MEMOIRS

OF THE

AMERICAN MATHEMATICAL SOCIETY

NUMBER 22

A GLOBAL FORMULATION OF THE LIE THEORY OF TRANSPORTATION GROUPS

BY

RICHARD S. PALAIS

1956

A GLOBAL FORMULATION OF THE LIE THEORY OF TRANSFORMATION GROUPS

Richard S. Palais
Harvard University

Preface

The goal of this memoir is to formulate in a modern global way the theory, due in its local form to Sophus Lie, which connects Lie algebras of vector fields on a differentiable manifold with local groups and groups of transformations acting on the manifold.

Chapter I is preliminary to the main trend of the memoir and is concerned with the question of giving a natural 'quotient' differentiable structure to the set of leaves of an involutive differential system. I have decided to develop this separately, rather than in context with its application to transformation groups, since I feel that it may be of some independent interest.

In chapter II we develop the theory of infinitesimal and local transformation groups in its greatest generality. Aside from proving the basic tool theorems that will be needed in the following mor special-ized chapters, we give a uniqueness theorem for a local transformation group with a given domain and given infinitesimal generator and also a global form of Lie's Second Fundamental Theorem (Hauptsatz der Gruppen-theorie).

In chapter III we characterize in a number of ways the class of infinitesimal transformation groups which generate global transformation groups. In chapter IV we use the results of chapter III to develop a Lie theory connecting the Lie algebra of differentiable vector fields on a

Received by the editors November 19, 1956.

manifold with the group of differentiable homeomorphisms of the manifold and use this to study the automorphisms of a structure given by a manifold and a set of tensor fields on the manifold.

Acknowledgments

I should like foremost to express my appreciation to Professor A. M. Gleason. My many discussions with him concerning the contents of this memoir were always helpful, and at several points his suggestions proved to be the factor that allowed me to surmount serious obstacles.

To my wife, Ellie, my thanks for her expert and dedicated work in typing manuscript.

Finally, I should like to thank the National Science Foundation for their fellowship support.

Contents

Chapter I

QUOTIENT MANIFOLDS DEFINED BY FOLIATIONS

A completely integrable differential system Θ on a differentiable manifold M defines a partitioning (foliation) of M into maximal connected integral manifolds (leaves) of Θ. In this chapter we investigate under what conditions the quotient space admits a natural manifold structure, and the elementary properties of the quotient manifolds that result.

1. Differentiable Manifolds.

We will use the word 'differentiable' as a substitute for 'C^∞' or 'analytic' in contexts where both of the latter would be appropriate, in order to avoid having to give separate proofs for the C^∞ case and the analytic case of various theorems.

In order to get a smooth theory of quotient manifolds it is expedient to drop the Hausdorff separation axiom in the definition of a manifold. When this is done it is possible to modify the definition of a manifold in terms of overlapping coordinate systems in such a way that the topology of the manifold is a derived concept. Since there are several novel points in this approach we will explain it briefly and at the same time develop the notation we will need.

The reader familiar with the work of Ehresmann will recognize the debt the author owes to this pioneer in manifold theory, both in concepts and in terminology. It is a debt which we gratefully acknowledge.

We denote real Euclidian n-space by R^n and by $u_1 \dots u_n$ we denote the natural coordinates on R^n. If M is a set, an <u>n-dimensional chart</u> in M is a one-to-one map φ of a subset of M onto an open subset of R^n. A real-valued function f with domain $S \subseteq M$ is said to be

<u>differentiable with respect to</u> φ at a point $p \in S$ if $p \in (\text{domain } \varphi)$ and there is a differentiable function g defined in a neighborhood N of $\varphi(p)$ such that $f \circ \varphi^{-1} \upharpoonright N = g \upharpoonright \varphi(S)$, i.e. $f \circ \varphi^{-1}$ and g agree where both are defined. Two n-dimensional charts in M, φ and ψ, with domains U and V respectively, are said to be <u>differentiably related</u> if each maps $U \cap V$ onto an open set and the mappings $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are differentiable. If this is the case and f is a real-valued function in M and $p \in U \cap V$ then the differentiability of f at p with respect to φ and with respect to ψ are equivalent.

An <u>n-dimensional differentiable atlas for</u> M is a set of mutually differentiably related n-dimensional charts in M whose domains cover M. An n-dimensional differentiable atlas for M is called <u>complete</u> if it is not a proper subset of an n-dimensional differentiable atlas for M. An <u>n-dimensional differentiable manifold</u> is a pair (M, Ψ) where M is a set (called the point set of the manifold) and Ψ is a complete n-dimensional differentiable atlas for M (called the atlas of the manifold). If Φ is any n-dimensional differentiable atlas for a set M, then the set Ψ of ψ such that $\Phi \cup \{\psi\}$ is an n-dimensional differentiable atlas for M is the unique complete n-dimensional differentiable atlas including Φ. It is called the <u>complete differentiable atlas associated with</u> Φ and (M, Ψ) is called the <u>differentiable manifold defined by</u> Φ.

If (M, Φ) is an analytic manifold then Φ is a C^∞ atlas for M. If Ψ is the complete C^∞ atlas associated with Φ then (M, Ψ) is called the C^∞ <u>manifold associated with</u> (M, Φ).

If (M, Ψ) is an n-dimensional differentiable manifold then the domains of the charts in Ψ form a base for a topology $\mathcal{J}$, called the <u>manifold topology</u> of (M, Ψ), and $(M, \mathcal{J})$ is called the underlying topological space of (M,Ψ). $\mathfrak{J}$ is the weakest topology for M rendering each $\psi \varepsilon \Psi$ continuous, and with respect to $\mathfrak{J}$ each $\psi \varepsilon \Psi$ is a homeomorphism. It follows that $\mathfrak{J}$ is a T_1 topology for M; it need not be a T_2 topology but if it is we call (M,Ψ) a Hausdorff differentiable manifold. Similarly all adjectives conventionally applied to $(M,\mathfrak{J})$ will be applied to (M,Ψ), e.g. (M,Ψ) will be called a compact or a connected differentiable manifold if $\mathfrak{J}$ is a compact or connected topology for M. A real-valued function in M with domain S is called differentiable at $p \varepsilon M$ if for some $\psi \varepsilon \Psi$ (and then automatically for all $\psi' \varepsilon \Psi$ with $p \varepsilon (\text{domain } \psi')$) f is differentiable at p with respect to ψ. We call f differentiable on $S' \subseteq M$ if it is differentiable at each point of S', and differentiable in M if it is differentiable on S. In the latter case f is continuous.

A coordinate system for the n-dimensional differentiable manifold (M,Ψ) is an ordered n+1-tuple $(x_1 \ldots x_n, \mathcal{O})$ such that $\mathcal{O}$ is the domain of a chart $\psi \varepsilon \Psi$ and the x_i are real-valued functions in M such that $x_i \restriction \mathcal{O} = u_i \circ \psi$. If f is a real-valued function in M then $f \circ \psi^{-1}$ is called the expression for f in terms of the coordinate system $(x_1 \ldots x_n, \mathcal{O})$. We shall say that $(x_1 \ldots x_n, \mathcal{O})$ is a cubical coordinate system of breadth 2a centered at $p \varepsilon M$ if $\psi(p) = (0 \ldots 0)$ and $\psi(\mathcal{O}) = \{(t_1 \ldots t_n) \varepsilon R^n : |t_i| < a\}$. In this case if $|t_{m+i}| < a$ $i = 1 \ldots n-m$ then we call $\Sigma_t = \{q \varepsilon \mathcal{O} : x_{m+i}(q) = t_{m+i}\}$ the m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$ defined by $t = (t_{m+1} \ldots t_n)$. The mapping $\varphi : p \to (x_1(p) \ldots x_m(p))$ is an m-dimensional chart in Σ_t and $\{\varphi\}$ is an m-dimensional differentiable atlas for Σ_t. We shall often refer to Σ_t as an m-dimensional differentiable manifold, meaning the manifold defined by $\{\varphi\}$.

Let (M,Ψ) be a differentiable manifold and $p \varepsilon M$. For the moment denote by $\mathcal{Q}(p)$ the class of real-valued functions with domain open in M and differentiable at p . Then the notion of tangent vector at p can be defined exactly as in [1 Chapter III] . Formula (1) of [1 page 77] is proved for the C^{∞} case in [2] . Except for this all properties of the tangent space etc. can be proved exactly as in [1] . The following additional elementary concepts are treated in [1] and the reader will be assumed to be familiar with them: (differentiable) vector field, bracket of two differentiable vector fields, differential of a differentiable function f (denoted by df), differentiable mapping F of one manifold into another and the differential of such a mapping (denoted by δF). F is called non-singular at p if δF maps the tangent space at p one-to-one.

Let (M,Ψ) and (N,Φ) be differentiable manifolds with $N \subseteq M$ and let i be the inclusion map of N in M . We say that (N,Φ) is a <u>differentiable submanifold</u> of (M,Ψ) if i is differentiable and everywhere non-singular. If moreover i is a homeomorphism into with respect to the respective manifold topologies then (N,Φ) is said to be <u>regularly imbedded in</u> (M,Ψ); and if further N is a closed subspace of M with respect to the manifold topology of (M,Ψ) then (N,Φ) is called a <u>closed submanifold of</u> (M,Ψ) . We identify the tangent space of the submanifold (N,Φ) at a point $p \varepsilon N$ with its image under δi (a subspace of the tangent space to (M,Ψ) at p) via the linear isomorphism given by δi .

If (M,Ψ) is a differentiable manifold, $\mathcal{O}$ a subset of M open with respect to the manifold topology and if $\Psi_{\mathcal{O}} = \{\psi \varepsilon \Psi : \text{domain } \psi \subseteq \mathcal{O}\}$ then $(\mathcal{O},\Psi_{\mathcal{O}})$ is a regularly imbedded differentiable submanifold of (M,Ψ) called the <u>open submanifold defined by</u> $\mathcal{O}$.

Let (M,Ψ) and (N,Φ) be manifolds. Following Ambrose we call a one-to-one map F of M onto N a diffeomorphism of (M,Ψ) onto (N,Φ) if F and F^{-1} are differentiable or, equivalently, if $\varphi \to \varphi \circ F$ is a one-to-one correspondence of Φ with Ψ. A mapping F defined near $p \in M$ and into N will be called a local diffeomorphism of (M,Ψ) into (N,Φ) at p if it maps an open submanifold of (M,Ψ) containing p diffeomorphically onto an open submanifold of (N,Φ). By the implicit function theorem a necessary and sufficient condition for this is that F be differentiable at p and δF map the tangent space to (M,Ψ) at p isomorphically onto the tangent space to (N,Φ) at $F(p)$. If $F : M \to N$ is a local diffeomorphism of (M,Ψ) into (N,Φ) at each point of M we call F a local diffeomorphism of (M,Ψ) into (N,Φ).

Whenever no confusion will result (i.e. when a single complete atlas Ψ is being considered) we will use the symbol M alone to denote a manifold (M,Ψ), its underlying point set and underlying topological space.

2. Foliations.

Let M be an n-dimensional differentiable manifold. We use the term m-dimensional differential system on M for what Chevalley [1 page 86] calls an m-dimensional distribution on M, i.e. a mapping Θ which assigns to each $p \in M$ an m-dimensional subspace Θ_p of the tangent space to M at p. A vector field L in M will be said to belong to Θ if for each p in the domain of L, $L_p \in \Theta_p$. The differential system Θ will be called differentiable if for each $p \in M$ there is a neighborhood $\mathcal{O}$ of p and m differentiable vector fields $L_1 \dots L_m$ defined in $\mathcal{O}$ such that $(L_1)_q \dots (L_m)_q$ is a base for Θ_q at each $q \in \mathcal{O}$. Θ is called involutive if it is differentiable and if whenever X and Y are two differentiable vector fields in M with the same domain, both belonging to Θ, their bracket $[X,Y]$ also belongs to Θ. A submanifold N of M will be called an integral manifold of the differential system Θ on M if for each point $p \varepsilon N$ the tangent space to N at p is included in Θ_p.

If Θ is an m-dimensional differential system on M, a coordinate system $(x_1 \ldots x_n, \mathcal{O})$ will be called <u>flat with respect to</u> Θ if for each $q \varepsilon \mathcal{O}$ $(X_1)_q \ldots (X_m)_q$ is a base for Θ_q, where $X_i = \partial / \partial x_i$. If $(x_1 \ldots x_n, \mathcal{O})$ is a cubical coordinate system for M then a necessary and sufficient condition that it be flat with respect to Θ is that each of its m-dimensional slices be an integral manifold of* Θ.

THEOREM I. *If Θ is an m-dimensional differential system on M then a necessary and sufficient condition that Θ be involutive is that for each $p \varepsilon M$ there is a cubical coordinate system centered at p and flat with respect to Θ.*

PROOF. Since the property of being involutive is local it suffices to prove the theorem in the case that M is Hausdorff. The proof is given in [1 page 89] for the analytic case and as the proof depends only on the implicit function theorem and the existence and uniqueness theorems for differential equations (which have exact C^∞ analogues), the same proof works in the C^∞ case.

COROLLARY. *Let Θ be an m-dimensional involutive differential system in the n-dimensional differentiable manifold M. If $p \varepsilon M$ then the set of domains of cubical coordinate systems centered at p and flat with respect to Θ form a basis of neighborhoods of p with respect to the manifold topology for M.*

PROOF. Let $(x_1 \ldots x_n, \mathcal{O})$ be a cubical coordinate system centered at p of breadth 2a. Then for any $b < a$ if $\mathcal{O}_b = \{q \varepsilon \mathcal{O} : |x_i(q)| < b\}$

then $(x_1 \dots x_n, \mathcal{O}_b)$ is a cubical coordinate system centered at p and flat with respect to θ, and the $\mathcal{O}_b$ are a basis of neighborhoods for p.

THEOREM II. *Let θ be an m-dimensional involutive differential system on an n-dimensional differentiable manifold M. Let $(x_1 \dots x_n, U)$ and $(y_1 \dots y_n, V)$ be cubical coordinate systems in M flat with respect to θ and let $p \varepsilon U \cap V$. Then there is a diffeomorphism*
$f : (t_{m+1} \dots t_n) \to (f_{m+1}(t_{m+1} \dots t_n) \dots f_n(t_{m+1} \dots t_n))$ of a neighborhood of $(y_{m+1}(p) \dots y_n(p))$ in R^{n-m} onto a neighborhood of $(x_{m+1}(p) \dots x_n(p))$ in R^{n-m} such that $x_{m+i}(q) = f_{m+i}(y_{m+1}(q) \dots y_n(q))$ for all $q \varepsilon \mathcal{O}$ = component of p in $U \cap V$. Moreover if Σ is the m-dimensional slice of $(x_1 \dots x_n, U)$ defined by $(x_{m+1}(p) \dots x_n(p))$ and Σ' is the m-dimensional slice of $(y_1 \dots y_n, V)$ defined by $(y_{m+1}(p) \dots y_n(p))$ then

$$\varphi : \Sigma \to R^m \quad q \to (x_1(q) \dots x_m(q)) \quad \text{and}$$
$$\psi : \Sigma' \to R^m \quad q \to (y_1(q) \dots y_m(q))$$

are differentiably related m-dimensional charts in M.

PROOF. Let g_i be the expression for x_i in terms of the coordinate system $(y_1 \dots y_n, V)$. Then

$$(dx_{m+i})_q = \sum_{j=1}^{n} (\partial g_{m+i} / \partial u_j)(y_1(q) \dots y_n(q))(dy_j)_q \quad \text{for } q \varepsilon U \cap V .$$

Since $(x_1 \dots x_n, U)$ and $(y_1 \dots y_n, V)$ are both flat with respect to θ, $((dx_{m+1})_q \dots (dx_n)_q)$ and $((dy_{m+1})_q \dots (dy_n)_q)$ are both bases for the annihilator of θ_q for $q \varepsilon U \cap V$ and hence
$(\partial g_{m+i} / \partial u_j)(y_1(q) \dots y_n(q)) = 0$ for $j \leq m$. If $\tilde{\mathcal{O}}$ is the image of $\mathcal{O}$ under the map $q \to (y_1(q) \dots y_n(q))$ it follows that the g_{m+i}

are independent of their first m arguments, that is if $\hat{\mathcal{O}}$ is the image of $\tilde{\mathcal{O}}$ under the map $\Pi : (t_1 \dots t_n) \to (t_{m+1} \dots t_n)$ then there are differentiable functions $f_{m+1} \dots f_n$ on $\hat{\mathcal{O}}$ such that $g_{m+i} = f_{m+i} \circ \Pi$. Then $x_{m+i}(q) = f_{m+i}(y_{m+1}(q) \dots y_n(q))$ for $q \varepsilon \mathcal{O}$ hence $(dx_{m+i})_p = \sum_{j=1}^{n-m} (\partial f_{m+i}/\partial u_j)(y_{m+1}(p) \dots y_n(p))(dy_{m+j})_p$ and since the $(dx_{m+i})_p$ are linearly independent it follows that $\det (\partial f_{m+i}/\partial u_j)(y_{m+1}(p) \dots y_n(p)) \neq 0$. By the implicit function theorem the mapping $f : (t_{m+1} \dots t_n) \to (f_{m+1}(t_{m+1} \dots t_n) \dots f_n(t_{m+1} \dots t_n))$ is a local diffeomorphism at $(y_{m+1}(p) \dots y_n(p))$.

In $\Sigma' \cap \mathcal{O}$ we have $x_{m+i}(q) = f_{m+i}(y_{m+i}(q) \dots y_n(q)) = f_{m+i}(y_{m+1}(p) \dots y_n(p)) = x_{m+i}(p)$ so $\Sigma' \cap \mathcal{O} \subseteq \Sigma$. Now ψ is an open mapping and $\Sigma' \cap \mathcal{O}$ is open in Σ' hence $\psi(\Sigma) \supseteq \psi(\Sigma' \cap \mathcal{O})$ is a neighborhood of $\psi(p)$. It follows that $\psi(\Sigma)$ is an open subset of R^m. Defining $\bar{g}_i(t_1 \dots t_m) = g_i(t_1 \dots t_m, y_{m+1}(p) \dots y_n(p))$ on $\psi(\Sigma' \cap \mathcal{O})$ we have for $q \varepsilon \Sigma' \cap \mathcal{O}$ $u_i \circ \varphi(q) = x_i(q) = \bar{g}_i(y_1(q) \dots y_m(q)) = \bar{g}_i(\psi(q))$ or $u_i \circ \varphi \circ \psi^{-1} = \bar{g}_i$. Since the $\bar{g}_i$ are clearly differentiable this shows that $\varphi \circ \psi^{-1}$ is a differentiable map. Similarly $\varphi(\Sigma')$ is open and $\psi \circ \varphi^{-1}$ is a differentiable map so φ and ψ are differentiably related.

DEFINITION I. Let Θ be an m-dimensional involutive differential system on an n-dimensional differentiable manifold (M, Ψ) and let $(x_1 \dots x_n, \mathcal{O})$ be a cubical coordinate system for M flat with respect to Θ. If Σ is any m-dimensional slice of $(x_1 \dots x_n, \mathcal{O})$ the mapping $q \to (x_1(q) \dots x_m(q))$ of Σ into R^m is called a <u>leaf chart for M with respect to Θ</u>. By theorems I and II the set of all leaf charts for M with respect to Θ form an m-dimensional differentiable atlas for M. Let (M,Φ) be the m-dimensional differentiable manifold it defines (i.e. Φ is the complete atlas containing all leaf charts). Then (M,Φ) is called the <u>maximum integral manifold of</u> Θ. A connected component of M with respect to the manifold topology of (M,Φ) regarded as an open submanifold of (M,Φ) is called a <u>leaf</u> of Θ. We call the set of leaves of Θ the <u>foliation defined by</u> Θ and denote it by M/Θ. We denote by Π_Θ the quotient mapping of M onto M/Θ which carries $p \in M$ onto the leaf of Θ containing p. A subset of M is called <u>saturated</u> (with respect to Θ) if it is the union of leaves of Θ, and if $S \subseteq M$ the <u>saturation</u> of S is $\Pi_\Theta^{-1}(\Pi_\Theta(S))$. The <u>quotient topology</u> for M/Θ is the strongest topology which makes Π_Θ continuous; equivalently its open sets are the images of saturated open sets of M under Π_Θ.

We note that it is almost immediate from the definition of (M,Φ) that (M,Φ) is an m-dimensional integral manifold of Θ and that any integral manifold of Θ is a submanifold of (M,Φ) so the name maximum integral manifold of Θ is justified. It follows that a connected m-dimensional integral manifold of Θ (and in particular an m-dimensional slice of a cubical coordinate system for (M,Ψ) flat with respect to Θ) is an open submanifold of a leaf of Θ. The fact that (M,Φ) is a submanifold of (M,Ψ) implies in particular that the manifold topology for (M,Φ) is stronger than the manifold topology for (M,Ψ), hence if (M,Ψ) is a Hausdorff manifold so is (M,Φ), however, even in this case M/Θ need not be a Hausdorff space in the quotient topology as we shall see later by example.

The task we set ourselves is, picturesquely, to 'factor' our n-dimensional manifold (M,Ψ) into an m-dimensional manifold (M,Φ) 'parallel' to Θ and an n-m dimensional quotient manifold M/Θ 'transverse' to Θ. The first part of this task, which is classical, or at least well-known [1 Chapt. III § VIII] and [3], has been accomplished above. The second part, namely putting a natural n-m dimensional differentiable manifold structure on M/Θ cannot always be accomplished and we investigate below the condition under which it can.

3. The Continuation Theorem.

THEOREM III. Let Θ be an involutive m-dimensional differential system on an n-dimensional differentiable manifold M and let $(x_1 \ldots x_n, \mathcal{O})$ be a cubical coordinate system centered at p and flat with respect to Θ. Let q be a point of the leaf $\Sigma = \Pi_\Theta(p)$ of Θ containing p and $(y_1 \ldots y_n, U)$ a cubical coordinate system flat with respect to Θ such that q is on the m-dimensional slice defined by $(0 \ldots 0)$. Then there is an $\varepsilon > 0$ and a diffeomorphism $f : (t_{m+1} \ldots t_n) \rightarrow (f_{m+1}(t_{m+1} \ldots t_n) \ldots f_n(t_{m+1} \ldots t_n))$ of $T_\varepsilon = \{(t_{m+1} \ldots t_n) \varepsilon R^{n-m} : |t_{m+i}| < \varepsilon\}$ into R^{n-m} such that for all $t \varepsilon T_\varepsilon$ the m-dimensional slice of $(y_1 \ldots y_n, U)$ defined by t and the m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$ defined by $f(t)$ are parts of the same leaf of Θ.

PROOF. Let Σ' be the set of $q \varepsilon \Sigma$ for which the conclusion of the theorem holds. It follows from Theorem II that $p \varepsilon \Sigma'$ so Σ' is not empty. Since Σ is connected it will suffice to show that if $\bar{q}$ is adherent to Σ' in Σ then $\bar{q}$ is interior to Σ' with respect to Σ.

Let $(z_1 \dots z_n, V)$ be a cubical coordinate system centered at $\bar{q}$ and flat with respect to Θ and let W be the m-dimensional slice of $(z_1 \dots z_n, V)$ defined by $(0 \dots 0)$. Then W is a neighborhood of $\bar{q}$ in Σ so we can find $q' \varepsilon W \cap \Sigma'$. By definition of Σ' we can find a $\delta < 0$ and functions $g_{m+1} \dots g_n$ defined on T_δ such that $t \rightarrow g(t) = (g_{m+1}(t) \dots g_n(t))$ is a diffeomorphism and for $t \varepsilon T_\delta$ the m-dimensional slice of $(z_1 \dots z_n, V)$ defined by t and the m-dimensional slice of $(x_1 \dots x_n, \mathcal{O})$ defined by $g(t)$ are parts of the same leaf of Θ. Let $q \varepsilon W$ and let $(y_1 \dots y_n, U)$ be a cubical coordinate system flat with respect to Θ containing q in its m-dimensional slice defined by $(0 \dots 0)$. By theorem II there are functions $h_{m+1} \dots h_n$ defined in a neighborhood of the origin in R^{n-m} such that $z_{m+i}(r) = h_{m+i}(y_{m+1}(r) \dots y_n(r))$ for r in an M neighborhood of q and moreover if ε is chosen sufficiently small $t \rightarrow h(t) = (h_{m+1}(t) \dots h_n(t))$ is a diffeomorphism of T_ε into T_δ. Define f on T_ε by $f = g \circ h$. Then f being the composition of two diffeomorphisms is a diffeomorphism. Moreover if $t \varepsilon T$ then the m-dimensional slice of $(x_1 \dots x_n, \mathcal{O})$ defined by $f(t) = g(h(t))$ is part of the same leaf of Θ as the m-dimensional slice of $(z_1 \dots z_n, V)$ defined by $h(t)$ which in turn is a part of the same leaf of Θ as the m-dimensional slice of $(y_1 \dots y_n, U)$ defined by t. This verifies that $q \varepsilon \Sigma'$ and hence that $W \subseteq \Sigma'$. Since W is a neighborhood of $\bar{q}$ in Σ, $\bar{q}$ is interior to Σ' with respect to Σ as was to be shown.

DEFINITION II. Let $(x_1 \dots x_n, \mathcal{O})$ be a cubical coordinate system of breadth $2a$ in a differentiable manifold M which is flat with respect to an m-dimensional involutive differential system Θ. A coordinate system $(y_1 \dots y_n, U)$ in M is said to be <u>subordinate to</u> $(x_1 \dots x_n, \mathcal{O})$ <u>with respect to</u> Θ

if it is flat with respect to Θ, cubical of breadth $2b < 2a$, and if $|t_{m+i}| < b$ $i = 1 \ldots n-m$ implies that the m-dimensional slices of $(x_1 \ldots x_n, \mathcal{O})$ and $(y_1 \ldots y_n, U)$ defined by $(t_{m+1} \ldots t_n)$ are parts of the same leaf of Θ.

COROLLARY 1. _Let $(x_1 \ldots x_n, \mathcal{O})$ be a cubical coordinate system centered at p on the differentiable manifold M which is flat with respect to an involutive differential system Θ. If $q \varepsilon \Pi_\Theta(p)$ then there is a coordinate system centered at q and subordinate to $(x_1 \ldots x_n, \mathcal{O})$ with respect to Θ._

PROOF. Let $(y_1 \ldots y_n, U)$ be any cubical coordinate system centered at q and flat with respect to Θ. Then, letting $f_{m+1} \ldots f_n$ be the functions given by the theorem, define functions $z_1 \ldots z_n$ near q by $z_i = y_i$ $i = 1 \ldots m$ $z_{m+i} = f_{m+i}(y_{m+1} \ldots y_n)$ $i = 1 \ldots n-m$. Then if W is a suitably chosen neighborhood of q $(z_1 \ldots z_n, W)$ is centered at q and is subordinate to $(x_1 \ldots x_n, \mathcal{O})$.

COROLLARY 2. _If Θ is an involutive differential system on a differentiable manifold M then Π_Θ is an open mapping of M onto M/Θ with respect to the quotient topology for M/Θ. Equivalently the saturation of an open set of M with respect to Θ is open._

PROOF. The equivalence of the two statements is clear. Let $\mathcal{O}$ be an open set of M and let q be in the saturation $\tilde{\mathcal{O}}$ of $\mathcal{O}$. Let p be a point of $\mathcal{O}$ belonging to the same leaf of Θ as q. Let $(x_1 \ldots x_n, U)$ be a cubical coordinate system centered at p and flat with respect to Θ with $U \subseteq \mathcal{O}$ (see corollary of theorem I). By corollary I we can find a coordinate system $(y_1 \ldots y_n, V)$ centered at q and subordinate to $(x_1 \ldots x_n, U)$ with respect to Θ. If $q' \varepsilon V$ then

q' belongs to the same leaf of Θ as does the m-dimensional slice of $(x_1 \ldots x_n, U)$ defined by $(y_{m+1}(q') \ldots y_n(q'))$, so in particular q' is in the saturation of U and hence of $\mathcal{O}$. Thus $V \subseteq \tilde{\mathcal{O}}$ so, as V is a neighborhood of q, q is interior to $\tilde{\mathcal{O}}$. Hence $\tilde{\mathcal{O}}$ is open.

Now in general if Π is a mapping of a topological space X onto a set Y there is clearly at most one topology for Y such that Π is both continuous and open. Hence:

COROLLARY 3. *If Θ is an involutive differential system on a differentiable manifold M then the quotient topology for M/Θ is uniquely characterized by the conditions that with respect to it Π_Θ is continuous and open.*

4. Regularity.

DEFINITION III. Let Θ be an involutive m-dimensional differential system on an n-dimensional differentiable manifold M. A coordinate system $(x_1 \ldots x_n, \mathcal{O})$ in M will be called *regular with respect to* Θ if it is cubical, flat with respect to Θ, and if each leaf of Θ intersects $\mathcal{O}$ in at most one m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$. A leaf of Θ will be called a *regular leaf of* Θ if it intersects the domain of a coordinate system regular with respect to Θ. We call Θ *regular* if every leaf of Θ is a regular leaf of Θ.

THEOREM IV. *If Θ is an involutive differential system on a differentiable manifold M and $(x_1 \ldots x_n, \mathcal{O})$ is a coordinate system in M regular with respect to Θ then any coordinate system in M subordinate* to $(x_1 \ldots x_n, \mathcal{O})$ *with respect to Θ is also regular with respect to Θ.*