SEVERAL COMPLEX VARIABLES

Local Theory

Second Edition

M. HERVÉ

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SEVERAL COMPLEX VARIABLES

Local Theory

TATA INSTITUTE OF FUNDAMENTAL RESEARCH STUDIES IN MATHEMATICS

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PREFACE

THIS book is an introduction to recent work in the theory of functions of several complex variables, especially on complex spaces. Many results of a local character, relating to the ring of germs of holomorphic functions at a given point, holomorphic mappings, analytic continuation, analytic sets and so on are usually assumed known, although they are not proved in the well-known books of Behnke-Thullen and Bochner-Martin and are available only in the original papers of H. Cartan, R. Remmert, K. Stein and others, or in seminar notes. (See [3], [5] in the references).

I thought that it might be useful to put all this material together, and that a new treatment might suggest fresh ideas. The treatment given here is as self-contained as was found possible. The reader needs only to be acquainted with the classical theory of holomorphic functions of a complex variable, and with a few results from Algebra, which are summarized in Chapter II, § 2.

The text is based on a course of lectures given early in 1962 at the Tata Institute of Fundamental Research, Bombay. I wish to express my gratitude to Professor K. Chandrasekharan who invited me to give these lectures and decided to have them printed.

The major part of the text was actually written by Mr. R. R. Simha. I think him warmly for his useful remarks and his whole-hearted co-operation. I am also indebted to Dr. Raghavan Narasimhan for important improvements.

Definitions and theorems are numbered in one sequence within each chapter, but lemmas and propositions within each section. A list of main definitions and results is given at the end of the book.

This second edition contains a few additions in close connection with the original text.

PARIS MICHEL HERVÉ

CONTENTS

I

BASIC PROPERTIES OF HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

In this chapter we present some basic properties of holomorphic functions of several complex variables, mostly without detailed proofs.

1. Holomorphic functions. The space we work in is an m-dimensional vector-space C^m over the field C of complex numbers, $m \geqslant 1$. In general we shall suppose a basis fixed for C^m, and identify C^m with the space of ordered m-tuples $x = (x_1, \ldots, x_m)$ of complex numbers.

Given a point $a = (a_1, \ldots, a_m)$ of C^m, and real numbers $r_1, \ldots, r_m > 0$, we call the set $P = \{(x_1, \ldots, x_m) \in C^m \mid |x_j - a_j| < r_j, j = 1, \ldots, m\}$ *the open polydisc with centre a and radii* r_j. Similarly, *the closed polydisc with centre a and radii* r_j is the set

$$\bar{P} = \{(x_1, \ldots, x_m) \in C^m \mid |x_j - a_j| \leqslant r_j, j = 1, \ldots, m\}.$$

The set

$$\Gamma = \{(x_1, \ldots, x_m) \in C^m \mid |x_j - a_j| = r_j, j = 1, \ldots, m\}$$

is called the *edge*, or *distinguished boundary*, of P (and of $\bar{P}$). The set of all open polydiscs is a basis for the "usual" topology on C^m. (The topology thus defined does not depend on the choice of the basis for C^m.)

Normal Convergence. A series $\sum f_n$ of complex-valued functions defined on a set E is said to converge normally on E if $\sum_n \|f_n\| < +\infty$; here $\|f_n\| = \sup_{x \in E} |f_n(x)|$.

Abel's Lemma. Suppose the power series

$$S = \sum_{k_1, \ldots, k_m \geqslant 0} a_{k_1 \ldots k_m} x_1^{k_1} \ldots x_m^{k_m}$$

in m variables $\in C$ converges* at the point $(b_1, \ldots, b_m)$ of C^m, and let $b_j \neq 0$ for $j = 1$ to m. Then S converges at every point of the

*Only absolute convergence will be considered in this book.

open polydisc P with centre the origin of C^m, and radii $|b_j|$; and S converges normally on every compact subset of P.

DEFINITION 1. *A complex-valued function f, defined on an open subset U of C^m, is holomorphic in U if for every point $b \in U$, there exist an open polydisc $P \subset U$ with centre b, and a power series*

$$\sum_{j_1, \ldots, j_m \geqslant 0} a_{j_1 \ldots j_m}(x_1 - b_1)^{j_1} \ldots (x_m - b_m)^{j_m}$$

converging to $f(x)$ at every point $x \in P$.

REMARK. Suppose f is holomorphic in U. Then by Abel's lemma, the power series S converges normally on compact subsets of P. Hence a holomorphic function is continuous.

PROPOSITION 1. *Let $f_1, \ldots, f_p$ be holomorphic functions on an open subset U of C^m, and suppose that for every x in U, $(f_1(x), \ldots, f_p(x))$ lies in a given open set V in C^p. Then for every holomorphic function g on V, the function $g(f_1(x), \ldots, f_p(x))$ is a holomorphic function of x on U.*

This follows from associative properties of normally convergent power series.

COROLLARY 1. *The holomorphy of a function on an open set in C^m does not depend on the choice of a basis for C^m.*

COROLLARY 2. *If f and g are holomorphic functions on an open set U in C^m, the functions $f + g$, fg are holomorphic in U. If g does not vanish anywhere in U, the function f/g is holomorphic in U.*

COROLLARY 3. *Let f be a holomorphic function on the open set U in C^m. Then for every point $a = (a_1, \ldots, a_m)$ of U, and every j $(1 \leqslant j \leqslant m)$, the function $f(a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_m)$ of one complex variable x_j, defined on the open set*

$$\{x_j \in C \mid (a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_m) \in U\}$$

in C, is holomorphic there.

REMARK. Suppose conversely that f is a complex-valued function on U such that each function of one complex variable obtained from f as above is holomorphic in the corresponding open set in C. Then *f is holomorphic in U* (Theorem of Hartogs-Osgood). This is a deep result, which we shall not prove in this book. For a proof, see Bochner and Martin ([1], p. 140).

PROPOSITION 2. *Let f be a holomorphic function on an open set U in C^m. For any integers $k_1, \ldots, k_m \geqslant 0$, the partial derivative $\dfrac{\partial^{k_1+\ldots+k_m} f}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}$ exists and is holomorphic in U. More precisely let $P \subset U$ be an open polydisc with centre b, and let S be a power series in the $x_j - b_j$, $j = 1, \ldots, m$, converging to f in P. Then the power series obtained by termwise differentiation of S, k_1 times with respect to $x_1, \ldots, k_m$ times with respect to x_m, converges to $\dfrac{\partial^{k_1+\ldots+k_m} f}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}$ in P.*

COROLLARY 1. *With the notation of Prop. 2, the coefficient of $(x_1 - b_1)^{k_1} \ldots (x_m - b_m)^{k_m}$ in S is $\dfrac{1}{k_1! \ldots k_m!} \dfrac{\partial^{k_1+\ldots+k_m} f(b)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}$. In particular, the power series S is uniquely determined by the values of f in a neighbourhood of b. We call it the Taylor expansion of f at b.*

COROLLARY 2. *Let f and g be holomorphic functions on a connected open set U in C^m. Suppose $f = g$ on a non-empty open subset of U. Then $f = g$ everywhere in U* (Principle of Analytic Continuation).

PROOF. Let $V = \{x \in U \mid f = g$ in a neighbourhood of $x\}$. V is non-empty by assumption, and by definition it is open. Plainly a point $x \in U$ belongs to V if and only if f and g have the same Taylor expansion at x. By Corollary 1, this means that

$$V = \bigcap_{k_1, \ldots, k_m \geqslant 0} \left\{ x \in U \,\middle|\, \frac{\partial^{k_1+\ldots+k_m} f(x)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} = \frac{\partial^{k_1+\ldots+k_m} g(x)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} \right\},$$

hence V is closed in U. U being connected, $V = U$, q.e.d.

REMARK. We shall see later (Chapter III, § 1) that if $f = g$ on a subset F of U such that $U - F$ is open and disconnected, or locally disconnected (i.e., there exists an open connected set $W \subset U$ such that $W \cap (U - F)$ is disconnected), then $f = g$ on U.

2. Germs of holomorphic functions. Let X be an arbitrary subset of C^m. We consider the set $E(X) = E$ of pairs (U, f), where U is an open set in C^m containing X and f a function holomorphic on U. We define a relation R on E as follows: $(U, f)\ R\ (V, g)$ if and only if $f = g$ on an open set $(\subset U \cap V)$ containing X. Clearly R is an equivalence relation.

DEFINITION 2. *A germ of a holomorphic function on X is an equivalence class of $E(X)$ with respect to the relation R.*

We denote by $\mathcal{H}(X)$ the set of germs of holomorphic functions on X. With the obvious addition and multiplication, $\mathcal{H}(X)$ is a commutative ring with identity. It is clear that each element of $\mathcal{H}(X)$ has a well-defined value at each point of X; however, in general, distinct elements of $\mathcal{H}(X)$ may have the same value at all points of X.

REMARK 1. Suppose the set X is the closure of its interior; or suppose each connected component of X has an interior point. Then any element of $\mathcal{H}(X)$ is uniquely determined by its values on X. In fact, suppose $(U, f), (V, g) \in E(X)$, and $f = g$ on X. Then, in both cases, it is clear from Corollary 2 to Prop. 2 that $f = g$ on all connected components of $U \cap V$ meeting X. Hence $(U, f)\ R\ (V, g)$.

REMARK 2. Suppose X contains just one point a. In this case we write $\mathcal{H}(X) = \mathcal{H}_a^m$. Two functions holomorphic in an open neighbourhood of a coincide in a neighbourhood of a if and only if they have the same Taylor expansion at a. Thus $\mathcal{H}_a^m$ is isomorphic to the ring of convergent power series in m complex variables. (A power-series $\sum_{k_1, \ldots, k_m \geqslant 0} a_{k_1 \ldots k_m} x_1^{k_1} \ldots x_m^{k_m}$ is said to be convergent if it converges in an open polydisc with centre the origin of C^m.) The value of an element of $\mathcal{H}_a^m$ at a is, of course, the constant term of its Taylor series.

3. Cauchy's integral formula. Let f be (a germ of) a holomorphic function on the closed polydisc $\bar{P}$ in C^m with centre a and radii r_j. Then for every point x of the open polydisc P with the same centre and radii, we have

$$f(x) = \frac{1}{(2\pi i)^m} \int_{|y_m - a_m| = r_m} dy_m \ldots \int_{|y_1 - a_1| = r_1} \frac{f(y_1, \ldots, y_m)}{(y_1 - x_1) \ldots (y_m - x_m)} dy_1$$

where, for the integrations, the circles $|y_j - a_j| = r_j$ are assumed positively oriented.

This follows immediately from Cauchy's integral formula for holomorphic functions of one complex variable.

Corollary 1. *With the above notation the Taylor series* $S = \sum_{k_1, \ldots, k_m \geqslant 0} a_{k_1 \ldots k_m}(x_1 - a_1)^{k_1} \ldots (x_m - a_m)^{k_m}$ *of* f *at a converges in* P.

In fact, the integrand in the Cauchy integral formula can be expanded in a power series in the $x_j - a_j$, converging in P, and with coefficients which are functions of y on the edge Γ of P. Since f is continuous, hence bounded, on Γ, this series (for each x in P) converges normally on Γ. The series can therefore be integrated termwise, and yields a power series in the $x_j - a_j$ converging to f in P.

Remark. Suppose that f is a holomorphic function on an open set U in C^m. The above result implies that the Taylor series of f at any point $a \in U$ converges in any polydisc with centre a, contained in U.

Corollary 2. *With the notation of Corollary* 1, *we have, for any* $k_1, \ldots, k_m \geqslant 0$,

$$|a_{k_1 \ldots k_m}| \leqslant \frac{1}{r_1^{k_1} \ldots r_m^{k_m}} \sup_{y \in \Gamma} |f(y)|,$$

where Γ *is the edge of* P (Inequalities of Cauchy).

In fact the integral formula yields

$$a_{k_1 \ldots k_m} = \frac{1}{(2\pi i)^m} \int_{|y_m - a_m| = r_m} dy_m \ldots \int_{|y_1 - a_1| = r_1} \frac{f(y_1, \ldots, y_m)}{(y_1 - a_1)^{k_1+1} \ldots (y_m - a_m)^{k_m+1}} dy_1,$$

leading to the given majorisations.

REMARK. Let $\mathcal{F}$ be a family of holomorphic functions on an open set U of C^m, uniformly bounded on compact subsets of U. Since the Taylor series of the partial derivatives of a holomorphic function may be obtained by termwise differentiation, the inequalities of Cauchy imply that for every $k_1, \ldots, k_m \geqslant 0$ the family $\left\{\frac{\partial^{k_1 + \ldots + k_m} f}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} \middle| f \in \mathcal{F}\right\}$ is also uniformly bounded on compact subsets of U. In particular, the family of all first partial derivatives of members of $\mathcal{F}$ is uniformly bounded on compact subsets of U, hence the family $\mathcal{F}$ is *equicontinuous*. It follows from Ascoli's Theorem (see [2], p. 43), that one may extract, from any infinite sequence of members of $\mathcal{F}$, a subsequence which converges uniformly on every compact subset of U.

COROLLARY 3. (The Maximum Principle.) *Let f be a holomorphic function on an open set U in C^m. Let ∂U be the boundary of U—if U is not relatively compact in C^m, ∂U is to include the point at infinity of C^m. Suppose that, for every point y of ∂U,* $\limsup_{x \to y,\, x \in U} |f(x)| \leqslant M$. *Then* (i) $|f| \leqslant M$ *in U,* (ii) *if* $|f(x_0)| = M$ *for a point x_0 in U, then $f(x) \equiv f(x_0)$ on the connected component of U containing x_0.*

This is proved as in the case of one complex variable, using the integral formula.

4. Weierstrass' Theorem. *If a sequence $\{f_n\}$ of functions, holomorphic on an open subset U of C^m, converges uniformly on every compact subset of U, then* (i) *the limit function f is holomorphic in U,* (ii) *for any $k_1, \ldots, k_m \geqslant 0$, the sequence* $\left\{\frac{\partial^{k_1 + \ldots + k_m} f_n}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}\right\}$ *converges to* $\frac{\partial^{k_1 + \ldots + k_m} f}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}$ *on U, uniformly on every compact subset of U.*

The first statement is proved by using Cauchy's integral formula, the second one then follows from Cauchy's inequalities.

COROLLARY 1. *If a power series in $x_1, \ldots, x_m$ converges in an open* *polydisc P with the origin of C^m as centre, then the sum is a holomorphic function on P.*

COROLLARY 2. *Let U be an open set in C^m, K a compact space, and μ a Radon measure on K. Suppose $(x, y) \to f(x, y)$ is a continuous function on $U \times K$, and $x \to f(x, y)$ is a holomorphic function on U for each fixed y in K. Then* (i) *the function $F(x) = \int_K f(x, y)\, d\mu(y)$ is holomorphic in U,* (ii) *for any $k_1, \ldots, k_m \geqslant 0$,*

$$\frac{\partial^{k_1 + \ldots + k_m} F(x)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} = \int_K \frac{\partial^{k_1 + \ldots + k_m} f(x, y)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}\, d\mu(y).$$

We thus obtain, in particular, the Cauchy integral formulas for the partial derivatives of a function holomorphic on a closed polydisc.

5. Holomorphic mappings.

DEFINITION 3. *A mapping $f(x) = (f_1(x), \ldots, f_p(x))$ of an open subset U of C^m into C^p is holomorphic if its coordinates $f_1(x), \ldots, f_p(x)$ are holomorphic functions on U. If $p = m$, then the Jacobian $J_f(x)$ of f at $x \in U$ is the determinant of the matrix $\left(\frac{\partial f_i(x)}{\partial x_j}\right)$.*

We shall give a complete proof of the following theorem.

THEOREM 1. *Let f be a holomorphic mapping of an open subset U of C^m into C^m, and suppose $J_f(a) \neq 0$ at a point $a \in U$. Then there exist open neighbourhoods $V(\subset U)$ and V' of a and $f(a)$ respectively such that* (i) *the restriction $f \mid V$ of f to V is one-one in V and maps V onto V',* (ii) *the inverse mapping of $f \mid V$ is holomorphic in V'.*

PROOF. We may assume that $a = f(a) = \mathfrak{o}$, the origin of C^m. Since $J_f(\mathfrak{o}) \neq 0$, we may also assume a basis for C^m so chosen that the matrix of $J_f(\mathfrak{o})$ is the identity matrix. For $x \in U$, let us write $f(x) = x - g(x)$. Then $g(x)$ defines a holomorphic mapping of U into C^m, and all the coordinates $g_j(x)$, with all their first partial derivatives, vanish at $\mathfrak{o}$. Using the mean-value theorem of the differential calculus, we can therefore find an open polydisc $P \subset U$, with centre $\mathfrak{o}$ and radii r, such that for all x, x' in P,

$$\sup_{1 \leqslant j \leqslant m} |g_j(x) - g_j(x')| < \tfrac{1}{2} \sup_{1 \leqslant j \leqslant m} |x_j - x'_j|. \qquad (*)$$

Let P' be the open polydisc with centre $\mathfrak{o}$ and radii $r/2$; (*) implies in particular that $g(P) \subset P'$. We shall show that the assertions of Theorem 1 are valid with $V = P \cap f^{-1}(P')$, and $V' = P'$.

By definition of V, V', we have $f(V) \subset V'$. We assert first that $f \mid V$ is one-one. Suppose in fact that $x, x' \in V$, $x \neq x'$ and $f(x) = f(x')$. Then $x - x' = g(x) - g(x')$, so that

$$\sup_{1 \leqslant j \leqslant m} |g_j(x) - g_j(x')| = \sup_{1 \leqslant j \leqslant m} |x_j - x'_j| \quad (>0, \text{ since } x \neq x').$$

However, $V \subset P$, so this is a contradiction. Now for any $y \in V$, we set $x^{(0)}(y) = y$, and define $x^{(n)}(y)$ for $n \geqslant 1$ inductively by $x^{(n)}(y) = y + g(x^{(n-1)}(y))$—since $g(P) \subset P'$, it is easily checked, by induction, that $x^{(n)}(y) \in P$ for all n. We have, for $n \geqslant 2$,

$$x^{(n)}(y) - x^{(n-1)}(y) = g(x^{(n-1)}(y)) - g(x^{(n-2)}(y)),$$

and hence (*) easily leads to the majorisation

$$\sup_{1 \leqslant j \leqslant m} |x_j^{(n)}(y) - x_j^{(n-1)}(y)| \leqslant \frac{r}{2^n}, \quad n \geqslant 1.$$

Hence the sequences $\{x_j^{(n)}(y)\}$ converge uniformly on V'. Plainly the $x_j^{(n)}(y)$ are holomorphic functions on V', hence the limit $x_j(y)$ of the $x_j^{(n)}(y)$ is a holomorphic function on V, for $j = 1, \ldots, m$. Set $x(y) = (x_1(y), \ldots, x_m(y))$. Then the mapping $x(y)$ of V into C^m is holomorphic. Since, for every n, $x^{(n)}(y) - y \in P'$, $x(y) - y$ lies in $\bar{P}'$. Since $y \in P'$, this means that $x(y)$ lies in P. Again, for every n, $x^{(n)}(y) - g(x^{(n-1)}(y)) = y$, hence $x(y) - g(x(y)) = y$, i.e., $f(x(y)) = y$. Since $f \mid V$ is one-one, this shows that $f(V) = V'$, and that $x(y)$ is the inverse of $f \mid V$, q.e.d.

REMARK. Conversely, suppose f is a one-one holomorphic mapping of an open set U of C^m onto an open subset of C^p. Then $p = m$, and the Jacobian of f never vanishes in U. This will be proved later (Chapter IV, § 5).

II

THE RING OF GERMS OF HOLOMORPHIC FUNCTIONS AT A POINT

In this chapter, we shall be mainly concerned with the "Preparation Theorem of Weierstrass" and some of its consequences. This theorem is an important tool in the local study of the zeros of holomorphic functions.

1. Preparation theorems. Let $\mathcal{H}_a^m$ denote, as before, the ring of germs of holomorphic functions at the point $a \in C^m$. If f is a holomorphic function on some open neighbourhood of a in C^m, we denote by $\mathbf{f}$ the element of $\mathcal{H}_a^m$ induced by f. In particular, $\mathbf{0}$ and $\mathbf{1}$ are respectively the zero element and the identity of $\mathcal{H}_a^m$.

PROPOSITION 1. *$\mathcal{H}_a^m$ is an integral domain.*

PROOF. As observed before, $\mathcal{H}_a^m$ is isomorphic to the ring of convergent power series in m variables over C, which is a subring of the ring $\mathcal{F}^m$ of formal power series in m variables over C. Since $\mathcal{F}^m$ is isomorphic (for $m \geqslant 2$) to the ring of formal power series in one variable over $\mathcal{F}^{m-1}$, we can deduce, by induction on m, that $\mathcal{F}^m$ is an integral domain, from the following fact: the ring $A[[X]]$ of formal power series in one variable X over an integral domain A is an integral domain. To prove this last fact, let $f = \sum_{k \geqslant p} a_k X^k$, $g = \sum_{k \geqslant q} b_k X^k$, $p, q \geqslant 0$, $a_p \neq 0$, $b_q \neq 0$, be two non-zero elements of $A[[X]]$. Then $fg = \sum_{k \geqslant p+q} c_k X^k$ with $c_{p+q} = a_p b_q \neq 0$, q.e.d.

REMARK. Proposition 1 may also be deduced from the principle of analytic continuation: if f is holomorphic on an open connected set U in C^m, and vanishes on a non-empty open subset of U, then f vanishes identically on U.

An element $\mathbf{f} \in \mathcal{H}_a^m$ is invertible if and only if $f(a) \neq 0$. Hence the set of non-invertible elements of $\mathcal{H}_a^m$ is an ideal (which is therefore the unique (proper) maximal ideal of $\mathcal{H}_a^m$). We denote it by $\mathcal{H}'^m_a$.

DEFINITION 1. *Two elements* $\mathbf{f}, \mathbf{g} \in \mathcal{H}_a^m$ *are equivalent*: $\mathbf{f} \sim \mathbf{g}$, *if there exists an invertible element* $\mathbf{h} \in \mathcal{H}_a^m$ *such that* $\mathbf{f} = \mathbf{hg}$.

Clearly $\sim$ is an equivalence relation in $\mathcal{H}_a^m$; and $\mathbf{f} \sim \mathbf{g}$ implies that f and g have the same zeros in a neighbourhood of a. The trivial equivalence-classes in $\mathcal{H}_a^m$ with respect to this relation are those of $\mathbf{0}$ and $\mathbf{1}$, consisting, respectively, of $\mathbf{0}$ alone and of all invertible elements of $\mathcal{H}_a^m$.

If $x=(x_1, \ldots, x_m)$ is any point of $C^m (m \geqslant 2)$, we shall denote by x' the point $(x_1, \ldots, x_{m-1}) \in C^{m-1}$; in particular, $\mathfrak{o}$ and $\mathfrak{o}'$ are respectively the origins of C^m and C^{m-1}. Conversely, if $x'=(x_1, \ldots, x_{m-1}) \in C^{m-1}$, and $x_m \in C$, (x', x_m) denotes the point $(x_1, \ldots, x_{m-1}, x_m) \in C^m$. We shall also write $\mathcal{H}_{\mathfrak{o}}^m = \mathcal{H}^m$, $\mathcal{H}'^m_{\mathfrak{o}} = \mathcal{H}'^m$.

DEFINITION 2. *A distinguished pseudo-polynomial in* x_m, *of degree* p, *is an expression of the form* $x_m^p + \sum_{k=1}^{p} c_k(x') x_m^{p-k}$, $p \geqslant 1$, *where the* $c_k(x')$ *are holomorphic functions on open neighbourhoods of* $\mathfrak{o}'$ *in* C^{m-1}, *vanishing at* $\mathfrak{o}'$.

A distinguished pseudo-polynomial induces a non-zero and non-invertible element in $\mathcal{H}^m$.

THEOREM 1. (The Weierstrass Preparation Theorem.) *Suppose given an element* $\mathbf{f} \in \mathcal{H}'^m$, $\mathbf{f} \neq \mathbf{0}$. *Then*: (i) *we can choose a basis for* C^m *in such a way that* $f(\mathfrak{o}', x_m)$ *does not vanish identically in any neighbourhood of* $x_m = 0$ *in* C^1; (ii) *if the basis for* C^m *is such that* $f(\mathfrak{o}', x_m)$ *does not vanish identically in any neighbourhood of* $x_m = 0$ *in* C^1, *there exists a distinguished pseudo-polynomial* $\phi = x_m^p + \sum_{k=1}^{p} c_k(x') x_m^{p-k}$ *such that* $\mathbf{f} \sim \boldsymbol{\varphi}$; (iii) *if, with respect to the same basis as in* (ii), $\psi = x_m^q + \sum_{k=1}^{q} d_k(x') x_m^{q-k}$ *is any distinguished pseudo-polynomial such that* $\boldsymbol{\psi} \sim \mathbf{f}$, *then* $q = p$, *and for every* k, $1 \leqslant k \leqslant p$, c_k *and* d_k *induce the same element of* $\mathcal{H}'^{m-1}_{\mathfrak{o}'} = \mathcal{H}'^{m-1}$.

PROOF. (i) *Choice of the basis for* C^m. Let U be an open convex