Relativistic Numerical Hydrodynamics

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Relativistic Numerical Hydrodynamics

This book presents an overview of the computational framework in which calculations of relativistic hydrodynamics have been developed. It summarizes the jargon and methods used in the field, and provides illustrative applications to real physical systems. The authors explain how to break down the complexities of Einstein's equations and fluid dynamics, stressing the viability of the Euler–Lagrange approach to astrophysical problems. The book contains techniques and algorithms enabling one to build computer simulations of relativistic fluid problems for various astrophysical systems in one, two, and three dimensions. It also shows the reader how to test relativistic hydrodynamics codes.

Suitable for use as a textbook for graduate courses on astrophysical hydrodynamics and relativistic astrophysics, this book also provides a valuable reference for researchers already working in the field.

JAMES WILSON is widely recognized as a pioneer in the field of numerical relativity and hydrodynamics. Most of the techniques currently in active use in the field today were developed by him at one stage or another. He is best known for having first solved the supernova explosion mechanism by delayed neutrino heating, as well as for developing simulations for accreting black holes, black hole and neutron star collisions, supernova jets, and binary neutron stars. In 1994 he was awarded the Marcel Grossman General Relativity Prize for his contributions to the development of the field of numerical relativity. He is also the author of numerous publications in numerical astrophysics.

GRANT MATHEWS is Professor of Theoretical Astrophysics and Cosmology at Notre Dame University, Indiana. He has been working together with Jim Wilson for the past 15 years on the development of techniques for relativistic hydrodynamics in three spatial dimensions. He has published over 200 papers in areas of theoretical and experimental astrophysics, cosmology, and relativity.

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Dedicated to our loving and patient wives Demetra and Eve

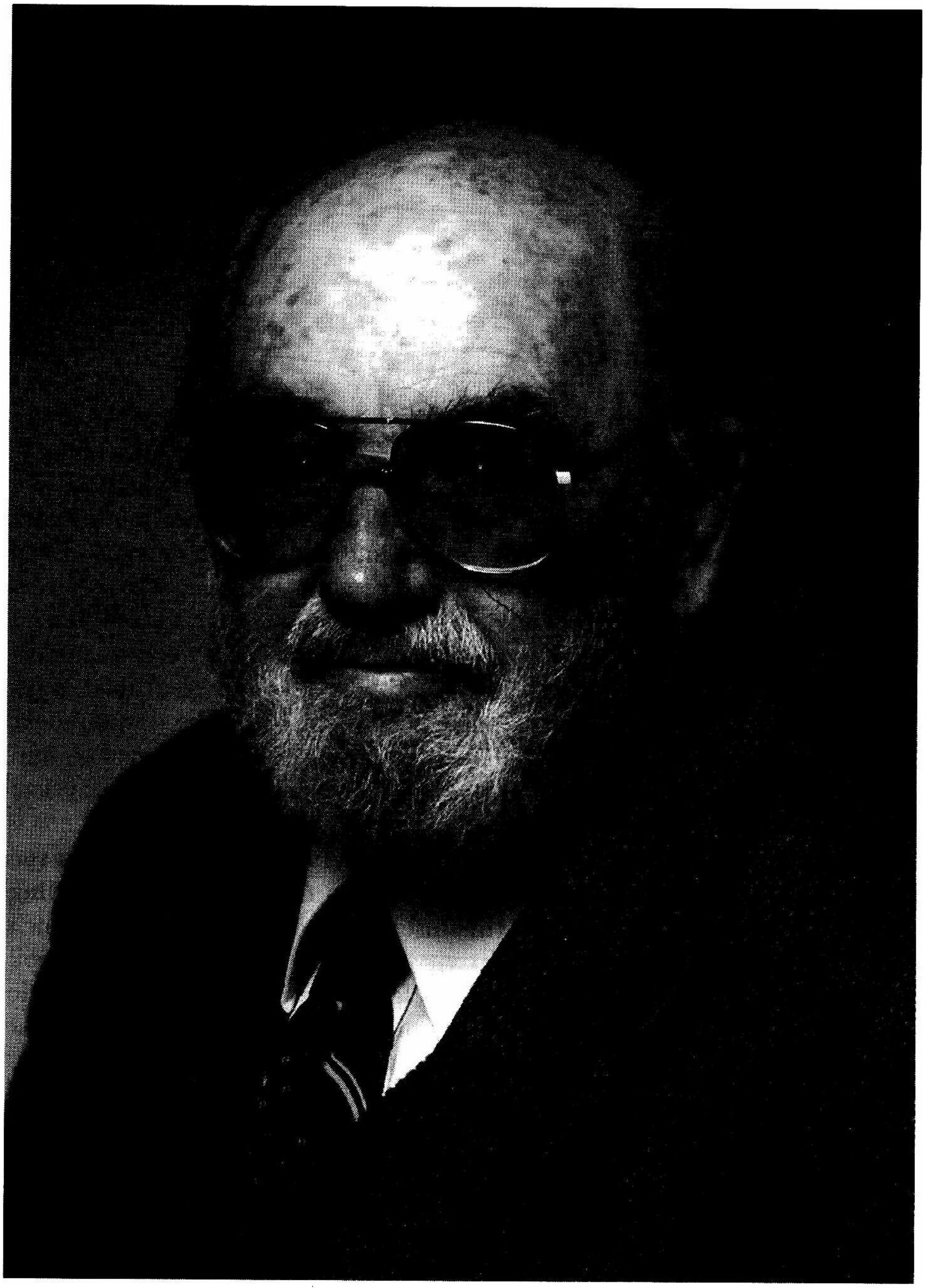
Preface

We are convinced of a genuine need for a monograph describing the many facets and new developments in numerical relativistic hydrodynamics. Such calculations are crucial to several areas of current research in the physics of stellar collapse, supernovae, and black hole formation, as well as the merging of the final orbits of coalescing binary neutron stars. Both problems are only now entering the level of sophistication where three-dimensional relativistic hydrodynamics simulations are both possible and necessary. In the former problem such calculations are crucial to understand the explosion mechanism. In the latter problem, a great deal of interest in such calculations has recently been inspired by the development of next-generation gravity wave detectors to search for such events, and as a possible explanation of the physics underlying observed astrophysical γ-ray bursts.

The field of numerical relativistic hydrodynamics has developed over the past 30 years, but there has not been written a technical text explaining the many techniques relevant to this discipline, many of which are much different than standard general relativity textbook approaches. This book will present such a review of techniques for numerical general relativistic hydrodynamics developed by one of the pioneers of this field over the past three decades.

We begin by developing the equations and differencing schemes for special relativistic hydrodynamics as an introduction to the metric formulation of the problems. Here, the basic numerical techniques and a number of test problems and applications will be discussed.

Following this, the formalism for matter flows in the curved spacetime of general relativity will be presented in the usual (3+1) formalism. With the techniques established, the next chapter will then summarize cosmological applications in one spatial dimension. This will also lead naturally to



a discussion of core-collapse supernovae in spherical symmetry including the many physical complexities due to neutrino interactions and a large range of dynamic timescales.

Next we will describe some important axisymmetric problems such as stellar and black hole rotation, accretion and the head-on collision of two neutron stars. This topic also naturally leads to a discussion of magnetohydrodynamics and its applications to axisymmetric problems.

The book then finally discusses an application in three spatial dimensions: the hydrodynamics of orbiting neutron stars. This chapter focuses on the development of the conformally flat approximation and techniques for analyzing the gravitational radiation generated by stellar collapse and binary mergers.

This book would best be described as a monograph. That is, it is a summary by experts in the field written for others at a similar level. Nevertheless, enough introductory material has been included that a graduate student or nonexpert can become familiar with the concepts without additional resource material. The main point of this book is to provide a summary of results and techniques for both the expert and nonexpert in one complete text. Some of this material has never been published and only exists in private notes. Most of the available material only exists in a number of journal publications and/or obscure conference proceedings, many of which are no longer in print.

The work described herein is of course the result of the efforts of many knowledgeable collaborators. We would particularly like to acknowledge the important contributions to the general relativistic hydrodynamics work discussed herein from Joan Centrella, Sam Dalhed, Steven Detweiler, Peter Dykema, Charles Evans, Chris Fragile, Hannu Kurki-Suonio, James LeBlanc, Pedro Marronetti, Richard Matzner, Ronald Mayle, Thomas McAbee, Jay Salmonson, and Larry Smarr. We would also like to acknowledge useful input from Peter Anninos and Dinshaw Balsara, along with help from Heidi Grantham in the preparation of some of the figures.

Contents

1
Introduction

Relativistic numerical hydrodynamics is currently a field of intense interest. On the one hand, the development of next-generation laser interferometric and cryogenic gravity wave detectors is opening a new window of astronomy, one which will peer into a world of multidimensional rapidly varying matter and gravity fields such as occur in and around neutron stars, black holes, supernovae, compact binary systems, dense clusters, collapsing stars, the early universe, etc. At the same time, X-ray and γ-ray observatories are providing (or will soon provide) a wealth of data on the evolution of matter in and around X-ray and γ-ray emitting compact objects such as accreting black holes and neutron stars. Such systems can only be realistically analyzed by a detailed numerical study of the spacetime and matter fields.

A quantitative understanding of these systems as well as a host of other astrophysical phenomena such as stellar collapse leading to supernovae, the evolution of massive stars, and the origin of γ-ray bursts, the origin and evolution of relativistic jets, all require multidimensional complex relativistic numerical simulations in three spatial dimensions. Since analytic and post-Newtonian methods are only applicable for systems of special symmetry and/or relatively weak fields, numerical relativistic hydrodynamics is the only viable method to model such highly dynamical asymmetrical strong field systems.

The technology for observing such energetic astrophysical phenomena has developed in concert with the development of high speed computing. Hence, it is perhaps no accident that the requirement for next-generation multi-dimensional relativistic hydrodynamics modeling is occurring at a time when computers are just now approaching the speed and memory capability needed to explore such systems. For these reasons, it is expected that there will be much research in relativistic numerical hydrodynamics calculations in the coming years, hence the need for a book reviewing the development of the subject.

The textbooks from which most of us learn general relativity usually emphasize a number of analytic solutions of some special cases, like that of an isotropic Schwarzschild or Friedmann metric. Indeed, one is hard pressed to think of a problem in relativity which can still be addressed with paper and pencil. The remaining real-world applications in astrophysics and cosmology cannot be seriously studied analytically, nor can one ignore the hydrodynamic evolution of the matter fields. Such systems must be studied numerically. Indeed, the solution of numerical problems often requires one to abandon some or all aspects of Newtonian or even post-Newtonian intuition. Our goal here will be to provide an overview of the computational framework in which such calculations have been done, along with illustrative applications to real physical systems.

This book does not, however, attempt to give a comprehensive overview of how to do numerical relativistic hydrodynamics calculations. It is rather a compilation of those projects with which one or both of the authors have had some involvement. An attempt at a comprehensive overview of a field in which there have been so many significant contributors would be difficult. Hence, although we shall refer here to a number of other works in the field, this text will for the most part only summarize the contributions of the authors and collaborators. These are the works with which we are most familiar. Nevertheless, in the process of reading this text, it is hoped that the reader will gain some understanding of the development of relativistic hydrodynamics which has occurred over the past 30 years.

In what follows we will assume that the reader has some familiarity with basic concepts in special and general relativity. We only provide enough introductory material so that the relativistic field and matter equations can be introduced in a context which is most easily applied to numerical problems, and not in the way they might be introduced in an introductory text in either relativity or hydrodynamics alone.

1.1 Notation and convention

As with any other intensely mathematical subject, a text on numerical relativity should contain a concise summary of notation and convention in one location. Hence, we begin with an overview of the notation and conventions which we have attempted to maintain throughout the book. By and large, these are the conventions widely adopted in the field, and as such, comprise useful introductory material.

In what follows we use the usual convention of Greek indices to denote components of four-dimensional spacetime ($\mu = 0, 1, 2, 3$). When referring to a specific coordinate system they will be identified according to normal convention (e.g. $\mu = t, x, y, z$ for Cartesian coordinates). We use Latin characters, $i, j, k, \ldots$ to denote spatial indices. Partial differentiation will be denoted in both the explicit and abbreviated form, e.g.

$$\frac{\partial}{\partial x^\mu} = \partial_\mu. \tag{1.1}$$

Partial differentiation along the time coordinate will also frequently be denoted by the familiar "dot" notation,

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial x^0} = \dot{A}. \tag{1.2}$$

We will also make use of geometrized units, $c = G = 1$. For convenience, Table 1.1 gives conversions from cgs units to geometrized units for various parameters in use in this text.

1.2 General relativity

A brief summary of general relativity is a necessary starting point for introducing concepts and notation to be encountered in subsequent chapters. General relativity derives from the principle of equivalence which asserts that at every spacetime point we can choose a coordinate system such that the laws of physics have the same form as they would in the absence of a gravitational field. This principle has led to the Einstein field equations which relate the curvature of spacetime to the distribution of mass–energy,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.3}$$

where $T_{\mu\nu}$ is the energy momentum (or stress energy) tensor.

The Einstein tensor $G_{\mu\nu}$ can be written in terms of the Ricci tensor $R_{\mu\nu}$, metric tensor $g_{\mu\nu}$, and Ricci scalar R,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{1.4}$$

where the Ricci tensor is a contraction of the Reimann tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \tag{1.5}$$

and

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.6}$$