GENERAL TOPOLOGY

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by

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AUTHOR'S PREFACE

THE theorems of any geometry (e.g., Euclidean) follow, as is well known, from a number of axioms, i.e., hypotheses about the space considered, and from accepted definitions. A given theorem may be a consequence of some of the axioms and may not require all of them. Such a theorem will be true not only in the space defined by all the axioms, but also in more general spaces. It will, therefore, be of importance to introduce axioms gradually and to deduce from them as many conclusions as possible.

We thus arrive at the concept of an abstract space (Fréchet). Theorems obtained for a given abstract space are true for each set of elements which satisfies the axioms of that space; however, the set may also satisfy other axioms. Herein lies the practical advantage of the study of abstract spaces. For, with a suitable choice of axioms for such a space, the theorems obtained from that space may be applied to different branches of mathematics, e.g., to various types of geometry, to the theory of functions, etc.

In the first chapter we develop a fairly detailed theory of the so-called Fréchet (V)spaces. A Fréchet (V)space is a set K whose elements are subject to only one condition, namely, that with each element p of K there is associated at least one subset of K called a neighbourhood of the element p. In chapter II we investigate (V)spaces which satisfy additional axioms, i.e., the so-called topological spaces; in chapters III, IV, and V we study topological spaces satisfying various additional axioms. Chapter VI is devoted to the study of very important particular topological spaces, namely, the so-called metric spaces, which find numerous applications, and chapter VII deals with the so-called complete metric spaces.

It may be said about chapters I, II, V, VI, and VII that in each of them new axioms are introduced about the space under consideration and theorems are derived from them. In general, the theorems of each of these chapters are not true in a space satisfying only the axioms of the preceding chapters.

Such an axiomatic treatment of the theory of point sets, apart from its logical simplicity, has also an advantage in that it supplies excellent material for exercise in abstract thinking and logical argument in the deduction of theorems from stated suppositions alone; i.e., in proving the theorems by drawing logical conclusions only and without any appeal to intuition, which is so apt to mislead one in the theory of sets.

The book differs to quite an extent from the *Introduction to General Topology* (Toronto, 1934). Apart from a different axiomatic treatment, which seems to us much more advantageous, the subject matter has been considerably enlarged and numerous problems added.

In conclusion I wish to express my thanks to the University of Toronto for making the publication of this book possible, and to Dr. Cecilia Krieger for translating it from the Polish manuscript.

WACLAW SIERPINSKI

Warsaw, October 1948

TRANSLATOR'S PREFACE

WHEN a new edition of *Introduction to General Topology* was being considered, Professor Sierpinski informed me that he had prepared a new manuscript on "General Topology" differing from the "*Introduction*" in both content and treatment. He expressed the hope that the University of Toronto Press would publish a translation of the new manuscript in preference to a revised edition of the "*Introduction.*"

The appendix appearing at the end of the *Introduction* is reprinted here with very slight changes. The numerous footnotes have, for economy in printing, been placed at the end of the book. For the same reason, the usual notation for analytic sets was changed. It is hoped that this change will not place any serious difficulties in the way of the reader.

I wish to take this opportunity to express my deep gratitude to all those who with their discussion and criticism contributed to the enjoyment of a task which might easily have proved tedious. My special thanks are due to Mr. L. W. Crompton and Mr. W. T. Sharp who read part of the manuscript and to Dr. R. G. Stanton who read all of it and offered valuable suggestions.

C. CECILIA KRIEGER

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CHAPTER I

FRECHET (V) SPACES

1. Fréchet (V)spaces. A Fréchet (V)space, or briefly a (V)space, is a set K of elements in which with each element a there is associated a certain class of subsets of K called *neighbourhoods* of a.

Thus the set of points in the plane is a (V)space if a neighbourhood of a point p is taken to be, e.g., the interior of an arbitrary circle with centre at p. Clearly, a neighbourhood in this case can be defined in many ways as, for instance, the interior and boundary of any square with centre at p. It would also be consistent with the definition to assume that each point p of the plane possesses only one neighbourhood, e.g., the set consisting of p itself.

The set of all real functions of a real variable is a (V)space, if a neighbourhood of $f(x)$ is defined to be the set of all functions $g(x)$ which, for a given positive c and for all values of x, satisfy the inequality

$$|f(x) - g(x)| < c.$$

In particular, an arbitrary set K is a (V)space if each element of K possesses only one neighbourhood, for instance, the set K itself, or if every subset of K is a neighbourhood of each element of K.

A Fréchet (V)space is thus defined by its system of neighbourhoods. A given set K for which there are defined two different systems of neighbourhoods gives rise to two different corresponding (V)spaces. It might seem that the concept of a (V)space without additional assumptions is too general to embrace many properties. It will be seen however that, with suitable definitions, a whole theory of (V)spaces can be developed and that certain of its results find an application in various branches of topology and of the theory of functions.

2. Limit elements and derived sets. Let K be a given (V)space. An element p of K is said to be a *limit element* of a set $E \subset K$ if every neighbourhood of p contains at least one element of E different from p. The set of all limit elements of a set E is called the *derived set* of E and is denoted by E'. It is clear that if $p \in E'$ then $p \in (E - \{p\})'$, and $p \in A'$, where $E - \{p\} \subset A \subset K$.[1]

If a set E has no limit elements its derived set is the null set. In particular, the derived set of the null set is empty. We thus have the following properties of the derived set:

(1) $$E' = 0, \qquad E = 0,$$

(2) $$E'_1 \subset E', \qquad E_1 \subset E \subset K,$$

(3) $$a \in (E - \{a\})', \qquad a \in E'.$$

Thus the function $f(E) = E'$ assigns to each set $E \subset K$ a set $f(E) \subset K$ which is subject to the following conditions:

(i) If $E = 0$, then $f(E) = 0$;

(ii) If $E_1 \subset E \subset K$, then $f(E_1) \subset f(E)$;

(iii) If $a \in f(E)$, then $a \in f(E - \{a\})$.

Suppose now that K is a given set and $f(E)$ a function which assigns to each set $E \subset K$ a set $f(E) \subset K$ which is subject to conditions (i), (ii), (iii). It is then possible to define neighbourhoods of the elements of K so that K is a (V)space in which

(4) $$E' = f(E), \qquad E \subset K.$$

For let a subset $H \subset K$ be a neighbourhood of the element $a \in K$ if and only if $a \in K - f(K - H)$. This condition is certainly satisfied by $H = K$ for then, from (i), $f(K - H) = 0$; consequently every element of K has at least one neighbourhood.

Suppose that E is a given subset of K and $a \in E'$. Every neighbourhood of a contains at least one element of E; consequently, the set $H = K - E$ cannot be a neighbourhood of a, i.e., $a \notin K - f(K - H)$. But $a \in K$; hence $a \in f(K - H) = f(E)$. This gives

(5) $$E' \subset f(E).$$

Next assume that $a \notin E'$. Then there exists a neighbourhood H of a such that $H(E - \{a\}) = 0$ and therefore $E - \{a\} \subset K - H$. By (ii)

(6) $$f(E - \{a\}) \subset f(K - H).$$

Since H is a neighbourhood of a we have $a \in K - f(K - H)$; hence $a \notin f(K - H)$ and therefore, from (6), $a \notin f(E - \{a\})$; by (iii) $a \notin f(E)$. This gives

(7) $$f(E) \subset E'.$$

Combining (5) and (7), we obtain (4).

It follows from the above argument that all properties of the derived set which can be proved to hold in every (V)space can be deduced from the properties (1), (2), and (3).

3. Topological equivalence of (V)spaces. Two (V)spaces consisting of the same elements are said to be *topologically equivalent* if the derived set of each subset in one space is the same as the derived set of the same subset in the other space. They are also said to possess the same topological structure or, more briefly, the same topology.

It is easily seen that every (V)space may be associated with a topologically equivalent (V)space in which each element is contained in each one of its own neighbourhoods. It may, therefore, be assumed without any loss of generality that, whatever the definition of neighbourhoods, each element is contained in each one of its neighbourhoods.

THEOREM 1. *Two (V)spaces K_1 and K_2 consisting of the same elements are topologically equivalent (we assume that each element is contained in each one of its neighbourhoods) if and only if to every neighbourhood U of an element in K_1 there exists a neighbourhood of that element in K_2 which is contained in U, and vice versa.*

Proof. Let K_1 and K_2 be two topologically equivalent (V)spaces consisting of the same elements. Let a be a given element of $K = K_1 = K_2$ and U_1 a neighbourhood of a in K_1. Put $E = K - U_1$; hence $E \,.\, U_1 = 0$ and so $a \notin E'$ and, of course, $a \notin E$. Since K_1 and K_2 are topologically equivalent the derived sets of E are the same in both spaces. There exists, therefore, a neighbourhood $U_2 \subset K_2$ of a such that $U_2(E - \{a\}) = U_2 \,.\, E = 0$; hence $U_2 \subset K - E = U_1$. Similarly, because of the symmetry of the conditions, to every neighbourhood $U_2 \subset K_2$ of a there exists a neighbourhood $U_1 \subset K_1$ of a such that $U_1 \subset U_2$. The condition of the theorem is therefore necessary.

Suppose the condition of the theorem satisfied and let E be a set contained in $K_1 = K_2$. If an element $a \notin E' \subset K_1$ there exists a neighbourhood U_1 such that $U_1(E - \{a\}) = 0$; but, by the condition of the theorem, there exists a neighbourhood $U_2 \subset K_2$ such that $U_2 \subset U_1$; hence $U_2(E - \{a\}) = 0$ and therefore $a \notin E' \subset K_2$. Thus every element of a derived set in K_2 is an element of the corresponding derived set in K_1 and conversely, because of the symmetry of the conditions. As a consequence we see that derived sets of a given set in the two spaces are identical and therefore the two spaces are topologically equivalent.

Examples

1. Given two elements a and b obtain all (V)spaces consisting of these two elements (assuming that each element is contained in each one of its neighbourhoods) and determine which of them are topologically equivalent.

	Neighbourhoods of a:	Neighbourhoods of b:
K_1	$\{a\}$	$\{b\}$
K_2	$\{a\}$	$\{a, b\}$
K_3	$\{a\}$	$\{b\}, \{a, b\}$
K_4	$\{a, b\}$	$\{b\}$
K_5	$\{a, b\}$	$\{a, b\}$
K_6	$\{a, b\}$	$\{b\}, \{a, b\}$
K_7	$\{a\}, \{a, b\}$	$\{b\}$
K_8	$\{a\}, \{a, b\}$	$\{a, b\}$
K_9	$\{a\}, \{a, b\}$	$\{b\}, \{a, b\}$

The following spaces are topologically equivalent: K_1, K_3, K_7, and K_9; K_2 and K_8; K_4 and K_6. But no two of K_1, K_2, K_4, and K_5 are topologically equivalent.

2. Show that the number of topologically non-equivalent (*V*)spaces consisting of three elements is 125.

Let $K = \{a, b, c\}$; there are 15 different sets of neighbourhoods of the element a:

1. $\{a\}$; 2. $\{a, b\}$; 3. $\{a, c\}$; 4. $\{a, b, c\}$; 5. $\{a\}, \{a, b\}$; 6. $\{a\}, \{a, c\}$; 7. $\{a\}, \{a, b, c\}$; 8. $\{a, b\}, \{a, c\}$; 9. $\{a, b\}, \{a, b, c\}$; 10. $\{a, c\}, \{a, b, c\}$; 11. $\{a\}, \{a, b\}, \{a, c\}$; 12. $\{a\}, \{a, c\}, \{a, b, c\}$; 13. $\{a, b\}, \{a, c\}, \{a, b, c\}$; 14. $\{a\}, \{a, b\}, \{a, b, c\}$; 15. $\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$.

Of these the systems 1, 5, 6, 7, 11, 12, 14, and 15 are topologically equivalent and so are the systems 2 and 9, 3 and 10, 8 and 13; but no two of 1, 2, 3, 4, and 8 are topologically equivalent. Corresponding to each element of K there are 5 topologically non-equivalent systems of neighbourhoods; consequently there are 5^3 topologically non-equivalent (*V*)spaces each consisting of the same three elements.

3. Show that there are 19^4 topologically non-equivalent (*V*)spaces each consisting of the same 4 elements.

4. Show that the number of different (*V*)spaces consisting of the same n elements is

$$(2^{2^{n-1}} - 1)^n.$$

5. Determine the number of different topologies in a (*V*)space consisting of (a) two elements, (b) three elements (see examples 1 and 2).

Given a set M of cardinal $\mathfrak{m}$ one may divide all (*V*)spaces obtained from M into disjoint classes assigning two (*V*)spaces to the same class if and only if they are topologically equivalent. How many of these classes are there? In other words, how many different topological structures can be induced into a space of cardinal $\mathfrak{m}$?

It can be shown that in an infinite space of cardinal $\mathfrak{m}$ there can be defined $2^{2^{\mathfrak{m}}}$ different topologies (hence as many as there are different (*V*)spaces obtained from a given set of cardinal $\mathfrak{m}$).

4. Closed sets. A set which contains all its limit elements is called *closed*. Thus E is closed if and only if $E' \subset E$.

THEOREM 2. *The intersection of any aggregate of closed sets is closed.*

Proof. Let $P = \prod E$ be the intersection of the closed sets E. Hence $P \subset E$ for every E of the aggregate; by property (2) of derived sets

$P' \subset E' \subset E$ since E is closed. Therefore, $P' \subset \prod E = P$; consequently P is closed.

Since, in a given (V)space, the derived set is uniquely defined it follows that the family Φ of all closed sets of this (V)space is also uniquely defined. Thus the families of closed sets in two topologically equivalent spaces are identical. But, as is shown in § 5, there are topologically non-equivalent spaces consisting of the same elements and having all closed sets in common.[2] Hence the family of all closed sets of a (V)space does not determine the topology of this space.

THEOREM 3. *If a set E is closed then every set contained in E and containing E' is closed.*

Proof. Let T be a set such that $E' \subset T \subset E$; then $T' \subset E' \subset T$ and therefore T is closed.

In particular, the derived set of a closed set is closed. However, the derived set of a set which is not closed may not be closed.[3]

5. The closure of a set. It follows from the definition of a closed set that *the null set is closed* and *the whole (V)space is closed.* Thus for every set $E \subset K$ there exist closed sets containing E (e.g., the set K). Denote by $\bar{E}$ the intersection of all closed sets containing E. By Theorem 2, $\bar{E}$ is closed; it is called the *closure* of the set E. Hence the closure of every set is a closed set. Moreover, it is the smallest closed set containing E, that is to say, it is contained in every closed set containing E. Consequently *E is closed if and only if $E = \bar{E}$.* In particular,

$$\bar{\bar{E}} = \bar{E} \qquad \text{(where } \bar{\bar{E}} = (\bar{\bar{E}})\text{)}.$$

From $E \subset \bar{E}$, since $\bar{E}$ is closed, we obtain at once that $E' \subset \bar{E}$ and so $E + E' \subset \bar{E}$ for every set $E \subset K$.

It follows immediately that the closure of a set possesses the following properties:[4]

1) $\bar{E} = 0, \qquad E = 0;$

2) $\bar{E}_1 \subset \bar{E}, \qquad E_1 \subset E \subset K;$

3) $E \subset \bar{E}, \qquad E \subset K;$

4) $\bar{\bar{E}} = \bar{E}, \qquad E \subset K.$

We have already defined the function $f(E) = E'$ for every $E \subset K$; we can now define the function $\phi(E) = \bar{E}$ in terms of the function f. For $\phi(E)$ is the intersection of all sets $F \subset K$ such that $E + f(F) \subset F$. But we cannot define the function f in terms of the function ϕ. Two (V)spaces with the same elements and having two different functions $f(E) = E'$ defined in them may have the same function $\phi(E) = \bar{E}$, as can be seen from the following example: Let V_1 be a (V)space with three elements a, b, c, each element having only one neighbourhood, namely:

$$U_1(a) = \{a, c\},\ U_1(b) = \{b, a\},\ U_1(c) = \{c, b\}.$$

The set $\{a\}$ has a single limit element b, hence $\{a\}' = \{b\}$; similarly, $\{b\}' = \{c\}$ and $\{c\}' = \{a\}$. These sets are obviously not closed; this proves incidentally that, in a (V)space, derived sets need not be closed. Nor are the sets consisting of two elements closed. For $\{a, b\}' = \{b, c\}$ which is obviously not contained in $\{a, b\}$. The only closed sets of V_1 are the null set and the set V_1. Hence for $E \subset V_1$ and $E \neq 0$, we have

$$\phi_1(E) = \bar{E} = V_1$$

while for $E = \{a\}$, we have

$$f_1(E) = E' = \{b\}.$$

Next, let V_2 be a (V)space with the same three elements a, b, c, each element having the same neighbourhood, namely,

$$U_2(a) = U_2(b) = U_2(c) = \{a, b, c\}.$$

Here $\{a\}' = \{b, c\}$, $\{b\}' = \{c, a\}$, $\{c\}' = \{a, b\}$, $\{b, c\}' = \{c, a\}' = \{a, b\}' = \{a, b, c\}$; hence the only closed sets of V_2 are the null set and V_2. Thus for $E \subset V_2$, $E \neq 0$, we find that

$$\phi_2(E) = V_2 = V_1 = \phi_1(E)$$

but for $E = \{a\}$ we have

$$f_2(E) = \{b, c\} \neq \{b\} = f_1(E).$$

It is thus seen that, even if the closures of a given set in two (V)spaces with the same elements be the same, the derived sets of that set may be different. Hence, if in a given (V)space the derived set is known, then the closure also is known, but not conversely. The function $\phi(E) = \bar{E}$ does not, therefore, define the topology of a (V)space.

The function $\phi(E) = \bar{E}$ associates with each set $E \subset K$ a definite set $\phi(E) \subset K$ subject to the conditions:

1. $\phi(E) = 0, \qquad E = 0;$
2. $\phi(E_1) \subset \phi(E), \qquad E_1 \subset E;$
3. $E \subset \phi(E);$
4. $\phi(\phi(E)) = \phi(E).$

Let now K be a given set, $\phi(E)$ a function defined for every $E \subset K$ and subject to the conditions 1, 2, 3, and 4; it is then possible to define neighbourhoods in K so that K becomes a (V)space in which

$$\bar{E} = \phi(E) \qquad \text{for all } E \subset K. \tag{8}$$

Thus, for example, let a set $H \subset K$ be a neighbourhood of $a \in K$ if and only if

$$a \in K - \phi((K - H) - \{a\}). \tag{9}$$

The set $H = K$ satisfies (9) hence every element of K has at least one neighbourhood. We first show that for every $E \subset K$ we have

$$\phi(E) = E + E'. \tag{10}$$

In fact, if $a \notin \phi(E)$ then, from 3, $a \notin E$; since $E - \{a\} \subset E$ we get from condition 2, $\phi(E - \{a\}) \subset \phi(E)$ and so $a \notin \phi(E - \{a\})$. But this gives $a \in K - \phi(E - \{a\})$. Let $H = K - E$, then $a \in K - \phi((K - H) - \{a\})$; consequently H is a neighbourhood of a and H contains no elements of E. Hence $a \notin E'$. This gives

$$E + E' \subset \phi(E). \tag{11}$$

Next, suppose that $a \notin E + E'$. Then $a \notin E'$; hence there exists a neighbourhood H of a such that $H \,.\, E = 0$. Therefore, $E \subset K - H$ and, since $a \notin E$, $E \subset (K - H) - \{a\}$; hence, from condition 2, we have

$$\phi(E) \subset \phi((K - H) - \{a\}). \tag{12}$$

But $a \in H$, i.e., $a \in K - \phi((K - H) - \{a\})$; hence, from (12), $a \notin \phi(E)$; this gives

$$\phi(E) \subset E + E'. \tag{13}$$

Relations (11) and (13) give (10).

From (10) and condition 4 we obtain, for every $E \subset K$, the relation

$$\phi(E + E') = \phi(\phi(E)) = \phi(E) = E + E',$$

that is,

$$\phi(E + E') = E + E'; \tag{14}$$

since, by (10), $E' \subset \phi(E)$ for every $E \subset K$, (14) implies that

$$(E + E')' \subset \phi(E + E') = E + E'.$$

Hence the set $E + E'$ is closed and since it contains E it must contain $\bar{E}$. Therefore,

$$\bar{E} \subset \phi(E). \tag{15}$$

On the other hand, we have $E + E' \subset \bar{E}$ and so by (10)

$$E + E' = \phi(E) \subset \bar{E}. \tag{16}$$

From (15) and (16), we obtain

$$\bar{E} = \phi(E) = E + E' \qquad \text{for every } E \subset K. \tag{17}$$

We have thus proved that for every function $\phi(E)$ defined for $E \subset K$ and subject to the conditions 1, 2, 3, and 4, neighbourhoods can be so defined that K becomes a (V)space in which (17) holds.

The relation $\bar{E} = E + E'$ holds in many important (V)spaces (§ 19), but need not be true in general, as may be seen from the example of the space $V_1 = \{a, b, c\}$ given in this section, where $E = \{a\}$ and $E + E' = \{a, b\} \neq \{a, b, c\} = \bar{E}$. It follows from the above established properties of the functions $\phi(E)$ that every property of the closure of a set which holds in all (V)spaces must result from conditions 1, 2, 3, and 4. It can be easily shown that these conditions are independent.

The closure $\bar{E}$ of a set $E \subset K$ (K a (V)space) can be obtained by means of transfinite construction as follows:

Let $E_0 = E$; for every ordinal number $\alpha > 0$ define by transfinite induction the set

$$E_\alpha = \Big(\sum_{0 < \xi < \alpha} E_\xi\Big)'. \tag{18}$$

Since

$$\sum_{0<\xi<\alpha} E_\xi \subset \sum_{0<\xi<\beta} E_\xi, \qquad 0 < \alpha < \beta,$$

we have

$$E_\alpha \subset E_\beta. \tag{19}$$

Suppose that the cardinal of K is $\aleph_\mu$; then there exists an ordinal number ν, where $0 < \nu < \omega_{\mu+1}$, such that

$$E_\nu = E_{\nu+1}. \tag{20}$$

For if not, assume that

$$E_\alpha \neq E_{\alpha+1}, \qquad 0 < \alpha < \omega_{\mu+1}. \tag{21}$$

By (19), $E_\alpha \subset E_{\alpha+1}$ and therefore for every ordinal α satisfying the inequality $0 < \alpha < \omega_{\mu+1}$, there exists, by (21), an element p_α such that $p_\alpha \in E_{\alpha+1}$ but $p_\alpha \notin E_\alpha$. Consequently $p_\alpha \notin E_{\xi+1}$ for $\xi < \alpha$. But $p_\xi \in E_{\xi+1}$; hence $p_\alpha \neq p_\xi$, for $\xi < \alpha$.

The transfinite sequence $\{p_\alpha\}$, $\alpha < \omega_{\mu+1}$, consisting of different elements has cardinal $\aleph_{\mu+1}$, contrary to the fact that it is a subset of K whose cardinal is $\aleph_\mu$. The existence of an ordinal number ν, where $0 < \nu < \omega_{\mu+1}$, such that (20) is true is thus established.

Furthermore,

$$\bar{E} = \sum_{0<\xi<\nu} E_\xi. \tag{22}$$