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COMPLEX ANALYSIS

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An Introduction to the Theory of Analytic Functions of One Complex Variable

Third Edition

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To Erna

Preface

Complex Analysis has successfully maintained its place as the standard elementary text on functions of one complex variable. There is, nevertheless, need for a new edition, partly because of changes in current mathematical terminology, partly because of differences in student preparedness and aims.

There are no radical innovations in the new edition. The author still believes strongly in a geometric approach to the basics, and for this reason the introductory chapters are virtually unchanged. In a few places, throughout the book, it was desirable to clarify certain points that experience has shown to have been a source of possible misunderstanding or difficulties. Misprints and minor errors that have come to my attention have been corrected. Otherwise, the main differences between the second and third editions can be summarized as follows:

1. Notations and terminology have been modernized, but it did not seem necessary to change the style in any significant way.

2. In Chapter 2 a brief section on the change of length and area under conformal mapping has been added. To some degree this infringes on the otherwise self-contained exposition, for it forces the reader to fall back on calculus for the definition and manipulation of double integrals. The disadvantage is minor.

3. In Chapter 4 there is a new and simpler proof of the general form of Cauchy's theorem. It is due to A. F. Beardon, who has kindly permitted me to reproduce it. It complements but does not replace the old proof, which has been retained and improved.

4. A short section on the Riemann zeta function has been included.

This always fascinates students, and the proof of the functional equation illustrates the use of residues in a less trivial situation than the mere computation of definite integrals.

5. Large parts of Chapter 8 have been completely rewritten. The main purpose was to introduce the reader to the terminology of germs and sheaves while emphasizing all the classical concepts. It goes without saying that nothing beyond the basic notions of sheaf theory would have been compatible with the elementary nature of the book.

6. The author has successfully resisted the temptation to include Riemann surfaces as one-dimensional complex manifolds. The book would lose much of its usefulness if it went beyond its purpose of being no more than an introduction to the basic methods and results of complex function theory in the plane.

It is my pleasant duty to thank the many who have helped me by pointing out misprints, weaknesses, and errors in the second edition. I am particularly grateful to my colleague Lynn Loomis, who kindly let me share student reaction to a recent course based on my book.

Lars V. Ahlfors

COMPLEX ANALYSIS

Contents

1 COMPLEX NUMBERS

1. THE ALGEBRA OF COMPLEX NUMBERS

It is fundamental that real and complex numbers obey the same basic laws of arithmetic. We begin our study of complex function theory by stressing and implementing this analogy.

1.1. Arithmetic Operations. From elementary algebra the reader is acquainted with the *imaginary unit* i with the property $i^2 = -1$. If the imaginary unit is combined with two real numbers α, β by the processes of addition and multiplication, we obtain a *complex number* $\alpha + i\beta$. α and β are the *real* and *imaginary part* of the complex number. If $\alpha = 0$, the number is said to be *purely imaginary;* if $\beta = 0$, it is of course *real.* Zero is the only number which is at once real and purely imaginary. Two complex numbers are equal if and only if they have the same real part and the same imaginary part.

Addition and multiplication do not lead out from the system of complex numbers. Assuming that the ordinary rules of arithmetic apply to complex numbers we find indeed

$$(\alpha + i\beta) + (\gamma + i\delta) = (\alpha + \gamma) + i(\beta + \delta) \tag{1}$$

and

$$(\alpha + i\beta)(\gamma + i\delta) = (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma). \tag{2}$$

In the second identity we have made use of the relation $i^2 = -1$.

It is less obvious that division is also possible. We wish to show that $(\alpha + i\beta)/(\gamma + i\delta)$ is a complex number, provided that $\gamma + i\delta \neq 0$. If the quotient is denoted by $x + iy$, we must have

$$\alpha + i\beta = (\gamma + i\delta)(x + iy).$$

By (2) this condition can be written

$$\alpha + i\beta = (\gamma x - \delta y) + i(\delta x + \gamma y),$$

and we obtain the two equations

$$\begin{aligned} \alpha &= \gamma x - \delta y \\ \beta &= \delta x + \gamma y. \end{aligned}$$

This system of simultaneous linear equations has the unique solution

$$\begin{aligned} x &= \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} \\ y &= \frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2}, \end{aligned}$$

for we know that $\gamma^2 + \delta^2$ is not zero. We have thus the result

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + i\,\frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2}. \tag{3}$$

Once the existence of the quotient has been proved, its value can be found in a simpler way. If numerator and denominator are multiplied with $\gamma - i\delta$, we find at once

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{(\alpha + i\beta)(\gamma - i\delta)}{(\gamma + i\delta)(\gamma - i\delta)} = \frac{(\alpha\gamma + \beta\delta) + i(\beta\gamma - \alpha\delta)}{\gamma^2 + \delta^2}.$$

As a special case the reciprocal of a complex number $\neq 0$ is given by

$$\frac{1}{\alpha + i\beta} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2}.$$

We note that i^n has only four possible values: $1, i, -1, -i$. They correspond to values of n which divided by 4 leave the remainders 0, 1, 2, 3.

EXERCISES

1. Find the values of

$$(1 + 2i)^3, \qquad \frac{5}{-3 + 4i}, \qquad \left(\frac{2 + i}{3 - 2i}\right)^2, \qquad (1 + i)^n + (1 - i)^n.$$

2. If $z = x + iy$ (x and y real), find the real and imaginary parts of

$$z^4, \qquad \frac{1}{z}, \qquad \frac{z-1}{z+1}, \qquad \frac{1}{z^2}.$$

3. Show that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1 \qquad \text{and} \qquad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of signs.

1.2. Square Roots. We shall now show that the square root of a complex number can be found explicitly. If the given number is $\alpha + i\beta$, we are looking for a number $x + iy$ such that

$$(x + iy)^2 = \alpha + i\beta.$$

This is equivalent to the system of equations

$$\begin{aligned} x^2 - y^2 &= \alpha \\ 2xy &= \beta. \end{aligned} \tag{4}$$

From these equations we obtain

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \alpha^2 + \beta^2.$$

Hence we must have

$$x^2 + y^2 = \sqrt{\alpha^2 + \beta^2},$$

where the square root is positive or zero. Together with the first equation (4) we find

$$\begin{aligned} x^2 &= \tfrac{1}{2}(\alpha + \sqrt{\alpha^2 + \beta^2}) \\ y^2 &= \tfrac{1}{2}(-\alpha + \sqrt{\alpha^2 + \beta^2}). \end{aligned} \tag{5}$$

Observe that these quantities are positive or zero regardless of the sign of α.

The equations (5) yield, in general, two opposite values for x and two for y. But these values cannot be combined arbitrarily, for the second equation (4) is not a consequence of (5). We must therefore be careful to select x and y so that their product has the sign of β. This leads to the general solution

$$\sqrt{\alpha + i\beta} = \pm\left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + i\frac{\beta}{|\beta|}\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}\right) \tag{6}$$

provided that $\beta \neq 0$. For $\beta = 0$ the values are $\pm\sqrt{\alpha}$ if $\alpha \geqq 0$, $\pm i\sqrt{-\alpha}$