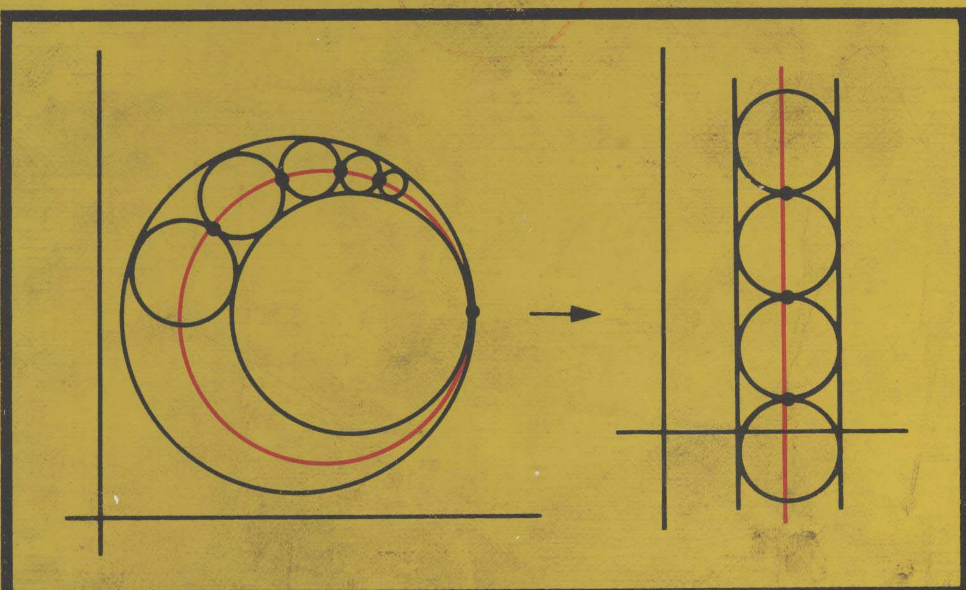
Undergraduate Texts in Mathematics

Joseph Bak
Donald J. Newman

Complex Analysis



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With 69 Illustrations

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Preface

One of our goals in writing this book has been to present the theory of analytic functions with as little dependence as possible on advanced concepts from topology and several-variable calculus. This was done not only to make the book more accessible to a student in the early stages of his mathematical studies, but also to highlight the authentic complex-variable methods and arguments as opposed to those of other mathematical areas. The minimum amount of background material required is presented, along with an introduction to complex numbers and functions, in Chapter 1.

Chapter 2 offers a somewhat novel, yet highly intuitive, definition of analyticity as it applies specifically to polynomials. This definition is related, in Chapter 3, to the Cauchy-Riemann equations and the concept of differentiability. In Chapters 4 and 5, the reader is introduced to a sequence of theorems on entire functions, which are later developed in greater generality in Chapters 6–8. This two-step approach, it is hoped, will enable the student to follow the sequence of arguments more easily. Chapter 5 also contains several results which pertain exclusively to entire functions.

The key result of Chapters 9 and 10 is the famous Residue Theorem, which is followed by many standard and some not-so-standard applications in Chapters 11 and 12.

Chapter 13 introduces conformal mapping, which is interesting in its own right and also necessary for a proper appreciation of the subsequent three chapters. Hydrodynamics is studied in Chapter 14 as a bridge between Chapter 13 and the Riemann Mapping Theorem. On the one hand, it serves as a nice application of the theory developed in the previous chapters, specifically in Chapter 13. On the other hand, it offers a physical insight into both the statement and the proof of the Riemann Mapping Theorem.

In Chapter 15, we use "mapping" methods to generalize some earlier results. Chapter 16 deals with the properties of harmonic functions and the related theory of heat conduction.

A second goal of this book is to give the student a feeling for the wide applicability of complex-variable techniques even to questions which initially do not seem to belong to the complex domain. Thus, we try to impart some of the enthusiasm apparent in the famous statement of Hadamard that "the shortest route between two truths in the real domain passes through the complex domain." The physical applications of Chapters 14 and 16 are good examples of this, as are the results of Chapter 11. The material in the last three chapters is designed to offer an even greater appreciation of the breadth of possible applications. Chapter 17 deals with the different forms an analytic function may take. This leads directly to the Gamma and Zeta functions discussed in Chapter 18. Finally, in Chapter 19, a potpourri of problems—again, some classical and some novel—is presented and studied with the techniques of complex analysis.

The material in the book is most easily divided into two parts: a first course covering the material of Chapters 1–11 (perhaps including parts of Chapter 13), and a second course dealing with the later material. Alternatively, one seeking to cover the physical applications of Chapters 14 and 16 in a one-semester course could omit some of the more theoretical aspects of Chapters 8, 12, 14, and 15, and include them, with the later material, in a second-semester course.

The authors express their thanks to Ms. Barbara Brown, who diligently reviewed the manuscript and made many useful suggestions. We are also indebted to the staff of Springer-Verlag Inc. for their careful and patient work in bringing the manuscript to its present form.

J.B.
D.J.N.

Contents

Chapter 1

The Complex Numbers

Introduction

Numbers of the form $a + b\sqrt{-1}$, where a and b are real numbers—what we call complex numbers—appeared as early as the 16th century. Cardan (1501–1576) worked with complex numbers in solving quadratic and cubic equations. In the 18th century, functions involving complex numbers were found by Euler to yield solutions to differential equations. As more manipulations involving complex numbers were tried, it became apparent that many problems in the theory of real-valued functions could be most easily solved using complex numbers and functions. For all their utility, however, complex numbers enjoyed a poor reputation and were not generally considered legitimate numbers until the middle of the 19th century. Descartes, for example, rejected complex roots of equations and coined the term "imaginary" for such roots. Euler, too, felt that complex numbers "exist only in the imagination" and considered complex roots of an equation useful only in showing that the equation actually has *no* solutions.

The wider acceptance of complex numbers is due largely to the geometric representation of complex numbers which was most fully developed and articulated by Gauss. He realized it was erroneous to assume "that there was some dark mystery in these numbers." In the geometric representation, he wrote, one finds the "intuitive meaning of complex numbers completely established and more is not needed to admit these quantities into the domain of arithmetic."

Gauss' work did, indeed, go far in establishing the complex number system on a firm basis. The first complete and formal definition, however, was given by his contemporary, William Hamilton. We begin with this definition, and then consider the geometry of complex numbers.

1.1 The Field of Complex Numbers

We will see that complex numbers can be written in the form $a + bi$, where a and b are real numbers and i is a square root of -1. This in itself is not a formal definition, however, since it presupposes a system in which a square root of -1 makes sense. The existence of such a system is precisely what we are trying to establish. Moreover, the operations of addition and multiplication that appear in the expression $a + bi$ have not been defined. The formal definition below gives these definitions in terms of ordered pairs.

1.1 Definition. The complex field $\mathbb{C}$ is the set of ordered pairs of real numbers (a,b) with addition and multiplication defined by

$$(a,b) + (c,d) = (a + c, b + d)$$
$$(a,b)(c,d) = (ac - bd, ad + bc).$$

The associative and commutative laws for addition and multiplication as well as the distributive law follow easily from the same properties of the real numbers. The additive identity, or *zero*, is given by $(0,0)$, and hence the additive inverse of (a,b) is $(-a,-b)$. The multiplicative identity is $(1,0)$. To find the multiplicative inverse of any nonzero (a,b) we set

$$(a,b)(x,y) = (1,0),$$

which is equivalent to the system of equations:

$$ax - by = 1$$
$$bx + ay = 0$$

and has the solution

$$x = \frac{a}{a^2 + b^2}, \qquad y = \frac{-b}{a^2 + b^2}.$$

Thus the complex numbers form a field.

Suppose now that we associate complex numbers of the form $(a,0)$ with the corresponding real numbers a. It follows that

$$(a_1,0) + (a_2,0) = (a_1 + a_2, 0) \quad \text{corresponds to } a_1 + a_2$$

and that

$$(a_1,0)(a_2,0) = (a_1 a_2, 0) \quad \text{corresponds to } a_1 a_2.$$

Thus the correspondence between $(a,0)$ and a preserves all arithmetic operations and there can be no confusion in replacing $(a,0)$ by a. In that sense, we say that the set of complex numbers of the form $(a,0)$ is isomorphic with the set of real numbers, and we will no longer distinguish between them. In this manner we can now say that $(0,1)$ is a square root of -1 since

$$(0,1)(0,1) = (-1,0) = -1$$

and henceforth (0, 1) will be denoted i. Note also that

$$a(b,c) = (a,0)(b,c) = (ab,ac),$$

so that we can rewrite any complex number in the following way:

$$(a,b) = (a,0) + (0,b) = a + bi.$$

We will use the latter form throughout the text.

Returning to the question of square roots, there are in fact two complex square roots of -1: i and $-i$. Moreover, there are two square roots of any nonzero complex number $a + bi$. To solve

$$(x + iy)^2 = a + bi$$

we set

$$\begin{aligned} x^2 - y^2 &= a \\ 2xy &= b \end{aligned}$$

which is equivalent to

$$\begin{aligned} 4x^4 - 4ax^2 - b^2 &= 0 \\ y &= b/2x. \end{aligned}$$

Solving first for x^2, we find the two solutions are given by

$$x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$$

$$y = \frac{b}{2x} = \pm\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}} \cdot (\operatorname{sign} b)$$

where

$$\operatorname{sign} b = \begin{cases} 1 & \text{if } b \geqslant 0 \\ -1 & \text{if } b < 0. \end{cases}$$

EXAMPLE

i) The two square roots of $2i$ are $1 + i$ and $-1 - i$.
ii) The square roots of $-5 - 12i$ are $2 - 3i$ and $-2 + 3i$.

It follows that any quadratic equation with complex coefficients admits a solution in the complex field. For by the usual manipulations,

$$az^2 + bz + c = 0 \qquad a,b,c \in \mathbb{C}, \qquad a \neq 0$$

is seen to be equivalent to

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2},$$

and hence has the solutions

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

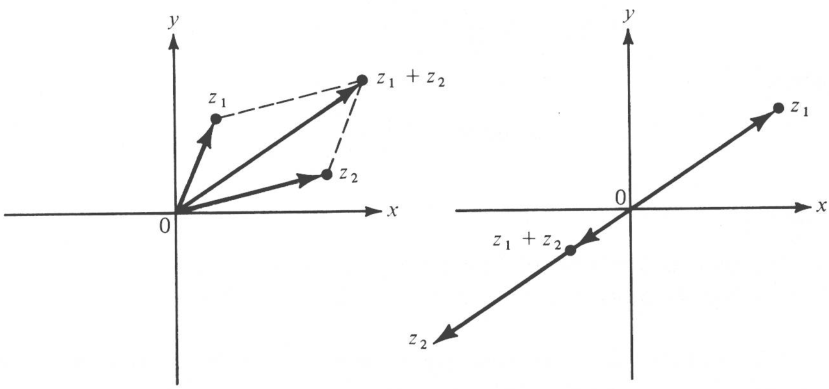
In Chapter 5, we will see that quadratic equations are not unique in this respect: every nonconstant polynomial with complex coefficients has a zero in the complex field.

One property of real numbers that does not carry over to the complex plane is the notion of *order*. We leave it as an exercise for those readers familiar with the axioms of order to check that the number i cannot be designated as either positive or negative without producing a contradiction.

1.2 The Complex Plane

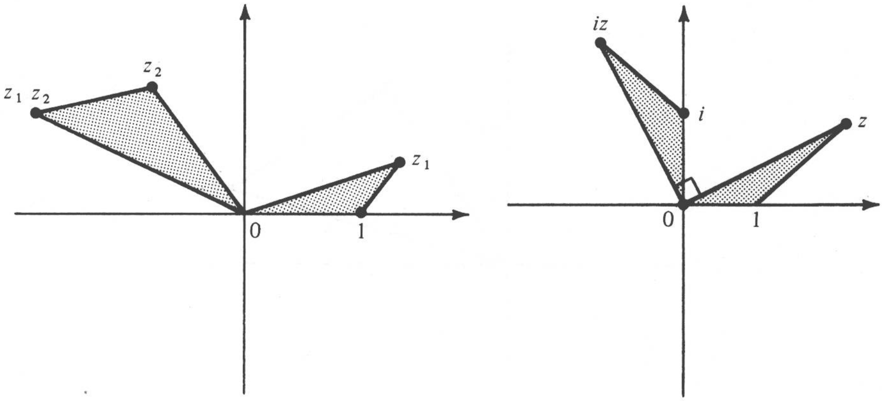
Thinking of complex numbers as ordered pairs of real numbers (a, b) is closely linked with the geometric interpretation of the complex field, discovered by Wallis, and later developed by Argand and by Gauss. To each complex number $a + bi$ we simply associate the point (a, b) in the Cartesian plane. Real numbers are thus associated with points on the x-axis, called the *real axis* while the purely imaginary numbers bi correspond to points on the y-axis, designated as the *imaginary axis*.

Addition and multiplication can also be given a geometric interpretation. The sum of z_1 and z_2 corresponds to the vector sum: If the vector from 0 to z_2 is shifted parallel to the x and y axes so that its initial point is z_1, the resulting terminal point is $z_1 + z_2$. If 0, z_1 and z_2 are not collinear this is the so-called parallelogram law; see below.



The geometric method for obtaining the product z_1z_2 is somewhat more complicated. If we form a triangle with two sides given by the vectors (originating from 0 to) 1 and z_1 and then form a similar triangle with the same orientation and the vector z_2 corresponding to the vector 1, the vector which then corresponds to z_1 will be z_1z_2.

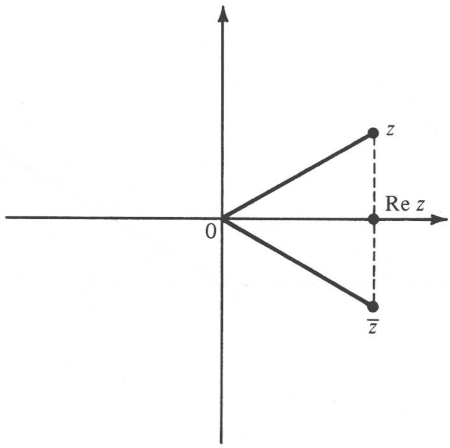
This can be verified geometrically but will be most transparent when we introduce polar coordinates later in this section. For the moment, we observe that multiplication by i is equivalent geometrically to a counter-clockwise rotation of 90°.



With $z = x + iy$, the following terms are commonly used:

Re z, the *real part* of z, is x;
Im z, the *imaginary part* of z, is y (note that Im z is a real number);
$\bar{z}$, the *conjugate* of z, is $x - iy$.

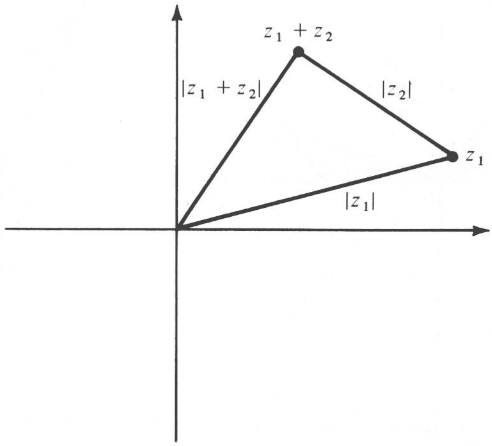
Geometrically, $\bar{z}$ is the mirror image of z reflected across the real axis.



$|z|$, the *absolute value* or *modulus* of z, is equal to $\sqrt{x^2 + y^2}$; that is, it is the length of the vector z. Note also that $|z_1 - z_2|$ is the (Euclidean) distance between z_1 and z_2. Hence we can think of $|z_2|$ as the distance between $z_1 + z_2$ and z_1 and thereby obtain a proof of the triangle inequality:

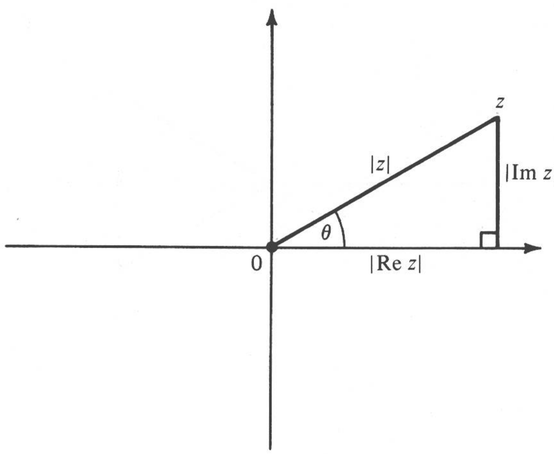
$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

An algebraic proof of the inequality is outlined in Exercise 8.



Arg z, the *argument* of z, defined for $z \neq 0$, is the angle which the vector (originating from 0) to z makes with the positive x-axis. Thus $\operatorname{Arg} z$ is defined (modulo 2π) as that number θ for which

$$\sin\theta = \frac{\operatorname{Im} z}{|z|}; \qquad \cos\theta = \frac{\operatorname{Re} z}{|z|}.$$



Examples

(i) The set of points given by the equation $\operatorname{Re} z > 0$ is represented geometrically by the right half-plane.
(ii) $\{z : z = \bar{z}\}$ is the real line.

(iii) $\{z : -\theta < \operatorname{Arg} z < \theta\}$ is an angular sector (wedge) of angle 2θ.
(iv) $\{z : |\operatorname{Arg} z - \pi/2| < \pi/2\} = \{z : \operatorname{Im} z > 0\}$.
(v) $\{z : |z + 1| < 1\}$ is the disc of radius 1 centered at -1.