular bacterial colony is known exactly:

$$P(t) = 1000e^{t \ln 2}$$
.

Therefore, we can predict the population at any future time; for example, the population at time t = 90 minutes (min) (1.5 h) will be $P(1.5) = 1000e^{(1.5)\ln 2}$, or about 2828 bacteria.

The condition P(0) = 1000 is called an **initial condition** because we normally write differential equations for which t = 0 is the starting time. Figure 1.1 shows a number of graphs of the form $P(t) = Ce^{kt}$ for which $k = \ln 2$. The graphs of all the solutions of $dP/dt = (\ln 2)P$ in fact fill up the entire two-dimensional plane, and no two intersect. Moreover, the selection of any point on the P-axis amounts to a determination of the value P(0). Because exactly one solution passes through each such point, we see in this case that an initial condition $P(0) = P_0$ may determine a unique solution agreeing with known data.

It is possible that none of these solutions fits the known information. In such a case we must suspect that the differential equation—a mathematical model of the physical phenomenon in question—may not adequately describe the real world. The solutions of Eq. (2) are of the form $P(t) = Ce^{kt}$ where C is a positive constant, but for no choice of the constants k and C does P(t) accurately describe the actual growth of the human population of the world over the past hundred years. We must therefore write a more complicated differential equation, one that takes into account the effects of population pressure on the birth rate, the declining food supply, and other factors. This should not be regarded as a failure of the model of Example 2, but as an insight into what additional factors must be considered in studying the growth of populations. Indeed, Eq. (2) is quite accurate under certain circumstances—for