GIAN-CARLO ROTA, *Editor*

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Volume 6

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**Section:** Linear Algebra

Marvin Marcus, *Section Editor*

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# Permanents

**Henryk Minc**

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With a Foreword by

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# Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive changes of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

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Ordinarily, a specialized volume, such as this one, would call for a prior work on determinants. It is hoped that, in time, a companion volume will be added to the ENCYCLOPEDIA. Meanwhile, Professor Minc's monograph is expected to remain the definitive treatment on permanents and, as the author wittily remarks, the only one in all probability.

A permanent is an improbable construction to which we might have given little chance of survival fifty years ago. Yet the numerous appearances it has made in physics and in probability betoken the mystifying usefulness of the concept, which has a way of recurring in the most disparate of circumstances. Thanks to Professor Minc's efforts, they are now all collected here.

GIAN-CARLO ROTA

# Foreword

In referring to Sir Thomas Muir and his monumental work *The Theory of Determinants in the Historical Order of Development*, Minc calls him the "master from Edinburgh." As a graduate of that venerable institution himself, Minc carries on with this high tradition of scholarship and masterly exposition with *Permanents*. The permanent function has been studied for more than a century. As Minc amusingly points out in Section 1.1, the word "permanent" originated with Cauchy in 1812, although a referee of one of Minc's earlier papers admonished him for daring to invent such a ludicrous name.

In the Carus monograph *Combinatorial Mathematics*, H. J. Ryser mentions that the permanent "appears repeatedly in the literature of combinatorics in connection with certain enumeration and extremal problems." As an example, if $D$ is the $n$-square matrix with 0's on the main diagonal, 1's elsewhere, then $\operatorname{per}(D)$ is a count of the total number of derangements—that is, permutations with no fixed points—of $1,\ldots,n$. The Laplace expansion theorem works equally well for permanents as for determinants —indeed it is simpler, since no sign changes arise. From this it follows immediately that

$$\operatorname{per}(A+B)=\sum_{r=0}^{n}\sum_{\alpha,\beta}\operatorname{per}A[\alpha|\beta]\operatorname{per}B(\alpha|\beta), \qquad (1)$$

where the inner summation is over all products of an $r\times r$ subpermanent of $A$ lying in rows $\alpha=(\alpha_1,\ldots,\alpha_r)$, columns $\beta=(\beta_1,\ldots,\beta_r)$, and the complementary subpermanent of $B$. If we take $A=J$, the matrix of all 1's, $B=-I_n$, then $D=A+B$, and the number of derangements is given by the remarkable formula

$$\operatorname{per}(J-I_n)=\sum_{r=0}^{n}\sum_{\alpha}\operatorname{per}J[\alpha|\alpha](-1)^{n-r}$$

$$=\sum_{r=0}^{n}(-1)^{n-r}\frac{n!}{(n-r)!}.$$

Let $U_n$ be the $n$th menage numbers; that is, $U_n$ is a count of the number of permutations $\sigma$ of $1, 2, \ldots, n$ such that $\sigma(i)$ is neither $i$ nor $i+1$ (mod $n$) $i=1,\ldots,n$. Analogously to the derangement problem we have

$$U_n = \mathrm{per}(J - I_n - P),$$

where $P$ has 1's in positions $(1,2)$, $(2,3), \ldots, (n-1,n)$, $(n,1)$, and 0's elsewhere. Thus, as Ryser suggests, the permanent *function* is the "correct" tool for dealing with a number of difficult enumeration problems for restricted permutations. Minc is certainly a master in making the computations required for such problems (see Section 3.4). In fact, I have personally watched while Minc punched some quite remarkable permanents of circulants out of one of the more primitive hand-held calculators of the early sixties.

Although a number of deep and interesting results about the permanent have been obtained by direct methods, there is a somewhat oblique approach to the function that has proved to be quite productive over the last two decades. Let $V$ be an $n$-dimensional inner product space. Then the Z-graded contravariant tensor space over $V, T_0(V) = C \dotplus V \dotplus V \otimes V \dotplus V \otimes V \otimes V \dotplus \ldots$, inherits an inner product from $V$ that satisfies the formula

$$(x_1 \otimes \ldots \otimes x_p, y_1 \otimes \ldots \otimes y_p) = \prod_{t=1}^{p} (x_t, y_t)$$

for homogeneous decomposable elements of degree $p$. The symmetric space, $\dot{V}$, is the range in $T_0(V)$ of the symmetry operator

$$\sum_{p=0}^{0} \mathbb{S}_p; \ldots$$

$\mathbb{S}_p = \frac{1}{p!}\Sigma\sigma$, and the summation is over the symmetric group of degree $p$ (the action of $\sigma$ on a decomposable tensor is defined by $\sigma(x_1 \otimes \ldots \otimes x_p) = x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(p)}$). Each $\mathbb{S}_p$ is a hermitian idempotent, so that, if $x_1 \ldots x_p = \mathbb{S}_p x_1 \otimes \ldots \otimes x_p$, we have

$$\begin{aligned}(x_1 \ldots x_p, y_1 \ldots y_p) &= (x_1 \otimes \ldots \otimes x_p, \mathbb{S}_p y_1 \otimes \ldots \otimes_p) \\ &= \frac{1}{p!} \sum_{\sigma} \prod_{t=1}^{p} (x_t, y_{\sigma(t)}) \\ &= \frac{1}{p!} \mathrm{per}((x_i, y_j)).\end{aligned}$$

Thus, the permanent function appears naturally as an analytical expression for the inner product in $V^{(p)} = \mathrm{im}\ \mathbb{S}_p$ in precisely the same way as the determinant does in the $p$th exterior space $\wedge^p V$. This means that the unitary geometry of $V^{(p)}$ is available for investigating per($A$), and it is this observation that has led to substantial progress in dealing with the function. Minc skillfully interweaves the combinatorial and multilinear approaches to the function throughout the book.

Certainly *Permanents* is the definitive treatise. The history, theory, and applications are completely surveyed, and the bibliography contains a reference to every book and paper written on the subject. No doubt the present book will result in renewed interest in this intractable and fascinating matrix function.

MARVIN MARCUS
*General Editor*, Section in Linear Algebra

# Preface

Permanents made their first appearance in 1812 in the famous memoirs of Binet and Cauchy. Since then 155 other mathematicians contributed 301 publications to the subject, more than three-quarters of which appeared in the last 19 years. The present monograph is an outcome of this remarkable re-awakening of interest in the permanent function.

The purpose of the book is to give a complete account of the theory of permanents, their history and applications, in a form accessible not only to mathematicians but also to workers in various applied fields, and to students of pure and applied mathematics. Here is the first complete account of the theory of permanents. It is a survey in the style of MacDuffy *The Theory of Matrices* and of *A Survey of Matrix Theory and Matrix Inequalities*, by Marcus and Minc. However, it differs from both works in several respects: the style is more leisurely, the proportion of theorems proved in the text is higher, and the scope is wider—the volume covers virtually the whole of the subject, a feature that no survey of the theory of matrices can even attempt. Apart from many theorems proved in detail, there are numerous results stated without proof. Due to limitation of space, not every known result could be mentioned in the text. The choice of the theorems included in the book reflects, of course, the author's predilections.

The first chapter of the monograph is a historical survey of the theory of permanents since its beginnings in 1812. Here only some classical results are discussed in detail. In Chapter 2 general properties of permanents are developed. Chapter 3 is devoted to combinatorial and structural properties of (0.1)-matrices. The next three chapters may be regarded as the heart of the monograph. They deal with inequalities involving permanents, and with lower and upper bounds for permanents. The latter are particularly important due to the lack of efficient methods for computing permanents. One of the three chapters, Chapter 5, contains an up-to-date survey of the literature on the famed van der Waerden conjecture. In Chapter 7 we discuss several methods for computing permanents and compare their efficiency. The concluding chapter contains a section on some important topics that do not fall under the headings of the preceding chapters, a section on applications of permanents to combinatorics, graph theory, and to statistical mechanics, and two sections in which we report on the present status of the conjectures and problems in the Marcus–Minc 1965 list, and compile a new list of unresolved conjectures and unsolved problems on permanents.

Every chapter concludes with a set of problems of varying difficulty. Thus the book can be used as a text for a course at the advanced undergraduate or graduate level. The only prerequisites are a standard undergraduate course in the theory of matrices and a measure of mathematical maturity.

A special feature of the monograph, and, in fact, its foundation is the Bibliography which contains every paper and book on permanents published before the end of 1977 or awaiting publication at that time. The Bibliography also includes some papers on cognate topics even if they make no explicit use of permanents but can be interpreted in terms of permanents. Thus several classical papers on the "problème des ménages" are listed. Papers on Schur functions are included if the specialization of the results to permanents produces new significant theorems. Articles on graphs and on combinatorial properties of matrices are excluded unless they are related to or make use of permanents. In general, only the most important result of papers are reviewed in the Bibliography. In case of papers or books covering more than one area, only the part related to permanents is reviewed.

The usual double numeration is used in references. Thus "Section 3.2" refers to Section 2 in Chapter 3. The fourth theorem in Section 3.2 is referred to as "Theorem 2.4, Chapter 3"; similarly with references to examples and exhibited formulas. Within a chapter the reference to the chapter is omitted; e.g., within Chapter 3 the above theorem is quoted simply as "Theorem 2.4".

I should like to express my appreciation to Mrs. Barbara Federman for her assistance in preparing the book, to the Director and the staff of the Institute for the Interdisciplinary Applications of Algebra and Combinatorics, U.C.S.B., for having the manuscript typed and assembled, and, in particular, to Mrs. Michelle Dunn for her excellent job of typing. The work on the book was supported in part by the Air Force Office of Scientific Research under Grants AFOSR-72-2164 and AFOSR-77-3166.

HENRYK MINC

# Contents

CHAPTER 1

# The Theory of Permanents in the Historical Order of Development

## 1.1 Introduction

Modern mathematicians have a proclivity to invent flippant names for newly introduced mathematical entities and concepts. They delight in talking about mobs, radicals, derogatory matrices, osculating planes, improper ideals, etc. It may appear that the term "permanent" was also invented by a waggish algebraist. In fact, a few years ago a well-meaning referee admonished the author for daring to invent this ludicrous name for a function that Schur himself introduced without designating it by any specific term. The fact of the matter is that the permanent function was studied and called by that name before Schur was even born.

In his famous memoir of 1812, Cauchy [2] developed the theory of determinants as a special type of alternating symmetric functions, which he distinguished from the ordinary symmetric functions by calling the latter "fonctions symétriques permanentes." He also introduced a certain subclass of symmetric functions, which were later named *permanents* by Muir [14] and which are nowadays known by this name. These functions can be defined by means of matrices and modern notation as follows.

Let $A=(a_{ij})$ be an $m\times n$ matrix over any commutative ring, $m\leqslant n$. The *permanent* of $A$, written Per$(A)$, or simply Per $A$, is defined by

$$\operatorname{Per}(A)=\sum_{\sigma} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{m\sigma(m)}, \tag{1.1}$$

where the summation extends over all one-to-one functions from $\{1,\ldots,m\}$ to $\{1,\ldots,n\}$. The sequence $(a_{1\sigma(1)},\ldots,a_{m\sigma(m)})$ is called a *diagonal* of $A$, and the product $a_{1\sigma(1)}\cdots a_{m\sigma(m)}$ is a *diagonal product* of $A$. Thus the permanent of $A$ is the sum of all diagonal products of $A$.

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For example, if

$$A=[3\,2\,4],$$

$$B=\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix},$$

$$C=\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \\ -1 & 2 & -2 \end{bmatrix},$$

then Per $A=9$, Per $B=44$, and Per $C=18$.

The special case $m=n$ is of particular importance. We denote the permanent of a square matrix $A$ by per($A$) instead of Per($A$). In fact, most writers restrict the designation "permanent" to the case of square matrices.

## 1.2 The Originators: Binet and Cauchy

Permanents were introduced in 1812 almost simultaneously by Binet [1] and Cauchy [2]. Binet in his memoir also gave formulas for computing the permanents of $m\times n$ matrices for $m<4$.

The permanent of an $m\times n$ matrix $A$, $m<n$, is the sum of all the diagonal products of $A$. In other words, Per $A$ is the sum of all products of $m$ elements of $A$, no two in the same row or the same column. It follows that all the terms of Per$A$, and many other superfluous terms, are contained in the set of terms obtained by multiplying the row sums of $A$. For example, if $A$ is a $2\times n$ matrix, then

$$\operatorname{Per}A=\sum_{s\neq t} a_{1s}a_{2t},$$

while the product of the row sums of $A$ is

$$\prod_{i=1}^{2}\sum_{j=1}^{n} a_{ij}=\sum_{s,t=1}^{n} a_{1s}a_{2t}$$

$$=\sum_{s\neq t} a_{1s}a_{2t}+\sum_{s=1}^{n} a_{1s}a_{2s}.$$

Hence

$$\operatorname{Per}A=\prod_{i=1}^{2}\sum_{j=1}^{n} a_{ij}-\sum_{s=1}^{n} a_{1s}a_{2s}, \tag{2.1}$$

which is Binet's formula for $m=2$.

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Let $Q_{t,k}$ denote the set of all strictly increasing sequences of integers $\omega=(\omega_1,\ldots,\omega_t)$ satisfying $1\leq\omega_1<\omega_2<\cdots<\omega_t\leq k$. For an $m\times n$ matrix $A=(a_{ij})$ and a sequence $(i_1,\ldots,i_s)\in Q_{s,m}$, define

$$r_{i_1*\cdots*i_s}=\sum_{j=1}^{n}a_{i_1j}a_{i_2j}\cdots a_{i_sj}.$$

In particular,

$$r_i=\sum_{j=1}^{n}a_{ij}$$

denotes the $i$th row sum of $A$. Then Binet's formula (2.1) can be written in the form

$$\operatorname{Per}A=r_1r_2-r_{1*2}.$$

Now consider a $3\times n$ matrix $A=(a_{ij})$, $n\geq 3$. The product $r_1r_2r_3$ contains all the terms of $\operatorname{Per}A$ and in addition $n^3-n(n-1)(n-2)$ "unwanted" terms such as $a_{11}a_{21}a_{3j}, a_{12}a_{22}a_{3j},\ldots,a_{11}a_{2j}a_{31},\ldots,a_{1j}a_{21}a_{31},\ldots$, etc., $j=1,\ldots,n$—that is, the terms of $r_{1*2}r_3$, $r_{1*3}r_2$, and $r_{2*3}r_1$. It seems, therefore, that if we subtract $r_{1*2}r_3+r_{1*3}r_2+r_{2*3}r_1$ from $r_1r_2r_3$, we should be left with the terms of $\operatorname{Per}A$. Unfortunately, this is not the case. What happens is that, although we subtract all the "unwanted" terms, we subtract some of them more than once. To be precise, the terms $a_{1j}a_{2j}a_{3j}$, $j=1,\ldots,n$, appear in all three products $r_{1*2}r_3$, $r_{1*3}r_2$, and $r_{2*3}r_1$, and therefore each of them is subtracted three times instead of once. But $r_{1*2*3}$ is the sum of all the $a_{1j}a_{2j}a_{3j}$. Thus we obtain Binet's second formula:

$$\operatorname{Per}A=r_1r_2r_3-(r_{1*2}r_3+r_{1*3}r_2+r_{2*3}r_1)+2r_{1*2*3}. \qquad (2.2)$$

We now introduce the following simplifying notation: For $2\times n$ matrices,

$$S(1,1)=r_1r_2,$$
$$S(2)=r_{1*2};$$

for $3\times n$ matrices,

$$S(1,1,1)=r_1r_2r_3,$$
$$S(1,2)=r_1r_{2*3}+r_2r_{1*3}+r_3r_{1*2},$$
$$S(3)=r_{1*2*3};$$

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for $4\times n$ matrices,

$$S(1,1,1,1)=r_1r_2r_3r_4,$$
$$S(1,1,2)=r_1r_2r_{3*4}+r_1r_3r_{2*4}+r_1r_4r_{2*3}+r_2r_3r_{1*4}$$
$$+r_2r_4r_{1*3}+r_3r_4r_{1*2},$$
$$S(2,2)=r_{1*2}r_{3*4}+r_{1*3}r_{2*4}+r_{1*4}r_{2*3},$$
$$S(1,3)=r_1r_{2*3*4}+r_2r_{1*3*4}+r_3r_{1*2*4}+r_4r_{1*2*3},$$
$$S(4)=r_{1*2*3*4},$$

etc. In general, if $A$ is an $m\times n$ matrix and $t_1,\ldots,t_k$ are integers, $1\leqslant t_1\leqslant\cdots\leqslant t_k, t_1+\cdots+t_k=m$, then $S(t_1,\ldots,t_k)$ is the symmetrized sum of all distinct products of the $r_{i_1*\cdots*i_s}, s=t_1,\ldots,t_k$, so that in each product the sequences $(i_1,\ldots,i_s)\in Q_{s,m}, s=t_1,\ldots,t_k$, partition the set $\{1,\ldots,m\}$.

Using this notation we can write equations (2.1) and (2.2) in the following form:

$$\operatorname{Per}A=S(1,1)-S(2), \tag{2.1'}$$

$$\operatorname{Per}A=S(1,1,1)-S(1,2)+2S(3). \tag{2.2'}$$

Both these formulas were proved by Binet by a very involved method. He also gave, without proof, a formula for the permanent of a $4\times n$ matrix $A$:

$$\operatorname{Per}A=S(1,1,1,1)-S(1,1,2)+S(2,2)+2S(1,3)-6S(4). \tag{2.3}$$

This formula can be established by the use of the principle of inclusion and exclusion which we used to prove formulas (2.1) and (2.2).

Let $A$ be an $4\times n$ matrix. The function $S(1,1,1,1)$ is the sum of all the terms of $\operatorname{Per}A$ and also some "superfluous" terms all of which are the terms of $S(1,1,2)$. We therefore subtract $S(1,1,2)$ from $S(1,1,1,1)$. However, we have "overreacted": Some of the terms, such as $a_{11}a_{21}a_{32}a_{42}$, $a_{11}a_{21}a_{31}a_{42}$, and $a_{11}a_{21}a_{31}a_{41}$, appear in $S(1,1,2)$ with multiplicity greater than 1. For example, $a_{11}a_{21}a_{32}a_{42}$ appears both in $r_{1*2}r_3r_4$ and in $r_1r_2r_{3*4}$. We compensate by adding $S(2,2)$. Now, we count the number of the terms of the form $a_{1i}a_{2j}a_{3j}a_{4j}, i\neq j$, in

$$S(1,1,1,1)-S(1,1,2)+S(2,2).$$

Each of them occurs once in $S(1,1,1,1)$, three times in $S(1,1,2)$ (for example, $a_{11}a_{22}a_{32}a_{42}$ appears once in each of $r_1r_2r_{3*4}$, $r_1r_3r_{2*4}$, and $r_1r_4r_{2*3}$) and does not appear in $S(2,2)$. Hence we compensate by adding twice $S(1,3)$:

$$S(1,1,1,1)-S(1,1,2)+S(2,2)+2S(1,3). \tag{2.4}$$

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It remains to account for the terms $a_{1j}a_{2j}a_{3j}a_{4j}, j=1,\ldots,n$. Each of them appears once in $S(1,1,1,1)$, six times in $S(1,1,2)$, three times in $S(2,2)$, and eight times in $2S(1,3)$. Therefore we must subtract $1-6+3+8=6$ times $S(4)$ from (2.4), and the formula (2.3) follows.

*Example* 2.1. Compute the permanent of matrix

$$A=\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

We compute:

$$S(1,1,1,1)=144, \qquad S(1,1,2)=151,$$
$$S(2,2)=13, \qquad S(1,3)=13, \qquad S(4)=0.$$

Hence,

$$\operatorname{Per}(A)=144-151+13+2\times 13-6\times 0=32.$$

Binet did not explain how he derived the coefficients in (2.4), nor did he give a general formula for the permanent of an $m\times n$ matrix for $m>4$.

Recently Binet's formula was generalized to any $m\times n$ matrices, $m\leqslant n$ [301]. This formula and a more efficient formula due to Ryser [87] will be given in the chapter on the evaluation of permanents.

## 1.3 The Continuators: Borchardt, Cayley, and the Master from Edinburgh—Sir Thomas Muir

During the century that followed the appearance of the memoirs of Binet and Cauchy, some twenty papers on permanents were published. Most of them dealt with identities involving determinants and permanents. The results that created the most interest are identities of Borchardt [4], Cayley [6], and Muir [14]. All three are formulas for the product of the permanent and the determinant of a matrix.

Let $A=(a_{ij})$ be an $n\times n$ matrix. Then

$$\operatorname{per}(A)\det(A)=\left(\sum_{\sigma\in E}\prod_{i=1}^{n}a_{i\sigma(i)}\right)^2-\left(\sum_{\sigma\in F}\prod_{i=1}^{n}a_{i\sigma(i)}\right)^2, \tag{3.1}$$

where $E$ and $F$ are the sets of even and odd permutations, respectively. The problem is how to express the difference on the right in a more

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attractive form; for example, it is clearly equal to

$$\sum_{\sigma\in E}\prod_{i=1}^{n} a^2_{i\sigma(i)} - \sum_{\sigma\in F}\prod_{i=1}^{n} a^2_{i\sigma(i)} + f(A) = \det(A^{(2)}) + f(A), \tag{3.2}$$

where $A^{(2)} = A * A$ is the matrix whose $(i,j)$ entry is $a^2_{ij}$, and $f(A)$ represents the remaining terms. If $n=2$, then actually $f(A)=0$. For $n=3$, Cayley expressed $f(A)$ in terms of the determinant of a related matrix.

THEOREM 3.1 (Cayley [6]). *Let $A=(a_{ij})$ be a $3\times 3$ matrix, $a_{ij}\neq 0$, and let $A^{(-1)}$ be the $3\times 3$ matrix whose $(i,j)$ entry is $a_{ij}^{-1}$. Then*

$$\operatorname{per}(A)\det(A) = \det(A^{(2)}) + 2\left(\prod_{i,j} a_{ij}\right)\det(A^{(-1)}). \tag{3.3}$$

The proof follows immediately from (3.1) and (3.2).

COROLLARY. *If $B=(b_{ij})$ is a singular matrix, $b_{ij}\neq 0$, then*

$$\operatorname{per}(B^{(-1)})\det(B^{(-1)}) = \det(B^{(-2)}). \tag{3.4}$$

Here $B^{(-2)}$ denotes the $3\times 3$ matrix whose $(i,j)$ entry is $b_{ij}^{-2}$.

Borchardt obtained a formula similar to (3.4) for any $n$, but only for a special type of matrix.

THEOREM 3.2 (Borchardt [4]). *Let $A$ be an $n\times n$ matrix whose $(i,j)$ entry is $(s_i - t_j)^{-1}$. Then*

$$\operatorname{per}(A)\det(A) = \det(A^{(2)}). \tag{3.5}$$

We shall not offer a separate proof of Borchardt's result, since it is an immediate consequence of a generalization of Cayley's theorem by Carlitz and Levine [63], which we give at the end of this section.

Sir Thomas Muir occupies a unique position in the history of permanents, and even more so in the history of determinants. In his monumental *The Theory of Determinants in the Historical Order of Development* [25,26,35,36,40], he gives *inter alia* an abstract of every paper on permanents published before 1920, a third of which were his own contributions. Muir's papers deal mostly with expressions and identities involving permanents and determinants. Of these we give below one of the results in his first paper on permanents.

THEOREM 3.3 (Muir [14]). *Let $A=(a_{ij})$ and $X=(x_{ij})$ be n-square matrices. Then*

$$\operatorname{per}(A)\det(X) = \sum_{\sigma\in S_n} \varepsilon(\sigma)\det(A * X_\sigma), \tag{3.6}$$

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