Olivier Vallée
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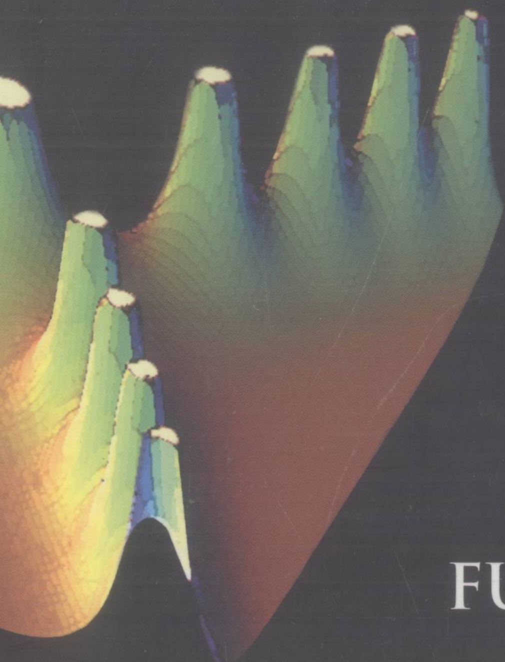

AIRY FUNCTIONS AND APPLICATIONS TO PHYSICS

Imperial College Press

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World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

Imperial College Press
57 Shelton Street
Covent Garden
London WC2H 9HE

Distributed by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401–402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

AIRY FUNCTIONS AND APPLICATIONS TO PHYSICS

ISBN 1-86094-478-7

Printed in Singapore by World Scientific Printers (S) Pte Ltd

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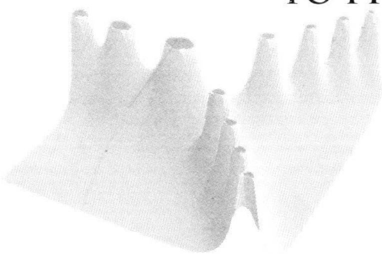

Preface

The use of special functions, and in particular of Airy functions, is rather common in physics. The reason may be found in the need, and even in the necessity, to express a physical phenomenon in terms of an effective and comprehensive analytical form for the whole scientific community. However, for almost the last twenty years, many physical problems have been resolved by computers. This trend is now becoming the norm as the importance of computers continues to grow. As a last resort, the special functions employed in physics will have, indeed, to be calculated numerically, even if the analytic formulation of physics is of first importance.

The knowledge on Airy functions was periodically the subject of many review articles. Generally these were about their tabulations for the numerical calculation of these functions which is particularly difficult. We shall quote the most known works in this field: the tables of J.C.P. Miller which are from 1946 and the chapter in the *Handbook of Mathematical Functions* by Abramowitz and Stegun whose first version appeared in 1954. No noteworthy compilation on Airy functions has been published since that time, in particular about the calculus implying these functions. For example, in the last editions of the tables of Gradshteyn and Ryzhik, they are hardly evoked. At the same time, many accumulated results in the scientific literature, remain extremely dispersed and fragmentary.

The Airy functions are used in many fields of physics, but the analytical outcomes that have been obtained are not (or weakly) transmitted between the various fields of research which after all remain isolated. Moreover the tables of Abramowitz and Stegun are still the only common reference to all the authors using these functions. Thus many of the results have been rediscovered, sometimes extremely old findings are the subject of publications and consequently a useless effort for researchers.

In this work, we would like to make a rather exhaustive compilation of the current knowledge on the analytical properties of Airy functions. In particular, the calculus implying the Airy functions is developed with care. This is, actually, one of the major objectives of this book. We are however aware of making a great number of repetitions regarding the previous compilations, but, it seemed necessary to ensure coherence. This book is addressed mainly to physicists (from undergraduate students to researchers). For the mathematical demonstrations, as one will see, we do not have any claim about the rigour.[1] The aim is the outcome, or the fastest way to reach it. Finally, in the second part of this work, the reader will find some applications to various fields of physics. These examples are not exhaustive. They are only given to succinctly illustrate the use of Airy functions in classical or in quantum physics. For instance, we point out to the physicist in fluid mechanics, that he can find what he is looking for, in the works of molecular physics or in physics of surfaces, and *vice versa.*

The authors would like to warmly thank Nick Rowswell who considerably improved the content of this book.

O. Vallée *&* M. Soares, Fall 2003

[1] As a matter of fact, the Airy function can be considered as a distribution (generalised function) whose Fourier transform is an imaginary exponential. Also most of the integrals evoked in this work should be evaluated with the help of a convergence factor.

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Chapter 1

A Historical Introduction : Sir George Biddell Airy

George Biddell Airy was born July 27, 1801 at Alnwick in Northumberland (North of England). His family was rather modest, but thanks to the generosity of his uncle Arthur Biddell, he went to study at Trinity College (University of Cambridge). He was a brilliant student although being a sizar,[1] and finally graduated in 1823 as a senior wrangler. Three years later, he was elected to occupy the celebrated Lucasian chair of mathematics. Nevertheless, his salary as Lucasian professor was too small to marry Richarda, as her father said. So he applied for a new position. In 1828, Airy obtained the Plumian chair becoming professor of Astronomy and director of the new observatory at Cambridge. His first works at this time were about the mass of Jupiter and also about the irregular motions of Earth and Venus.

In 1834, Airy started his first mathematical studies on the diffraction phenomenon and optics. Due to the diffraction phenomenon, the image of a point through a telescope is actually a spot surrounded by rings of smaller intensity, this spot is now called the "Airy spot", the associated Airy function has nothing to do with the purpose of the present book.

In June 1835, Airy became the 7$^{\text{th}}$ Royal Astronomer and director of the Greenwich observatory, succeeding John Pond. Under his administration, modern equipment was installed, leading the observatory to its worldwide fame assisted by the quality of its published data. Airy also introduced the study of sun spots and the magnetism of Earth, he built a new apparatus for the observation of the Moon, and also for cataloging the stars. The question of absolute time was also a broad challenge, Airy defined the "Airy Transit Circle", that became in 1884: Greenwich Mean Time. But the

[1]With the meaning that he paid a reduced fee but worked as a servant to richer students.

renown of Airy is also due to the "Neptune affair". During the decade 1830–40, astronomers were interested in the perturbations of Uranus that were discovered in 1781. In France, François Arago suggested to Urbain Le Verrier finding a new planet that might cause the perturbations of Uranus. In England, the young John Adams was doing the same calculations with a slight advance, however Airy was doubtful on the issue of such a work. Adams tried twice to meet Airy in 1845 but was unsuccessful: the first time Airy was away, the second time Airy was taking dinner and did not like to be disturbed. Finally, Airy entrusted the astronomer James Challis with the observation of the new planet from the calculations of Adams. Unfortunately, Challis failed in his task. At the same time, Le Verrier asked the German astronomer Johann Galle in Berlin to locate the planet from his data: the new planet was discovered on September 20, 1846. A polemic started then between Airy and Arago, between France and England, and also against Airy himself. The polemic spread out with the name of the planet itself: Airy wanting to name the new planet Oceanus. The name of Neptune was finally given. The story goes that in the end, Adams and Le Verrier became good friends.

In 1854 Airy attempted to determine the mean density of the Earth. The experiment stood in the comparison of gravity forces on a single pendulum at the entrance of a pit and at its ground. This experiment was carried out near South Shields in a mine of 1250 feet in depth. Taking into account the elliptical form and the rotation of the Earth, Airy found a density of 6.56, which is not so far – considering the epoch – from the usually admitted density 5.42.

Airy was knighted in 1872, and so became Sir George Biddell.[2] At this time, Airy started a lunar theory. The results were published in 1886, but in 1890 he found an error in his calculations. The author was eighty-nine years old and was unwilling to revise his calculations. Late in 1881, Sir George left his astronomer position at Greenwich for retirement. He died January 2, 1892.

The autobiography of Sir George, edited by his son Wilfred, was published in 1896 ("*Autobiography of G.B.* AIRY", W. Airy ed., 1896). The name of Airy is associated with many phenomena such as the Airy spiral (optical phenomenon visible in quartz crystals), the Airy spot in diffraction phenomena or the Airy stress function he introduced in his work on elasticity, different as well from the Airy functions that we shall discuss in this

[2] After he declined the offer on three occasions, arguing the fees.

Fig. 1.1 Sir George Biddell Airy (after the Daily Graphic, January 6, 1892).

book. Among of the most-known books he wrote, we may quote "*Mathematical tracts on physical astronomy*" (1826) and "*Popular astronomy*" (1849).

Airy was particularly involved in optics, for instance he made special glasses to correct his own astigmatism. For the same reason, he was also interested by the calculation of light intensity in the neighbourhood of a caustic [Airy (1838), (1849)]. For this purpose, he introduced the function defined by the integral

$$W(m) = \int_0^\infty \cos\left[\frac{\pi}{2}\left(\omega^3 - m\omega\right)\right] \mathrm{d}\omega,$$

which is now called the Airy function. This is the object of the present book. W is the solution of the differential equation

$$W'' = -\frac{\pi^2}{12} mW.$$

The numerical calculation of Airy functions is somewhat tricky, even today!

However in 1838, Airy gave a table of the values of W for m varying from -4.0 to $+4.0$. Thence in 1849, he published a second table for m varying from -5.6 to $+5.6$, for which he employed the ascending series. The problem is that this series is slowly convergent as m increases. A few years after, Stokes (1851, 1858) introduced the asymptotic series of $W(m)$, of its derivative and of the zeros. Practically no reseach was endeavoured on Airy function until the work of Nicholson (1909), Brillouin (1916) and Kramers (1926) who contributed broadly to a better knowledge of this function.

In 1928 Jeffreys introduced the notation used nowadays

$$Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{t^3}{3} + xt\right) \mathrm{d}t,$$

which is the solution of the homogeneous differential equation, called the Airy's equation

$$y'' = xy.$$

Clearly, this equation may be considered as an approximation of the differential equation of the second order

$$y'' + F(x)y = 0,$$

where F is any function of x. If $F(x)$ is expanded in a neighbourhood of a point $x = x_0$, we have to the first order ($F'(x_0) \neq 0$)

$$y'' + [F(x_0) + (x - x_0)F'(x_0)]\, y = 0.$$

Then with a change of variable, we find the Airy's equation. This method is particularly useful in a neighbourhood of a zero of $F(x)$. The point x_0 defined by the relation $F(x_0) = 0$ is called a transition point by mathematicians and a turning point by physicists. Turning points are involved in the asymptotic solutions of linear differential equations of the second order [Jeffreys (1942)], such as the stationary Schrödinger equation.

Finally we can note that Airy functions are Bessel functions (or linear combinations of these functions) of order 1/3. The relation between both of the Airy's equation and the Bessel equation is done with the change of variable $\xi = \frac{2}{3}x^{3/2}$, yielding Jeffreys (1942) to say: "*Bessel functions of order* 1/3 *seem to have no application except to provide an inconvenient way of expressing this function*"!

Chapter 2

Definitions and Properties

This chapter is devoted to general definitions and properties of Airy functions as they can be, at least partially, found in the chapter concerning these functions in the "*Handbook of Mathematical Functions*" by Abramowitz & Stegun (1965).

2.1 The Homogeneous Airy Functions

2.1.1 *The Airy's equation*

We consider the following homogeneous second order differential equation called the Airy's equation

$$y'' - xy = 0. \tag{2.1}$$

This differential equation may be solved by the method of Laplace, *i.e.* in seeking a solution as an integral

$$y = \int_{\mathrm{C}} \mathrm{e}^{xz} v(z) \mathrm{d}z,$$

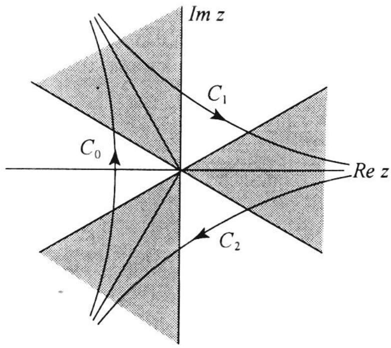
this is equivalent to solve the first order differential equation

$$v' + z^2 v = 0.$$

We thus obtain the solution to the equation (2.1), except a normalisation constant,

$$y = \int_{\mathrm{C}} \mathrm{e}^{xz - z^3/3} \mathrm{d}z.$$

The integration path C is chosen such that the function $v(z)$ must vanish at the boundaries. This is the reason why the extremities of the path must go into the regions of the complex plane z, where the real part of z^3 is positive (shading regions of the complex plane).



From symmetry considerations, it is useful to work with the paths C_0, C_1 and C_2. Clearly the integration paths C_1 and C_2 lead to solutions that tend to infinity when x goes to infinity. When we consider the path C_0 and the associated solution, we can deform this curve until it joins the imaginary axis. Now we define the Airy function Ai by

$$Ai(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{e}^{\,xz - z^3/3} \mathrm{d}z. \tag{2.2}$$

If 1, j, j^2 are the cubic roots of unity (that is to say $\mathrm{j} = \mathrm{e}^{\,\mathrm{i}2\pi/3}$) the functions defined by the paths C_1 and C_2 are respectively the functions $Ai(\mathrm{j}x)$ and $Ai(\mathrm{j}^2 x)$. We have between these solutions, two by two linearly independents for they satisfy the same second order differential equation, the relation

$$Ai(x) + \mathrm{j}Ai(\mathrm{j}x) + \mathrm{j}^2 Ai(\mathrm{j}^2 x) = 0. \tag{2.3}$$

Now, in place of the functions $Ai(\mathrm{j}x)$ and $Ai(\mathrm{j}^2 x)$, we define the function $Bi(x)$, linearly independent of $Ai(x)$, which has the interesting property to be real when x is real

$$Bi(x) = \mathrm{i}\mathrm{j}^2 Ai(\mathrm{j}^2 x) - \mathrm{i}\mathrm{j} Ai\,(\mathrm{j}x)\,. \tag{2.4}$$

Similarly to $Ai(x)$ (cf. formula (2.3)), we have the relation

$$Bi(x) + \mathrm{j}Bi(\mathrm{j}x) + \mathrm{j}^2 Bi(\mathrm{j}^2 x) = 0. \tag{2.5}$$

On Figs. 2.1 and 2.2, the plots of the functions $Ai(x)$, $Bi(x)$, and of their derivatives $Ai'(x)$ and $Bi'(x)$ are given.

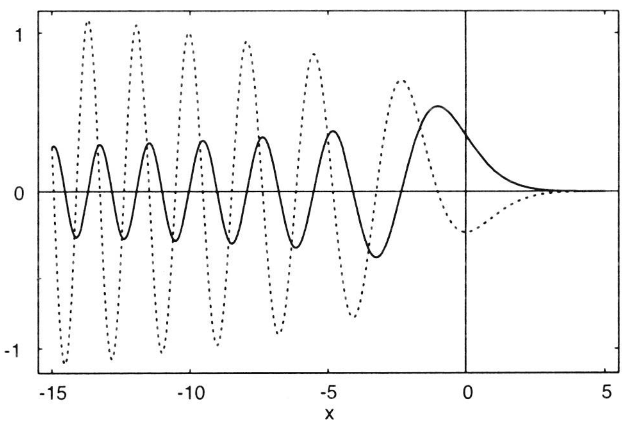


Fig. 2.1 Plot of the Airy function Ai (full line) and its derivative (dotted line).

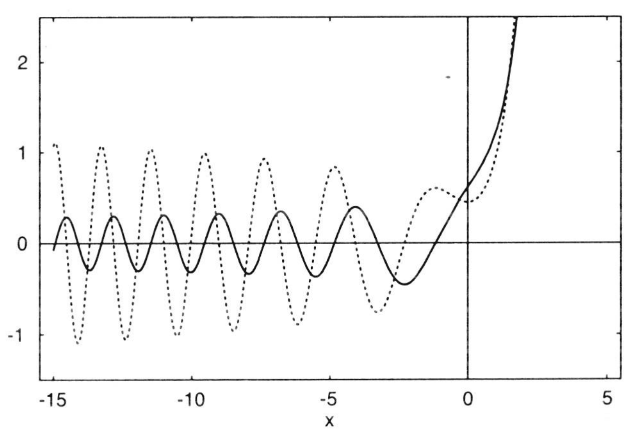


Fig. 2.2 Plot of the Airy function Bi (full line) and its derivative (dotted line).