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J. J. STOKER
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Introduction

During the past twenty years the nonlinear problems of mechanics and mathematical physics have been vigorously attacked by scientific workers having a wide variety of aims and purposes: one need only recall, for example, the advances made in fluid dynamics—particularly in gas dynamics—and in plasticity and nonlinear elasticity. The purpose of the present book is to give an account of another small segment of the general field of nonlinear mechanics, i.e., the field of nonlinear vibrations. Actually, the title of the book is rather too inclusive since only systems with one degree of freedom are treated, but this is rather natural because of the fact that more general systems have been studied very little.

It is perhaps worth while to consider for a moment the reasons why one should be interested particularly in the nonlinear problems of mechanics. Basically the reason is, of course, that practically all of the problems in mechanics simply *are* nonlinear from the outset, and the linearizations commonly practiced are an approximating device which is often simply a confession of defeat in the face of the challenge presented by the nonlinear problems as such. It should be added at once that such linearizations as a means of approximation have been and always will be valuable—in fact completely sufficient—for many purposes. However, there are also many cases in which linear treatments are not sufficient. For example, if the oscillations of an elastic system result in amplitudes which are not very small, then the linear treatment may be simply too inaccurate for the purposes in view. In such cases the accuracy can often be improved sufficiently by carrying out further approximations of the same sort as are involved in linearizations. However, it happens frequently that *essentially new phenomena* occur in nonlinear systems which can not in principle occur in linear systems. A familiar example of such an essentially nonlinear phenomenon in the field of gas dynamics is the building up of a discontinuous shock wave from a smooth wave. In nonlinear vibrations examples of the same sort are the occurrence of *subharmonic* forced oscillations in a wide variety of systems, the occurrence of systems (the so-called self-excited systems) in which a *unique* periodic free oscillation occurs, and the occurrence of what are called *combination tones*. In this book, the principal aim is not so much to introduce methods of improving the accuracy obtainable by linearization, but rather to focus the attention sharply on those features of the problems in which the nonlinearity results in distinctive new phenomena. One of the most attractive features of the subject of nonlinear vibrations is the existence of a surprisingly wide variety of such distinctive new phenomena; and, what is perhaps still more surprising, these phenomena can be treated by methods which are interesting and instructive in themselves without being difficult, and which do not require the use of sophisticated mathematics.

This book has been written with the needs and interests of several classes of readers in mind. To begin with, the author has wished to present the underlying principles and theory in such a way that they can be easily understood by engineers and physicists whose primary interest is in applying the ideas and methods to concrete physical problems. The author has the impression that the existing fund of knowledge in nonlinear vibration theory has not been used in practice as much as it could and would be used if engineers and physicists were more familiar with it. In view of the author's own general interests—not to say biases—it would be strange if the needs and interests of applied mathematicians were to be neglected. For this class of readers the author has emphasized the various known types of physical problems in the field which lead up to the questions of mathematical interest, and on the other hand carried out detailed treatments, particularly in the Appendices, of a variety of important problems of a mathematical character, some of which constitute results achieved only in the last few years. Thus the book might be hoped to serve those readers who wish to be brought up to the borderline where new discoveries are being made. At the same time these readers are supplied both implicitly and explicitly with hints regarding new problems to be tackled and with a number of ideas and methods that could perhaps be used to solve them. The author's acquaintance with the material of this book arose through seminars and lecture courses conducted from time to time at New York University over a period of nearly ten years. The subject has invariably proved to be interesting and stimulating to the students. Consequently the author hopes that the book may prove useful to other colleagues in the teaching profession who may wish to conduct similar seminar or lecture courses in nonlinear vibrations, or who may wish to supplement a course—even a quite elementary course—on ordinary differential equations with some of the striking illustrative material which occurs in such profusion in the field of nonlinear vibrations.

In attempting to please too many classes of readers the author incurs the well-known risk of pleasing none of them. In the present case, however, this risk is to a large extent obviated by the character of the material itself. While the main types of problems can be treated rather satisfactorily with the knowledge and use of little more mathematics than elementary differential equations, the problems, if investigated thoroughly and deeply, lead at once to questions of great subtlety and interest from the mathematical point of view—a case in point is the so-called "difficulty with the small divisors" which occurs as soon as one treats the problem of combination tones.

Before outlining more precisely the actual contents of the book, it should be said that the book is not intended to furnish a complete survey of the more recent literature in the field. This is particularly true with regard to the Russian literature, which is well known to be important and extensive. Fortunately, the recent book by N. Minorsky (*Introduction to Non-linear Mechanics*), and the translations by S. Lefschetz of books by Kryloff and Bogoliuboff and by Andronow and Chaikin, go far toward making this literature available. One omission which the author regrets is the theory of Liapounoff for the discussion of stability questions; but to deal adequately with this theory requires more space than would be reasonable in a book like the present one.

The source of practically all the basic mathematical ideas and also the techniques in nonlinear vibrations is the work of Poincaré, while the specific basic physical problems treated at present in nonlinear vibrations were introduced by Rayleigh, van der Pol, and Duffing. The object of the present book is to give a connected and systematic account of this work, which includes most of what had been done up to about 1930. The author would not presume to give a survey of outstanding work done after 1930 in the field. Nevertheless, a considerable amount of quite recent work is discussed in the book: the work of Levinson and Smith on the existence and uniqueness of the periodic solution in a very general case of the self-excited type, and the quite recent work of Haag and Dorodnitsyn on asymptotic developments for the period and other quantities associated with relaxation oscillations, are examples in point.

There are six chapters in the main body of the book, and six Appendices. The first five chapters of the book are elementary in character, the sixth chapter is rather less elementary, while the Appendices are on the whole not elementary, containing, as they do, rigorous existence and uniqueness proofs.

Chapter I consists of a short, though fairly complete, summary of the theory of linear vibrations for a system of one degree of freedom having constant characteristics. This chapter serves both for reference and for contrast with the results of the nonlinear theory. Chapter II treats easily integrable nonlinear systems in which no external forces depending on the time occur. A considerable number of physical problems illustrating the material is given, and a first glimpse of the advantages to be gained by working geometrically in the phase plane is achieved. In Chapter III free oscillation problems of a type like those of Chapter II, but not so easily integrable, are studied in detail by working in the phase plane, again with reference to a variety of physical problems. In the course of this chapter the graphical method of Liénard is introduced and applied, the theory due to Poincaré of singularities of first order differential equations is developed in order to obtain the criteria for their classification into types, and the notion of the index of a singularity is introduced. Finally, the usefulness of these ideas is illustrated by solving concretely a number of physical problems. One of these is a problem in elastic stability, which is treated dynamically. Another is the interesting problem of the pull-out torques of a synchronous motor, which is given a detailed treatment.

In Chapter IV problems of nonlinear *forced* oscillations are taken up and specialized to the cases in which the nonlinearity is provided solely by the "elastic" restoring force. This is the type of problem first treated with significant results by Duffing. Such problems arise any time that a system with elastic restoring forces is subjected to periodic external forces which cause sufficiently large amplitudes. The "response curves" are first studied—that is, the curves showing the amplitude of the forced oscillation as a function of the frequency for a given amplitude of the external force. In this connection the problem of hunting of a synchronous motor is considered. A curious "jump phenomenon" is studied which has often been observed experimentally. The effect of viscous damping is considered. The occurrence of subharmonic oscillations is discussed. The problem of the occurrence of combination tones is considered, and the general problem of the stability of the periodic oscillations is formulated. However, this chapter has another purpose aside from presenting the solutions to certain specific problems, and which is perhaps even more important. That purpose is the presentation and discussion of a considerable *variety of analytic methods* useful in treating the periodic solutions. In particular, iteration and perturbation methods are explained and applied to obtain directly the solutions of the differential equations; both methods are also applied more indirectly as a means of determining the coefficients of the Fourier series developments of the solutions. Thus the same problem is sometimes treated a number of times by different methods in this chapter. Special mention might be made of the iteration method of Rauscher, which is also explained in this chapter. At the end of the chapter a table (similar to one in the little book of Duffing) is given in which the contrasts between linear and nonlinear systems are pointed out. In the more recent literature on nonlinear vibrations the problems of the type discussed in this chapter are usually treated rather summarily, if at all, consequently it was felt that a detailed treatment might be found useful.

Chapter V is devoted entirely to problems in which the nonlinearity occurs in the "damping" terms (i.e., the terms depending on the velocity, rather than on the displacement) in such a way as to cause what are called self-excited or self-sustained oscillations. Systems of this kind are very common in nature: they occur always in fact when a periodic motion is maintained through absorption of energy from a constant flow of energy. The best known and most important technical applications occur in electrical systems containing vacuum tubes, in which the energy for the oscillations is supplied by a direct current source. Oscillations of the same type occur frequently also in mechanical and acoustical systems. In fact, Rayleigh probably pointed out the first example of the sort in the case of the production of a sustained note from a violin string caused by bowing it. The failure of the Tacoma bridge a few years ago is generally ascribed to a particularly heavy self-excited oscillation in which the constant energy source was the wind. The flutter of airplane wings is another example of the same kind.

Chapter V is divided into two parts. In the first part a number of electrical and mechanical systems which lead to self-excited oscillations are studied in some detail, and the corresponding differential equations are derived. The remainder of this part is concerned with an analysis of the free oscillations, i.e., those which occur without the action of external exciting forces that depend on the time. This part could thus have been placed in Chapter III, but the importance of the specific problems warrants a separate treatment. The basic occurrence, from the mathematical point of view, is that of limit cycles in the phase plane in the sense of Poincaré. In the simplest case just one such stable limit cycle occurs, and this in turn means that all motions tend to a unique periodic motion. This is in the strongest contrast with the behavior of the free oscillations in systems with nonlinear restoring forces, in which a whole family of free oscillations occurs. If the nonlinearity is small (in other words, if the oscillation is in the neighborhood of the linear oscillation) the problems can be and are treated by the perturbation method. However, the cases in which the departure from linearity is large—even very large—are of particular interest in this class of problems. In such cases the resulting oscillations are of a jerky, not to say discontinuous, character. They are often given the name relaxation oscillations. It is comparatively easy to obtain the lowest order term in an asymptotic development for the period of such a relaxation oscillation in terms of a parameter characterizing the departure from nonlinearity, but it is not easy to obtain higher order terms. Unfortunately it turns out that the higher order terms yield rather large contributions in cases in which the oscillation is markedly of the relaxation type, so that it is important to have a means of calculating them. Such a means has been devised quite recently by Dorodnitsyn, for the van der Pol equation, and more generally by Haag. The details of the complete asymptotic development are carried out in the first part of Chapter V for one relatively simple case.

The second part of Chapter V is concerned with forced oscillations of systems whose free oscillations are of the self-sustained type. It is assumed in all cases that the oscillations do not depart too much from the linear oscillations. This theory was created by van der Pol, who invented a special analytical mode of attacking the problems which is different from any of the several methods treated in Chapter IV. Van der Pol's method is the method used almost exclusively by the Russian writers—particularly Kryloff and Bogoliuboff—for solving all of the various types of problems that involve the time explicitly in the differential equation. The author believes, however, that this method, while it is particularly well suited to deal with the problems treated in this part of Chapter V, is not necessarily the simplest or the most straightforward method of treating other problems. The theory of van der Pol is developed with reference to a particular electric circuit containing a triode vacuum tube and a source of alternating current. In treating the response phenomena, i.e., the amplitude of the periodic response as a function of the frequency and amplitude of the excitation, the elegant variant of van der Pol's method introduced by Andronow and Witt is employed in order to study the stability of all possible periodic oscillations having the frequency of the excitation. This is done by reducing the problem to one of classifying singularities of the first order differential equation of van der Pol in accordance with the criteria of Poincaré derived in Chapter III, and this in turn is made feasible by the fact that to each periodic oscillation there is a corresponding type of singularity. This idea of Andronow and Witt also yields more than the criteria for the stability of the oscillations. By means of it one is led to the possibility of the occurrence of combination oscillations, i.e., oscillations that are the sum of two oscillations, one with the frequency of the excitation, the other with a frequency close to that of the free nonlinear oscillation. Such combination oscillations are correlated with the presence of Poincaré limit cycles of the first order differential equation just mentioned above. The methods used can be extended to prove that the combination oscillations for sufficiently large detuning, i.e., for sufficiently large differences between the frequencies of the free and the forced oscillations, are unique and stable. The circumstances which may occur when the detuning is neither very small nor very large are quite complicated; some description of the phenomena in such cases—which include jump phenomena of various sorts reminiscent of similar phenomena studied in Chapter IV—is given following recent work of Cartwright and Littlewood.

The final Chapter VI of the book returns once more to linear oscillations, but this time the characteristics of the systems treated are assumed to be not constant, as was the case in Chapter I, but rather periodic in the time. As a consequence the differential equations become of the type called Hill's equations. There are several reasons which dictated the inclusion of a lengthy chapter on linear systems in a book devoted primarily to nonlinear systems. In the first place, the treatment of the important question of stability of any periodic nonlinear oscillation leads inevitably to such Hill's equations. Second, the vibration phenomena encountered in systems of this type have features—the occurrence of vibrations somewhat like the subharmonics, for example—which place them, in a sense, in a position between those of nonlinear systems and of linear systems having constant characteristics. A few mechanical and electrical problems leading to Hill's equation are first discussed. Then follows an account of the Floquet theory for linear differential equations with periodic coefficients. For the study of the stability of a given periodic nonlinear oscillation it is necessary to determine whether the solutions of a certain Hill's equation are all bounded or not when certain parameters in the Hill's equation are given. The question of separating the "stable" from the "unstable" parameter values is discussed in detail for the most important special case, the Mathieu equation. These results are then applied to test the stability of the forced oscillations of the Duffing equation, which were treated in Chapter IV, with results the same as were advanced in Chapter IV as the result of plausible physical arguments.

As has been indicated earlier, the Appendices are devoted to a number of mathematical questions which, however, are in some instances also of interest from a practical point of view. Appendix I gives a rigorous treatment of the perturbation method in general as applied to periodic oscillations in the neighborhood of a linear oscillation. Again the basic idea used is due to Poincaré. The general theory is then applied to prove the existence of the perturbation series for all cases treated in this book. In the course of satisfying the conditions needed to ensure the existence of the solutions, one finds that important clues regarding the manner of interpreting the results are uncovered, and that insights are gained regarding the appropriate means of calculating the solutions concretely in the various cases. In Appendix II the existence of combination oscillations of certain systems with nonlinear restoring forces and with viscous damping is proved by obtaining a convergent perturbation series. This result is included as a contrast to the corresponding case in which no damping occurs, when the famous "difficulty of the small divisors" alluded to above makes it practically certain that no convergent perturbation series could be obtained. In Appendix III the existence of a limit cycle in certain rather general systems of the self-sustained type is proved by using a topological method involving the proof of the existence of a fixed point of a certain mapping. This Appendix is modeled somewhat on the work of Levinson and Smith, but a less general case is treated. Appendix IV gives a rigorous proof for the commonly conjectured character of the relaxation oscillation when the departure from nonlinearity becomes very large. Appendix V gives a derivation of the criterion of Poincaré for the stability of a limit cycle, or orbital stability, as it is called. Finally, in Appendix VI, the uniqueness of the limit cycle for the system treated in Appendix III is proved (following the idea of Levinson and Smith) by showing that all possible limit cycles are stable.

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The greater part of what the author has put into this book was learned in the course of seminars conducted in collaboration with his friend and colleague, Professor K. O. Friedrichs. It has been the author's hope that the writing of the book would be a joint enterprise, but this proved impossible. Nevertheless, the author has had the benefit of critical comments by Professor Friedrichs on many parts of the book.

This book is dedicated to R. Courant as a token of esteem and friendship, and as an acknowledgment of the strong influence he has had in the author's scientific development.

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