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Wavelets, Frames and Operator Theory

Focused Research Group Workshop on Wavelets, Frames and Operator Theory

January 15–21, 2003
University of Maryland
College Park, Maryland

Christopher Heil
Palle E.T. Jorgensen
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This volume contains papers from an AMS Special Session on "Wavelets, Frames and Operator Theory" from the 2003 Annual Meeting of the AMS in Baltimore, Maryland, January 15–18, 2003, and an NSF-sponsored workshop at the University of Maryland, January 19–21, 2003. Both events were associated with the NSF Focused Research Group (FRG) with support from NSF grant DMS-0139759.

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Wavelets, Frames and Operator Theory

Preface

This book grew out of the Special Session on *Wavelets, Frames, and Operator Theory* that we organized at the 2003 Annual Meeting of the AMS in Baltimore, January 15–18, 2003, and an immediately following NSF-sponsored workshop organized by John Benedetto at The University of Maryland, January 19–21, 2003. Both events were associated with the NSF Focused Research Group (FRG) of which we are a part, and whose other members are Akram Aldroubi, Lawrence W. Baggett, John J. Benedetto, Gestur Ólafsson, and Yang Wang. The speakers in the Special Session and the Maryland workshop were invited to contribute papers, and this volume is the very pleasant result.

We hope that those events and more like them that have since taken place or are planned for the future, and the present book itself, will act as a catalyst, encouraging members of our community to work on one or more of the many facets of the intertwined subjects of wavelets, frames, and operator theory. Some of the papers included here focus more on one of the three areas than the other two, but all hint at exciting connections and interrelationships. They stand at the crossroads of harmonic analysis, operator theory, and applied mathematics. Some papers are theoretical, some applied, but most are a mix of theory and applications, each inspiring the other. Wavelets and frames have emerged as significant tools in mathematics and in technology over the past two decades. They interact with harmonic analysis, with operator theory, and with a host of applications. In their primitive form, both wavelets and frames originate with the vector space notion of a basis. They are used in the analysis of functions, and the functions make up infinite-dimensional spaces, typically Hilbert spaces. While many wavelet constructions yield orthonormal bases, frames by their very nature, including many interesting classes of wavelets, satisfy conditions which are more general than the familiar orthogonality relations. Historically, operator theory has played a big part in the analysis of both wavelets and frames, and we hope to highlight this feature in our collection of papers.

The workshops, the research, and the publication of this volume were supported in part by our FRG grant from the National Science Foundation.[1] It is also a pleasure to thank Brian Treadway, whose assistance was essential to the smooth

[1]DMS-0139759 Collaborative Research: Focused Research on Wavelets, Frames, and Operator Theory. Description: In this project, fundamental problems are addressed in wavelet theory, non-uniform sampling, frames, and the theory of spectral-tile duality. These problems are inextricably interwoven by concept and technique. Operator theory provides the major unifying framework, combined with an integration of ideas from a diverse spectrum of mathematics including classical Fourier analysis, noncommutative harmonic analysis, representation theory, operator algebras, approximation theory, and signal processing. For example, the construction, implementation, and ensuing theory of single dyadic orthonormal wavelets in Euclidean space requires significant input from all of these disciplines as well as deep spectral-tile results.

completion of this volume. Brian managed the correspondence with referees and authors, organized the many drafts of papers, helped bring them into the TEX format for the book series, and managed and assisted us in numerous typesetting issues.

Christopher Heil, Palle E. T. Jorgensen, and David R. Larson
September 21, 2003

Speakers at the two sessions of the 2003 Annual Meeting of the American Mathematical Society, held in Baltimore, and the Wavelet Workshop held in College Park, Maryland, out of which the papers in this volume arose

Symbols are used in the list to designate the specific sessions, as follows:

(S): AMS Special Session on "Wavelets, Frames and Operator Theory", at the 2003 Annual Meeting of the American Mathematical Society (AMS–MAA–ASL–AWM–NAM–SIAM Joint Mathematics Meetings, Baltimore, MD, January 15–18, 2003);
(C): AMS Session on "Operator Theory", at the same Annual Meeting;
(W): NSF Focused Research Group Workshop on the theme "Modulation Spaces and the Continuous Wavelet Transform" (University of Maryland, College Park, MD, January 19–21, 2003).

The ten speakers marked (C) had requested to be part of our Special Session, but unfortunately could not be included due to lack of space. The AMS incorporated these speakers into a related session of contributed talks. We thank the AMS for kindly honoring our request to include them in this way, and we consider them informally included in our session for the purpose of this volume.

Affiliations follow the session symbols for each speaker. Where more than one affiliation is listed, the first is from the time of the sessions, and the others are later locations.

Akram Aldroubi (S), Vanderbilt University
    http://www.math.vanderbilt.edu/~aldroubi/
    "Non uniform sampling and reconstruction in irregular spaces"
Radu Balan (S), Siemens Corporate Research
    "Measure function and redundancy of Weyl-Heisenberg multiframes and superframes"
Robert Benedetto (S), Amherst College
    http://www.cs.amherst.edu/~rlb/
    "Wavelets on p-adic fields and related groups"
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    http://www.math.uio.no/~bratteli/
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Ingrid Daubechies (S), Princeton University
http://www.princeton.edu/~icd/
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Dorin Dutkay (S), University of Iowa
http://www.math.uiowa.edu/~ddutkay/
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http://www.univie.ac.at/NuHAG/FEI/
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http://www.math.cornell.edu/People/Postdocs/fickus.html
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"A physical interpretation for finite tight frames" (W)
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"Embeddings of Fourier-Lebesgue spaces into modulation spaces: Optimality of sufficient conditions"
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"Frequency estimation and vortex analysis using wavelet coefficients"
Karlheinz Gröchenig (S, W—plenary talk), University of Connecticut
http://www.math.uconn.edu/~groch/
"Localization of frames" (S, W)
Christopher Hammond (C), University of Virginia
"On the norm of a composition operator with linear fractional symbol"
Deguang Han (S), University of Central Florida
http://pegasus.cc.ucf.edu/~dhan/main.html
"Operator parametrization and tight frame approximations"
Doug Hardin (S), Vanderbilt University
http://math.vanderbilt.edu/~hardin/
"Continuous orthogonal wavelets on semi-regular triangulations"
Denise Jacobs (C), United States Military Academy
http://www.dean.usma.edu/math/people/jacobs/
"Orthogonal wavelets in higher dimensions"
Brody Johnson (W), Georgia Institute of Technology; Saint Louis University
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"Oversampling wavelet frames"
Norbert Kaiblinger (S), University of Vienna
http://www.mat.univie.ac.at/~kaib
"Varying the lattice of Gabor frames"
Keri Kornelson (S), Texas A & M University
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"Ellipsoidal tight frames"
Gitta Kutyniok (S, W), University of Paderborn
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"A qualitative uncertainty principle for functions generating a Gabor frame on LCA groups" (W)

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How to construct wavelet frames on irregular grids and arbitrary dilations in $\mathbb{R}^d$

Akram Aldroubi, Carlos Cabrelli, and Ursula M. Molter

Abstract. In this article, we present a method for constructing wavelet frames of $L^2(\mathbb{R}^d)$ of the type $\{|\det A_j|^{1/2}\psi(A_j x - x_{j,k}) : j \in J,\ k \in K\}$ on irregular lattices of the form $X = \{x_{j,k} \in \mathbb{R}^d : j \in J,\ k \in K\}$, and with an arbitrary countable family of invertible $d \times d$ matrices $\{A_j \in GL_d(\mathbb{R}) : j \in J\}$. Possible applications include image and video compression, speech coding, image and digital data transmission, image analysis, estimations and detection, and seismology.

1. Introduction

In this article we present a general method for constructing well-localized wavelet frames $\{|\det A_j|^{1/2}\psi(A_j x - x_{j,k}) : j \in J,\ k \in K\}$ of $L^2(\mathbb{R}^d)$ on arbitrary grids $X = \{x_{j,k} \in \mathbb{R}^d : j \in J,\ k \in K\}$, and with arbitrary dilation matrices $\{A_j\}_{j\in J}$. The construction presented here is a special case of a more general method for constructing time-frequency frame atoms in several variables discussed in [**ACM03**]. Although there has been considerable interest in trying to obtain wavelet frame decompositions of the space $L^2(\mathbb{R}^d)$, on irregular grids and with unstructured dilation matrices (see [**Bal97**], [**BCHL03**], [**Chr96**], [**Chr97**], [**CH97**], [**CDH99**], [**FZ95**], [**Fei87**], [**FG89**], [**FW01**], [**Grö91**], [**Grö93**], [**HK03**], [**OS92**], [**RS95**], [**SZ00**], [**SZ01**], [**SZ02**], [**SZ03**], [**SZ03**]), most of the methods that have been developed are small perturbations of wavelet frames on a regular grid and with a fixed dilation matrix. In contrast, our approach presented in [**ACM03**] is not a perturbation method and is very general, allowing quite general constructions. The setting includes as particular cases, wavelet frames on irregular lattices and with a set of dilations or transformations that do not have a group structure. For this paper, we will be mainly concerned with an even more particular case consisting of wavelet frames on irregular lattices and with an arbitrary but fixed expansive matrix A (A is said to be expansive if $|\lambda| > 1$ for every eigenvalue λ of A). The

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case of regular lattices can also be obtained by our system, producing a substantial part of the systems recently characterized by the fundamental work of Guido Weiss and his group [**HDW02, HDW03, Lab02**] on the decomposition of $L^2(\mathbb{R}^d)$. The wavelet frames obtained by Chui, He, Stöeckler and Sun [**CHS**], [**CHSS03**], [**CS00**] are also included in our setting. Wavelet sets and wavelet frame sets studied in [**BMM99**], [**BL99**], [**BL01**], [**BS03**], [**DLS97**], [**DLS98**], [**HL00**], [**Ola03**], [**OS03**] can also be produced by our methods. Furthermore we can obtain wavelet sets with translations on irregular grids.

The method we present relies on combining ideas from four related, but different subjects: 1) Sampling theory; 2) Frame theory; 3) Wavelet theory; and 4) Geometry of $\mathbb{R}^d$. The approach can be considered in the spirit of the classic construction in 1 dimension of smooth regular tight frames done by Daubechies, Grossmann and Meyer in [**DGM86**]. (See also [**HW89**] for an expository treatment.) We will say that a set $X = \{x_k \in \mathbb{R}^d : k \in K\}$ is *separated* if

$$\inf_{k,s \in K, k \neq s} |x_k - x_s| > 0.$$

Throughout the paper J and K will denote countable index sets. One of the main ingredients in sampling theory is the notion of *lower Beurling density* $D^-(X)$ [**Beu66**] of a separated set $X = \{x_k \in \mathbb{R}^d : k \in K\}$, which is defined as:

$$D^-(X) = \lim_{r \to \infty} \frac{\nu^-(r)}{(2r)^d}$$

where $\nu^-(r) := \min_{y \in \mathbb{R}^d} \# \left(X \cap (y + [-r, r]^d) \right)$. $\#(Z)$ denotes the cardinal of the set Z.

The *upper Beurling Density* $D^+(X)$ is defined in a similar way:

$$D^+(X) = \lim_{r \to \infty} \frac{\nu^+(r)}{(2r)^d}$$

where $\nu^+(r) := \max_{y \in \mathbb{R}} \# \left(X \cap (y + [-r, r]^d) \right)$. If $D^-(X) = D^+(X) = D(X)$, then X is said to have *uniform Beurling density* $D(X)$.

Remark. Since X is separated, the limits in the definitions of $D^+(X)$ and $D^-(X)$ exist (see [**BW99**]).

Beurling [**Beu66**] introduced also the following notion of density: The *gap* ρ of the set $X = \{x_k : k \in K\}$ is defined as

$$\rho = \rho(X) = \inf \left\{ r > 0 : \bigcup_{k \in K} B_r(x_k) = \mathbb{R}^d \right\}.$$

Equivalently, the gap ρ can be defined as

$$\rho = \rho(X) = \sup_{x \in \mathbb{R}^d} \inf_{x_k \in X} |x - x_k|.$$

A family $\{Q_j : j \in J\}$ is a *covering* of $\mathbb{R}^d$ if $\mathbb{R}^d \setminus \bigcup_j Q_j$ has measure zero. A covering $\{Q_j : j \in J\}$ has *finite index* if every $x \in \mathbb{R}^d$ is at most in i sets of the covering for some fixed positive integer i. The minimum i with this property is called the *covering index*. We will denote by $e_x(w)$ the exponential of frequency x at w, that is $e_x(w) = e^{-2\pi i x w}$.

Let us now state a general theorem on wavelet frames:

THEOREM 1.1 (Wavelets). *Let A be an expansive matrix and $V \subset \mathbb{R}^d$ be any measurable bounded set containing 0 in its interior and such that its boundary ∂V has measure zero. Set $Q = \overline{A^T V \setminus V}$, and choose any function $h \in C^r(\mathbb{R}^d), r > 0, h \neq 0$ on Q such that* $\operatorname{Supp} h \in Q_\varepsilon$ *where $Q_\varepsilon := \{x \in \mathbb{R}^d : \operatorname{dist}(x, Q) \leq \varepsilon\}$. If the set $X = \{x_k \in \mathbb{R}^d : k \in K\}$ is separated and such that $\rho(X) < \frac{1}{4\delta}$ where $\delta = \operatorname{Diameter}(Q)$, then the following collection is a wavelet frame for $L^2(\mathbb{R}^d)$*

$$\{|\det A|^{j/2}\psi(A^j x - x_k) : k \in K,\ j \in \mathbb{Z})\}, \tag{1.1}$$

where ψ is the inverse Fourier transform of h.

REMARKS.

(1) The result of the theorem remains valid even if the matrix A is not expansive. For example let Q be any closed subset of $\mathbb{R}^d$, and A any invertible matrix. If $\mathbb{R}^d = \cup_j A^j Q_\varepsilon$ with finite covering index, then (1.1) is a frame for $L^2(\mathbb{R}^d)$.

(2) Instead of taking the powers A^j of a single matrix A we can choose a set of invertible matrices $\{A_j \in GL_d(\mathbb{R}) : j \in J\}$ without a particular group structure. In particular the index j can be a multi-index. For example, the set $J = \mathbb{Z} \times \{1, \cdots, N\}$, and the matrices $A_{(i,j)} = D^i R^j$ where R is a rotation and D a dilation matrix, will be used to construct directional wavelets.

(3) The wavelet can be constructed to have polynomial decay of any order by choosing r sufficiently large.

(4) The sets of translations $X_j = \{A^{-j}x_k : k \in K\}$ for each resolution level are not nested. However, the theorem can be easily modified to produce nested sets of translations $X_{j+1} \subset X_j$ for all $j \in \mathbb{Z}$ (c.f. [**ACM03**]).

(5) For the one dimensional case, ρ can be replaced by the Beurling density $D^-(X)$ which is a weaker condition and allows for arbitrary gaps between sampling points.

(6) If we choose h to be the characteristic function of the set Q_ε, then we obtain a wavelet frame set, and our construction (1.1) gives wavelet sets with translations on irregular grids.

Although the set $\{|\det A|^{j/2}\psi(A^j x - x_k) : j \in J,\ k \in K\}$ in Theorem 1.1 is a wavelet frame for $L^2(\mathbb{R}^d)$, it is not in general true that for a fixed j the set $\{\psi_{j,x_k}(x) = |\det A|^{j/2}\psi(A^j x - x_k) : k \in K\}$ is a frame. Thus, it appears at first, that the reconstruction of a function $f \in L^2(\mathbb{R}^d)$ from the wavelet coefficients $\{< f, \psi_{j,x_k} >: j \in J,\ k \in K\}$ cannot be obtained in a stable way by first reconstructing at each level j and then obtaining f by summing over all levels j. But in fact it is always possible to reconstruct each f_j in a stable way and then obtain f by summing up over all levels j, as is described in [**ACM03**].

2. Examples of wavelet frames on irregular lattices and with arbitrary set of dilation matrices and other transformations

2.1. Examples of Wavelet frames in $\mathbb{R}^d$.

(1) *Isotropic, well-localized wavelets*: Let V be the ball of radius 1/2 centered at the origin. Let $A = 2I$, then $Q = \overline{AV \setminus V} = \{x \in \mathbb{R}^2 : 1/2 \leq \|x\| \leq 1\}$. Let $\varepsilon = 1/4$, $h(\xi_1, \xi_2) = n\beta_{n-1}\left((\xi_1^2 + \xi_2^2 - 1/4)n\right)$, where β_n is the B-spline of degree n, i.e., the $\beta_n = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ is the n-fold convolution