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THE THEORY OF RINGS

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PREFACE

The theory that forms the subject of this book had its beginning with Artin's extension in 1927 of Wedderburn's structure theory of algebras to rings satisfying the chain conditions. Since then the theory has been considerably extended and simplified. The only exposition of the subject in book form that has appeared to date is Deuring's *Algebren* published in the Ergebnisse series in 1935. Much progress has been made since then and this perhaps justifies a new exposition of the subject.

The present account is almost completely self-contained. That this has been possible in a book dealing with results of the significance of Wedderburn's theorems, the Albert-Brauer-Noether theory of simple algebras and the arithmetic ideal theory is another demonstration of one of the most remarkable characteristics of modern algebra, namely, the simplicity of its logical structure.

Roughly speaking our subject falls into three parts: structure theory, representation theory and arithmetic ideal theory. The first of these is an outgrowth of the structure theory of algebras. It was motivated originally by the desire to discover and to classify "hypercomplex" extensions of the field of real numbers. The most important names connected with this phase of the development of the theory are those of Molien, Dedekind, Frobenius and Cartan. The structure theory for algebras over a general field dates from the publication of Wedderburn's thesis in 1907; the extension to rings, from Artin's paper in 1927. The theory of representations was originally concerned with the problem of representing a group by matrices. This was extended to rings and was formulated as a theory of modules by Emmy Noether. The study of modules also forms an important part of the arithmetic ideal theory. This part of the theory of rings had its origin in Dedekind's ideal theory of algebraic number fields and more immediately in Emmy Noether's axiomatic foundation of this theory.

Throughout this book we have placed particular emphasis on the study of rings of endomorphisms. By using the regular representations the theory of abstract rings is obtained as a special case of the more concrete theory of endomorphisms. Moreover, the theory of modules, and hence representation theory, may be regarded as the study of a set of rings of endomorphisms all of which are homomorphic images of a fixed ring $\mathfrak{o}$. Chapter 1 lays the foundations of the theory of endomorphisms of a group. The concepts and results developed here are fundamental in all the subsequent work. Chapter 2 deals with vector spaces and contains some material that, at any rate in the commutative case, might have been assumed as known. For the sake of completeness this has been included. Chapter 3 is concerned with the arithmetic of non-commutative principal ideal domains. Much of this chapter can be regarded as a special case of the general arithmetic ideal theory developed in Chapter 6. The methods of Chapter 3 are, however, of a much more elementary character and this fact may be of interest to the student of geometry, since the results of this chapter have many applications in that field. A reader who is primarily interested in structure theory or in representation theory may omit Chapter 3 with the exception of **3**. Chapter 4 is devoted to the development of these theories and to some applications to the problem of representation of groups by projective transformations and to the Galois theory of division rings. In Chapter 5 we take up the study of algebras. In the first part of this chapter we consider the theory of simple algebras over a general field. The second part is concerned with the theory of characteristic and minimum polynomials of an algebra and the trace criterion for separability of an algebra.

In recent years there has been a considerable interest in the study of rings that do not satisfy the chain conditions but instead are restricted by topological or metric conditions. We mention von Neumann and Murray's investigation of rings of transformations in Hilbert space, von Neumann's theory of regular rings and Gelfand's theory of normed rings. There are many important applications of these theories to analysis. Because of the conditions that we have imposed on the rings considered in this work, our discussion is not directly applicable to these problems in topological algebra. It may be hoped, however, that the methods and results of the purely algebraic theory will point the way for further development of the topological algebraic theory.

This book was begun during the academic year 1940–1941 when I was a visiting lecturer at Johns Hopkins University. It served as a basis of a course given there and it gained materially from the careful reading and criticism of Dr. Irving Cohen who at that time was one of the auditors of my lectures. My thanks are due to him and also to Professors Albert, Schilling and Hurewicz for their encouragement and for many helpful suggestions.

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CHAPTER 1

GROUPS AND ENDOMORPHISMS

1. Rings of endomorphisms. With any commutative group $\mathfrak{M}$ we may associate a ring $\mathfrak{E}(\mathfrak{M})$, the ring of endomorphisms (homomorphisms of $\mathfrak{M}$ into itself) of $\mathfrak{M}$. On the other hand, as we shall see, any ring with an identity may be obtained as a subring of the ring of endomorphisms of its additive group. Because of this fact, we may use the theory of rings of endomorphisms to obtain the theory of abstract rings. This method of studying rings is one of the most important ones that we shall use in this book. It will therefore be well to begin our discussion with a brief survey of that part of the theory of groups and endomorphisms that will be required later.

Our primary concern in the sequel is with commutative groups. However, since most of the results of this chapter are valid for an arbitrary group $\mathfrak{M}$, we shall not assume at the outset that $\mathfrak{M}$ is commutative. Nevertheless, we shall find it convenient to use the additive notation in $\mathfrak{M}$: The group operation will be denoted as $+$, the identity element as 0, the inverse of a as $-a$, etc.

Consider the collection $\mathfrak{T}(\mathfrak{M})$ of single-valued transformations of $\mathfrak{M}$ into itself, i.e. onto a subset of $\mathfrak{M}$. As always for transformations, we regard $A = B$ if the images xA and xB are identical for all x in $\mathfrak{M}$. Now we shall turn $\mathfrak{T}$ into an algebraic system by introducing into it two fundamental operations. First, if A and B are in $\mathfrak{T}$, the *sum* $A + B$ is defined as the transformation whose effect on any x in $\mathfrak{M}$ is obtained by adding the images xA and xB. In other terms

$$x(A + B) = xA + xB.$$

The *product* AB is the resultant of A and B:

$$x(AB) = (xA)B.$$[1]

The following facts concerning the algebraic system $\mathfrak{T}$ are readily verified:

1) $\mathfrak{T}$ is a group relative to $+$. The identity element of this group is the transformation 0 that is defined by the equation $x0 = 0$. The negative of A, $-A$, is given by the defining equation $x(-A) = -xA$.

2) $\mathfrak{T}$ is a semi-group with an identity relative to multiplication, i.e. $(AB)C = A(BC)$ and the identity element of $\mathfrak{T}$ is the identity transformation 1 ($x1 = x$).

3) The distributive law

$$A(B + C) = AB + AC$$

holds.

The system $\mathfrak{T}$ is therefore very nearly a ring. It fails to be one since the

[1] This equation justifies our notation xA. For by using it, the order of writing corresponds to the order of performance of the transformations.

relations $A + B = B + A$ and $(B + C)A = BA + CA$ are not universally valid. We may satisfy the first of these conditions if we suppose that $\mathfrak{M}$ is commutative, but even in this case, the second condition fails.

Example. Let $\mathfrak{M}$ be the cyclic group of order 2 with elements 0, 1 where $1 + 1 = 0$. $\mathfrak{T}$ contains four elements

$$0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

where, in general, $\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$ denotes the transformation $0 \to a$, $1 \to b$. The addition and multiplication tables in $\mathfrak{T}$ are, respectively,

	0	1	A	B
0	0	1	A	B
1	1	0	B	A
A	A	B	0	1
B	B	A	1	0

	0	1	A	B
0	0	0	B	B
1	0	1	A	B
A	0	A	1	B
B	0	B	0	B

Since $0A \neq 0$, it is clear that the second distributive law does not hold.

We consider next the subset $\mathfrak{E}(\mathfrak{M})$ of $\mathfrak{T}$ consisting of the endomorphisms of $\mathfrak{M}$ (an arbitrary group). We recall the definition: A transformation A of a group is an *endomorphism* if it is a homomorphism of the group into itself, that is,

$$(x + y)A = xA + yA.$$

It is clear that $\mathfrak{E}$ is closed relative to the multiplication defined in $\mathfrak{T}$. Moreover, if B and C are arbitrary elements of $\mathfrak{T}$ and A is in $\mathfrak{E}$, then

$$(B + C)A = BA + CA.$$

From our point of view the system $\mathfrak{E}$ is not particularly interesting when $\mathfrak{M}$ is an arbitrary group, for then $\mathfrak{E}$ need not be closed relative to the addition that we defined in $\mathfrak{T}$. However, the situation is quite different when $\mathfrak{M}$ is commutative. In this case it is readily seen that if A and B are in $\mathfrak{E}$, then $A + B = B + A$, 0 and $-A$ all belong to $\mathfrak{E}$. Since the associative and distributive laws for multiplication hold, $\mathfrak{E}$ is a ring. This is the fundamental

THEOREM 1. *If $\mathfrak{M}$ is a commutative group, then the set $\mathfrak{E}(\mathfrak{M})$ of endomorphisms of $\mathfrak{M}$ is a ring relative to the operations $A + B$ and AB that are defined by the equations $x(A + B) = xA + xB$, $x(AB) = (xA)B$.*

Examples. 1) Let $\mathfrak{M}$ be the group of rational integers under ordinary addition. Since $\mathfrak{M}$ is a cyclic group with 1 as a generator, any endomorphism A is determined by its effect on 1. For if $1A = a$ and $x = \overbrace{1 + \cdots + 1}^{x}$, then $xA = xa$ the ordinary product of the integers x and a. Since $(-x)A = -xA$, this equation holds also for negative x's and since $0A = 0 = 0a$, it holds for 0. Thus any endomorphism A of $\mathfrak{M}$ is a transformation that multiplies the element

x of $\mathfrak{M}$ by a fixed element a. The element a is uniquely determined by A, and it is clear that every integer a arises from some endomorphism in this way. Hence $\mathfrak{E}$ is in $(1-1)$ correspondence with $\mathfrak{M}$. If $A \rightarrow a$ and $B \rightarrow b$ in our correspondence, then $x(A+B) = xA + xB = xa + xb = x(a+b)$ and similarly $x(AB) = x(ab)$. Hence $A + B \rightarrow a + b$ and $AB \rightarrow ab$, i.e. $\mathfrak{E}$ is isomorphic to the *ring* of rational integers $\mathfrak{M}$.

2) As a generalization of 1) we let $\mathfrak{M}$ be a direct sum of n infinite cyclic groups. If $e_1, \cdots, e_n$ are generators of $\mathfrak{M}$, any endomorphism A is completely determined by the images $e_iA = f_i$. On the other hand, we may choose elements f_i arbitrarily in $\mathfrak{M}$ and define $(\Sigma e_i x_i)A = \Sigma f_i x_i$, x_i integers. Then A is an endomorphism. If

$$e_iA = e_1a_{1i} + \cdots + e_na_{ni}, \qquad (i = 1, \cdots, n),$$

a_{ij} rational integers, then the correspondence $A \rightarrow (a_{ij})$ is $(1-1)$ between $\mathfrak{E}$ and the ring of $n \times n$ matrices with rational integral elements. If $B \rightarrow (b_{ij})$, we may verify that $A + B \rightarrow (a_{ij}) + (b_{ij})$ and $AB \rightarrow (b_{ij})(a_{ij})$. Hence the correspondence is an anti-isomorphism between $\mathfrak{E}$ and the ring of rational integral matrices.[2] It may be remarked that the associative and distributive laws for these matrices may be deduced by means of our correspondence from the associative and distributive laws for endomorphisms.

3) If $\mathfrak{M}$ is a direct sum of cyclic groups of order m, a similar discussion shows that the ring of endomorphisms of $\mathfrak{M}$ is anti-isomorphic to the ring of matrices with elements in the ring of rational integers reduced modulo m.

We return to the consideration of an arbitrary group $\mathfrak{M}$. Let $\mathfrak{G}(\mathfrak{M})$ be the set of $(1-1)$ transformations of $\mathfrak{M}$ onto itself. It is clear that if A is in $\mathfrak{G}(\mathfrak{M})$, then the inverse transformation A^{-1} is defined. It follows that $\mathfrak{G}(\mathfrak{M})$ is a group under multiplication.

Now if A is an endomorphism, A^{-1} is also an endomorphism. Hence the intersection $\mathfrak{A}(\mathfrak{M}) = \mathfrak{E}(\mathfrak{M}) \wedge \mathfrak{G}(\mathfrak{M})$ is also a group under multiplication. The elements of this group, the $(1-1)$ endomorphisms of $\mathfrak{M}$ onto itself, are the *automorphisms* of $\mathfrak{M}$. Of particular interest among these transformations are the *inner automorphisms*. If $s \in \mathfrak{M}$, then the inner automorphism corresponding to s is the transformation S defined by the equation $xS = -s + x + s$. If A is an arbitrary automorphism, then $x(A^{-1}SA) = -sA + x + sA$, i.e. $A^{-1}SA$ is the inner automorphism associated with the element sA. This shows that the totality of inner automorphisms constitutes an invariant subgroup of the complete group of automorphisms.

We recall that in any ring with an identity, an element u is a *unit* if it has both a left and a right inverse relative to the identity. It follows immediately that these two inverses are equal and that no other element in the ring can satisfy either of the equations $ux = 1$ or $xu = 1$. As usual we denote the inverse of

[2] If we use the correspondence $A \rightarrow (a_{ij})^*$, the transposed matrix of (a_{ij}), we obtain an isomorphism. However, in a similar situation that will be encountered later, it is impossible to effect this passage from an anti-isomorphism to an isomorphism. For this reason we prefer to emphasize the correspondence $A \rightarrow (a_{ij})$.

u by u^{-1}. It may be proved directly that the set of units of any ring is a group relative to the multiplication defined in the ring. Now consider any commutative group $\mathfrak{M}$, its ring of endomorphisms $\mathfrak{E}$ and its group of automorphisms $\mathfrak{A}$. Since the $(1 - 1)$ transformations of a set are the only ones that possess two-sided inverses, it is evident that $\mathfrak{A}$ is the group of units of $\mathfrak{E}$. As an application of this fact, we see that the group of automorphisms of the direct sum $\mathfrak{M}$ of n infinite cyclic groups is isomorphic to the extended unimodular group of $n \times n$ rational integral matrices having determinants $+1$ or -1. For we have seen that the ring of endomorphisms of $\mathfrak{M}$ is isomorphic to the ring of $n \times n$ rational integral matrices, and by using the multiplicative property of determinants, we see that the units of the latter ring are the matrices of determinants ± 1.

2. Groups relative to a set of endomorphisms. In many algebraic problems we are interested in studying a group $\mathfrak{M}$ relative to a fixed set of endomorphisms Ω acting in $\mathfrak{M}$. We fix our attention on the subgroups, called *Ω-subgroups* (allowable), which are transformed into themselves by every endomorphism belonging to Ω. Although, in our applications, $\mathfrak{M}$ will usually be an infinite group, the following examples indicate that this point of view is fruitful even in the study of finite groups.

Examples. 1) Ω is vacuous. All subgroups are allowable. 2) Ω consists of the inner automorphisms. Here the Ω-subgroups are the invariant subgroups. 3) Ω is the complete set of automorphisms. The Ω-subgroups are the characteristic subgroups of $\mathfrak{M}$.

We suppose now that $\mathfrak{M}$ and Ω are fixed. If $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are Ω-subgroups, evidently the intersection $\mathfrak{N}_1 \wedge \mathfrak{N}_2$ is also an Ω-subgroup. The join $(\mathfrak{N}_1, \mathfrak{N}_2)$, defined as the smallest subgroup containing $\mathfrak{N}_1$ and $\mathfrak{N}_2$, may be characterized as the set of finite sums of elements in $\mathfrak{N}_1$ and $\mathfrak{N}_2$. It follows that $(\mathfrak{N}_1, \mathfrak{N}_2)$ is an Ω-subgroup. If $\mathfrak{N}_1$ is invariant, $(\mathfrak{N}_1, \mathfrak{N}_2) = \mathfrak{N}_1 + \mathfrak{N}_2 = \mathfrak{N}_2 + \mathfrak{N}_1$ where $\mathfrak{N}_1 + \mathfrak{N}_2$ denotes the set of elements $x_1 + x_2$, x_i in $\mathfrak{N}_i$.

If $\mathfrak{N}$ is an Ω-subgroup, the endomorphism α of Ω induces in $\mathfrak{N}$ an endomorphism which we shall also denote as α. Of course, distinct mappings α and β in $\mathfrak{M}$ may coincide when regarded as mappings in $\mathfrak{N}$. We note that if $\alpha\beta = \gamma \epsilon \Omega$ or $\alpha + \beta = \delta \epsilon \Omega$, then these relations hold also for the induced transformations in $\mathfrak{N}$.

Now suppose that $\mathfrak{N}$ and $\mathfrak{P}$ are Ω-subgroups and that $\mathfrak{P}$ is invariant in $\mathfrak{N}$. We consider the difference group consisting of the cosets $\mathfrak{P} + y$, y in $\mathfrak{N}$. If $\alpha \epsilon \Omega$, α determines a transformation in $\mathfrak{N} - \mathfrak{P}$ in the following way. If $\mathfrak{P} + y$ is an arbitrary coset, then the coset $\mathfrak{P} + y\alpha$ does not depend on the choice of the representative y and so it is uniquely determined by the coset $\mathfrak{P} + y$ and by the endomorphism α. Hence the correspondence $\mathfrak{P} + y \to \mathfrak{P} + y\alpha$ is a single-valued transformation. Again, we denote this transformation in $\mathfrak{N} - \mathfrak{P}$ by α, i.e. $(\mathfrak{P} + y)\alpha = \mathfrak{P} + y\alpha$. It is clear that α is an endomorphism in $\mathfrak{N} - \mathfrak{P}$. As in the case of subgroups, $\alpha\beta = \gamma$ or $\alpha + \beta = \delta$ in $\mathfrak{N}$ implies the same relation for the induced transformations in $\mathfrak{N} - \mathfrak{P}$. We may repeat these processes, forming difference groups of difference groups, subgroups of difference groups, etc. In this way a whole hierarchy $\mathfrak{K}$ of groups is generated in which the original endomorphisms α induce uniquely defined endomorphisms. We shall call the members of $\mathfrak{K}$, *Ω-groups.*

Let $\mathfrak{N}$ and $\bar{\mathfrak{N}}$ be any two Ω-groups. A mapping A of $\mathfrak{N}$ into the whole of $\bar{\mathfrak{N}}$ is an *Ω-homomorphism* if it is an ordinary homomorphism and $\alpha A = A\alpha$ for all α in Ω. Then $\mathfrak{N}$ and $\bar{\mathfrak{N}}$ are *Ω-homomorphic.*[3] If A is $(1 - 1)$, it is an *Ω-isomorphism* and then $\mathfrak{N}$ and $\bar{\mathfrak{N}}$ are *Ω-isomorphic.* If $\bar{\mathfrak{N}} \leqq \mathfrak{N}$, we use the term *Ω-endomorphism* for Ω-homomorphism and if $\bar{\mathfrak{N}} = \mathfrak{N}$, we use the term *Ω-automorphism* for Ω-isomorphism.

3. The isomorphism theorems. Let $\mathfrak{N}$ and $\mathfrak{P}$ be Ω-groups, $\mathfrak{P}$ invariant in $\mathfrak{N}$. It is well known that the correspondence $x \rightarrow \mathfrak{P} + x$ is a homomorphism A between $\mathfrak{N}$ and $\mathfrak{N} - \mathfrak{P}$. Since $(\mathfrak{P} + x)\alpha = \mathfrak{P} + x\alpha$, $A\alpha = \alpha A$ and A is an Ω-homomorphism. Now suppose that $\mathfrak{N}$ and $\bar{\mathfrak{N}}$ are two Ω-groups and that $x \rightarrow \bar{x} = xA$ is an Ω-homomorphism between them. If $\mathfrak{P}$ is the set of elements of $\mathfrak{N}$ sent into 0, we know that $\mathfrak{P}$ is an invariant subgroup of $\mathfrak{N}$ and that the correspondence $\mathfrak{P} + x \rightarrow \bar{x} = xA$ is an isomorphism between $(\mathfrak{N} - \mathfrak{P})$ and $\bar{\mathfrak{N}}$. Since $(y\alpha)A = (yA)\alpha = 0\alpha = 0$ if $y \in \mathfrak{P}$, $\mathfrak{P}$ is an Ω-subgroup and since $(\mathfrak{P} + x)\alpha = (\mathfrak{P} + x\alpha) \rightarrow (x\alpha)A = (xA)\alpha$, the isomorphism is an Ω-isomorphism between $\mathfrak{N} - \mathfrak{P}$ and $\bar{\mathfrak{N}}$. This proves the fundamental theorem on Ω-homomorphisms:

THEOREM 2. *If $\mathfrak{N}$ and $\mathfrak{P}$ are Ω-groups and $\mathfrak{P}$ is invariant in $\mathfrak{N}$, then $\mathfrak{N}$ and $\mathfrak{N} - \mathfrak{P}$ are Ω-homomorphic. Conversely if $\mathfrak{N}$ is Ω-homomorphic to an Ω-group $\bar{\mathfrak{N}}$ and $\mathfrak{P}$ is the set of elements mapped into 0 by the homomorphism, $\mathfrak{P}$ is an invariant Ω-subgroup of $\mathfrak{N}$ and $\mathfrak{N} - \mathfrak{P}$ and $\bar{\mathfrak{N}}$ are Ω-isomorphic.*

If A is an Ω-homomorphism between $\mathfrak{N}$ and $\bar{\mathfrak{N}}$ and $\mathfrak{R}$ is an Ω-subgroup of $\mathfrak{N}$, then its image $\mathfrak{R}A$ is an Ω-subgroup of $\bar{\mathfrak{N}}$. If $\mathfrak{R}$ is invariant in $\mathfrak{N}$, $\mathfrak{R}A$ is invariant in $\mathfrak{N}A = \bar{\mathfrak{N}}$. On the other hand, if $\bar{\mathfrak{R}}$ is an Ω-subgroup of $\bar{\mathfrak{N}}$ and $\mathfrak{R}$ is the set of elements y of $\mathfrak{N}$ such that $yA \in \bar{\mathfrak{R}}$, then $\mathfrak{R}$ is an Ω-subgroup of $\mathfrak{N}$ containing $\mathfrak{P}$, the set of elements mapped into 0 by the homomorphism. Again, the invariance of $\bar{\mathfrak{R}}$ implies that of $\mathfrak{R}$. If $\mathfrak{R}$ is an Ω-subgroup containing $\mathfrak{P}$, any element of $\mathfrak{N}$ mapped into an element of $\mathfrak{R}A$ is in $\mathfrak{R}$. For if $xA = yA$ for x in $\mathfrak{N}$ and y in $\mathfrak{R}$, $(x - y)A = 0$ and $x - y \in \mathfrak{P}$. Hence $x = (x - y) + y \in \mathfrak{R}$. These results may be stated as follows:

THEOREM 3. *Let $\mathfrak{N}$ be Ω-homomorphic to $\bar{\mathfrak{N}}$ under the Ω-homomorphism A and let $\mathfrak{P}$ be the set of elements mapped into 0 by A. Then the correspondence $\mathfrak{R} \rightarrow \mathfrak{R}A = \bar{\mathfrak{R}}$ is $(1 - 1)$ between the Ω-subgroups $\mathfrak{R}$ containing $\mathfrak{P}$ and the Ω-subgroups of $\bar{\mathfrak{N}}$. The group $\mathfrak{R}$ is invariant in $\mathfrak{N}$ if and only if $\bar{\mathfrak{R}}$ is invariant in $\bar{\mathfrak{N}}$.*

[3] If $\mathfrak{M}_i$ $(i = 1, 2)$ is a group and Ω_i a fixed set of endomorphisms, we may define $\mathfrak{M}_1$ and $\mathfrak{M}_2$ to be (Ω_1, Ω_2)-homomorphic if there is a single-valued mapping $x_1 \rightarrow x_2$ of $\mathfrak{M}_1$ into the whole of $\mathfrak{M}_2$ and a single-valued mapping $\alpha_1 \rightarrow \alpha_2$ of Ω_1 into the whole of Ω_2 such that $x_1 + y_1 \rightarrow x_2 + y_2$, $x_1\alpha_1 \rightarrow x_2\alpha_2$ if $x_1 \rightarrow x_2$, $y_1 \rightarrow y_2$ and $\alpha_1 \rightarrow \alpha_2$. This differs from the definition of Ω-homomorphism, for in the latter the mapping between the transformations is completely determined by the original group $\mathfrak{M}$. The concept of Ω-homomorphism is the important one for our purposes.

Now let $\bar{\mathfrak{K}}$ be an invariant Ω-subgroup of $\bar{\mathfrak{N}}$. If we apply the Ω-homomorphism between $\bar{\mathfrak{N}}$ and $\bar{\mathfrak{N}} - \bar{\mathfrak{K}}$ after that between $\mathfrak{N}$ and $\bar{\mathfrak{N}}$, we obtain an Ω-homomorphism between $\mathfrak{N}$ and $\bar{\mathfrak{N}} - \bar{\mathfrak{K}}$. The elements mapped into 0 of $\bar{\mathfrak{N}} - \bar{\mathfrak{K}}$ are those in $\mathfrak{K}$. Hence we have the

First Isomorphism Theorem. *Suppose that $\mathfrak{N}$ is Ω-homomorphic to $\bar{\mathfrak{N}}$, and let $\bar{\mathfrak{K}}$ be an invariant Ω-subgroup of $\bar{\mathfrak{N}}$ and $\mathfrak{K}$ the totality of elements mapped into $\bar{\mathfrak{K}}$. Then $\mathfrak{N} - \mathfrak{K}$ and $\bar{\mathfrak{N}} - \bar{\mathfrak{K}}$ are Ω-isomorphic.*

Evidently this implies the

Corollary. *If $\mathfrak{K}$ is an Ω-subgroup of $\mathfrak{N}$ containing the invariant Ω-subgroup $\mathfrak{P}$ of $\mathfrak{N}$ and $(\mathfrak{K} - \mathfrak{P})$ is invariant in $(\mathfrak{N} - \mathfrak{P})$, then $\mathfrak{K}$ is invariant in $\mathfrak{N}$ and $\mathfrak{N} - \mathfrak{K}$ is Ω-isomorphic to $(\mathfrak{N} - \mathfrak{P}) - (\mathfrak{K} - \mathfrak{P})$.*

Suppose that $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{M}_1$ are Ω-groups; $\mathfrak{N}_i \leqq \mathfrak{M}_1$ and $\mathfrak{N}_2$ invariant in $\mathfrak{M}_1$. Then the smallest subgroup containing $\mathfrak{N}_1$ and $\mathfrak{N}_2$ is $\mathfrak{N} \equiv \mathfrak{N}_1 + \mathfrak{N}_2 = \mathfrak{N}_2 + \mathfrak{N}_1$. The group $\mathfrak{N}_2$ is invariant in $\mathfrak{N}$ and the cosets in the difference group $\mathfrak{N} - \mathfrak{N}_2$ have the form $\mathfrak{N}_2 + x_1$, x_1 in $\mathfrak{N}_1$. It follows that the correspondence $x_1 \rightarrow \mathfrak{N}_2 + x_1$ is an Ω-homomorphism between $\mathfrak{N}_1$ and $\mathfrak{N} - \mathfrak{N}_2$. Since the elements mapped into 0 are those of $\mathfrak{N}_1 \wedge \mathfrak{N}_2$, we have the

Second Isomorphism Theorem. *If $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{M}_1$ are Ω-groups, $\mathfrak{N}_i \leqq \mathfrak{M}_1$ and $\mathfrak{N}_2$ is invariant in $\mathfrak{M}_1$, then* 1) $\mathfrak{N}_1 + \mathfrak{N}_2 = \mathfrak{N}_2 + \mathfrak{N}_1$, 2) $\mathfrak{N}_1 \wedge \mathfrak{N}_2$ *is invariant in* $\mathfrak{N}_1$ *and* 3) $(\mathfrak{N}_1 + \mathfrak{N}_2) - \mathfrak{N}_2$ *is Ω-isomorphic to* $\mathfrak{N}_1 - (\mathfrak{N}_1 \wedge \mathfrak{N}_2)$.

4. The Jordan-Hölder-Schreier theorem. A chain of Ω-groups $\mathfrak{M}_1 \geqq \mathfrak{M}_2 \geqq \cdots \geqq \mathfrak{M}_{s+1} = 0$ is a *normal series* for $\mathfrak{M}_1$ if each $\mathfrak{M}_i$ is invariant in $\mathfrak{M}_{i-1}$. The difference groups $\mathfrak{M}_{i-1} - \mathfrak{M}_i$ are called the *factors* of the series while a second chain is a *refinement* of the first if it contains all of the $\mathfrak{M}_i$. We shall call two normal series *equivalent* if there is a $(1 - 1)$ correspondence between their factors such that the paired factors are Ω-isomorphic.

Theorem 4 (Schreier). *Any two normal series for $\mathfrak{M}_1$ have equivalent refinements.*

Let $\mathfrak{M}_1 \geqq \cdots \geqq \mathfrak{M}_{s+1} = 0$ and $\mathfrak{M}_1 = \mathfrak{N}_1 \geqq \cdots \geqq \mathfrak{N}_{t+1} = 0$ be the two normal series. Define $\mathfrak{M}_{ij} = \mathfrak{M}_{i+1} + (\mathfrak{M}_i \wedge \mathfrak{N}_j)$ for $j = 1, \cdots, t + 1$ and $i = 1, \cdots, s$, $\mathfrak{M}_{s+1,1} = 0$. Then $\mathfrak{M}_{i,t+1} = \mathfrak{M}_{i+1,1}$ and $(\mathfrak{M}_1 =)\ \mathfrak{M}_{11} \geqq \cdots \geqq \mathfrak{M}_{1t} \geqq \mathfrak{M}_{21} \geqq \cdots \geqq \mathfrak{M}_{2t} \geqq \cdots \geqq \mathfrak{M}_{s,t} \geqq 0$. Similarly, set $\mathfrak{N}_{ji} = \mathfrak{N}_{j+1} + (\mathfrak{N}_j \wedge \mathfrak{M}_i)$ for $i = 1, \cdots, s + 1$ and $j = 1, \cdots, t$, $\mathfrak{N}_{t+1,1} = 0$ and obtain $\mathfrak{N}_{j,s+1} = \mathfrak{N}_{j+1,1}$ and $(\mathfrak{N}_1 =)\ \mathfrak{N}_{11} \geqq \cdots \geqq \mathfrak{N}_{1s} \geqq \mathfrak{N}_{21} \geqq \cdots \geqq \mathfrak{N}_{2s} \geqq \cdots \geqq \mathfrak{N}_{t,s} \geqq 0$. Thus in each chain we have $st + 1$ terms. We may pair $\mathfrak{M}_{ij} - \mathfrak{M}_{i,j+1}$ with $\mathfrak{N}_{ji} - \mathfrak{N}_{j,i+1}$ to obtain the theorem as a consequence of the following

Lemma (Zassenhaus). *Let $\mathfrak{N}_1$, $\mathfrak{N}_1'$, $\mathfrak{N}_2$, $\mathfrak{N}_2'$, $\mathfrak{M}_1$ be Ω-groups where $\mathfrak{N}_i \leqq \mathfrak{M}_1$, $\mathfrak{N}_i' \leqq \mathfrak{N}_i$ and $\mathfrak{N}_i'$ is invariant in $\mathfrak{N}_i$. Then $\mathfrak{N}_1' + (\mathfrak{N}_1 \wedge \mathfrak{N}_2')$ is invariant in $\mathfrak{N}_1' + (\mathfrak{N}_1 \wedge \mathfrak{N}_2)$, $\mathfrak{N}_2' + (\mathfrak{N}_2 \wedge \mathfrak{N}_1')$ is invariant in $\mathfrak{N}_2' + (\mathfrak{N}_2 \wedge \mathfrak{N}_1)$ and the corresponding difference groups are Ω-isomorphic.*

By the Second Isomorphism Theorem, $\mathfrak{N}_2 \wedge \mathfrak{N}_1' = (\mathfrak{N}_2 \wedge \mathfrak{N}_1) \wedge \mathfrak{N}_1'$ is invariant in $\mathfrak{N}_1 \wedge \mathfrak{N}_2$ and $(\mathfrak{N}_1 \wedge \mathfrak{N}_2) - (\mathfrak{N}_1' \wedge \mathfrak{N}_2)$ and $(\mathfrak{N}_1' + (\mathfrak{N}_1 \wedge \mathfrak{N}_2)) - \mathfrak{N}_1'$ are Ω-isomorphic. Similarly, $\mathfrak{N}_1 \wedge \mathfrak{N}_2'$ is invariant in $\mathfrak{N}_1 \wedge \mathfrak{N}_2$ and hence $(\mathfrak{N}_1' \wedge \mathfrak{N}_2) + (\mathfrak{N}_1 \wedge \mathfrak{N}_2')$ is invariant. In the homomorphism between $\mathfrak{N}_1 \wedge \mathfrak{N}_2$ and $(\mathfrak{N}_1' + (\mathfrak{N}_1 \wedge \mathfrak{N}_2)) - \mathfrak{N}_1'$, the group $((\mathfrak{N}_1' \wedge \mathfrak{N}_2) + (\mathfrak{N}_1 \wedge \mathfrak{N}_2'))$ is mapped into $((\mathfrak{N}_1' \wedge \mathfrak{N}_2) + (\mathfrak{N}_1 \wedge \mathfrak{N}_2') + \mathfrak{N}_1') - \mathfrak{N}_1' = ((\mathfrak{N}_1 \wedge \mathfrak{N}_2') + \mathfrak{N}_1') - \mathfrak{N}_1'$. Hence by the above corollary $(\mathfrak{N}_1 \wedge \mathfrak{N}_2') + \mathfrak{N}_1'$ is invariant in $(\mathfrak{N}_1 \wedge \mathfrak{N}_2) + \mathfrak{N}_1'$ and $(\mathfrak{N}_1' + (\mathfrak{N}_1 \wedge \mathfrak{N}_2)) - (\mathfrak{N}_1' + (\mathfrak{N}_1 \wedge \mathfrak{N}_2'))$ and $(\mathfrak{N}_1 \wedge \mathfrak{N}_2) - ((\mathfrak{N}_1 \wedge \mathfrak{N}_2') + (\mathfrak{N}_1' \wedge \mathfrak{N}_2))$ are Ω-isomorphic. By symmetry $(\mathfrak{N}_1 \wedge \mathfrak{N}_2) - ((\mathfrak{N}_1 \wedge \mathfrak{N}_2') + (\mathfrak{N}_1' \wedge \mathfrak{N}_2))$ and $(\mathfrak{N}_2' + (\mathfrak{N}_1 \wedge \mathfrak{N}_2)) - (\mathfrak{N}_2' + (\mathfrak{N}_2 \wedge \mathfrak{N}_1'))$ are Ω-isomorphic. Comparing the second members of these isomorphic pairs, we obtain the lemma.

5. Chain conditions. If $\mathfrak{N}$ is an Ω-group, we shall at various times assume one or both of the following finiteness conditions:

Descending chain condition. If $\mathfrak{N} = \mathfrak{N}_1 > \mathfrak{N}_2 > \cdots$ where $\mathfrak{N}_i$ is an invariant Ω-subgroup of $\mathfrak{N}_{i-1}$, then the sequence has only a finite number of terms.

Ascending chain condition. If $\mathfrak{N} = \mathfrak{N}_1 > \cdots > \mathfrak{N}_k = \mathfrak{P} > 0$ is a normal series for $\mathfrak{N}$, then any chain of Ω-subgroups $0 < \mathfrak{P}_1 < \mathfrak{P}_2 < \cdots$ all of which are invariant in $\mathfrak{P}$ is finite.

Of course both chain conditions hold if $\mathfrak{N}$ is of finite order. On the other hand, we shall see that these conditions may be used in place of the assumption of finiteness of order to obtain extensions of some of the classical theorems on finite groups to infinite Ω-groups. The following examples prove the independence of the two chain conditions.

Examples. 1) *The additive group of integers.* This group satisfies the ascending chain condition but not the descending chain condition. This is also true for the direct sum of a finite number of infinite cyclic groups (Cf. Chapter 3, 3).

2) *The direct sum $\mathfrak{M}$ of an infinite number of cyclic groups of order a prime p.*[4] Let $x_1, x_2, \cdots$ be a basis for $\mathfrak{M}$ and let A be the endomorphism determined by the equations $x_1A = 0$, $x_iA = x_{i-1}$. Then $\mathfrak{M}$ satisfies the descending chain condition relative to $\Omega = \{A\}$ but not the ascending chain condition. Another example of this type is furnished by the commutative group with generators $x_1, x_2, \cdots$ satisfying the relations $px_1 = 0$, $px_i = x_{i-1}$. Here we take Ω to be vacuous.

It should be noted that if $\mathfrak{N}$ is commutative, the ascending chain condition assumes the simpler form that any chain $0 < \mathfrak{P}_1 < \mathfrak{P}_2 < \cdots$ of Ω-subgroups of $\mathfrak{N}$ is finite in length. If either chain condition holds for an (arbitrary) $\mathfrak{N}$, then it holds also for any invariant Ω-subgroup $\mathfrak{P}$ and for any difference group $\mathfrak{N} - \mathfrak{P}$. If both chain conditions hold, $\mathfrak{N}$ has a *composition series*, i.e. a normal series $\mathfrak{N} = \mathfrak{N}_1 > \cdots > \mathfrak{N}_h > 0$ that has no proper refinements. Thus a normal series is a composition series if $\mathfrak{N}_{i-1} > \mathfrak{N}_i$ and $\mathfrak{N}_{i-1} - \mathfrak{N}_i$ is Ω-*irreducible* in the sense that it has no proper invariant Ω-subgroups. To prove our assertion let $\mathfrak{N}'$ be a proper invariant Ω-subgroup. If $\mathfrak{N} - \mathfrak{N}'$ is reducible, there is an $\mathfrak{N}''$ invariant in $\mathfrak{N}$ such that $\mathfrak{N} > \mathfrak{N}'' > \mathfrak{N}' > 0$. Continuing in this way we

[4] Note that this group relative to the vacuous set of endomorphisms satisfies neither chain condition.

obtain, after a finite number of steps, an invariant Ω-subgroup $\mathfrak{N}_2$ of $\mathfrak{N} = \mathfrak{N}_1$ such that $\mathfrak{N}_1 - \mathfrak{N}_2$ is Ω-irreducible. If we repeat this process for $\mathfrak{N}_2$, we obtain an $\mathfrak{N}_3$, etc. Then we have a normal series $\mathfrak{N}_1 > \mathfrak{N}_2 > \cdots$, and by the descending chain condition this breaks off after a finite number of steps, yielding a composition series for $\mathfrak{N}$.

If Ω is the set of inner automorphisms, a composition series for $\mathfrak{N}$ is called a *principal series* and if Ω is the complete set of automorphisms, we have a *characteristic series*. The following extension of the Jordan-Hölder theorem implies, in particular, the uniqueness (in the sense of isomorphism) of the factors of these series as well as of ordinary composition series (Ω vacuous).

THEOREM 5. *Any two composition series for an Ω-group $\mathfrak{N}$ are equivalent.*

This is an immediate consequence of Schreier's theorem.

THEOREM 6. *A necessary and sufficient condition that an Ω-group have a composition series is that it satisfy both chain conditions.*

The sufficiency of this condition has already been proved. Now suppose that $\mathfrak{N}$ has a composition series of h terms. If $\mathfrak{N} = \mathfrak{N}_1 > \mathfrak{N}_2 > \cdots$ is a descending chain of Ω-subgroups, then there are at most h terms in this chain since $\mathfrak{N}_1 > \mathfrak{N}_2 > \cdots > \mathfrak{N}_k > 0$ is a normal chain and may be refined into a composition series having h terms. A similar argument applies to ascending chains.

If $\mathfrak{N}_1 > \cdots > \mathfrak{N}_h > 0$ is a composition series for $\mathfrak{N}_1$, then h is the *length* of the group $\mathfrak{N}_1$. Hence a group is Ω-irreducible if and only if it has length one. If $\mathfrak{N}'$ is an invariant Ω-subgroup of $\mathfrak{N}_1$, we may suppose that $\mathfrak{N}'$ is the term $\mathfrak{N}_{k+1}$ in a composition series. Then $\mathfrak{N}_{k+1}$ has length $h - k$. By the First Isomorphism Theorem, $(\mathfrak{N}_1 - \mathfrak{N}_{k+1}) > \cdots > (\mathfrak{N}_k - \mathfrak{N}_{k+1}) > 0$ is a composition series for $\mathfrak{N}_1 - \mathfrak{N}_{k+1}$, and so the difference group has length k.

An Ω-endomorphism A of $\mathfrak{N}$ is *normal* if it commutes with all the inner automorphisms of $\mathfrak{N}$. Then for any a and x, $-aA + xA + aA = -a + xA + a$. Thus $aA = a + c(a)$ where $c(a)$ is an element that commutes with every element of $\mathfrak{N}A$. If $\mathfrak{P}$ is an invariant Ω-subgroup, then $\mathfrak{P}A$ is invariant in $\mathfrak{N}$ for any normal A. We note also that the product of normal endomorphisms is normal.

If A is any Ω-endomorphism, the set $\mathfrak{Z}_A$ of elements z such that $zA = 0$ is an Ω-subgroup. Evidently $0 \leqq \mathfrak{Z}_A \leqq \mathfrak{Z}_{A^2} \leqq \cdots$. If $\mathfrak{Z}_{A^k} = \mathfrak{Z}_{A^{k+1}}$, we have $\mathfrak{Z}_{A^{k+1}} = \mathfrak{Z}_{A^{k+2}} = \cdots$. Thus in the chain $0 \leqq \mathfrak{Z}_A \leqq \mathfrak{Z}_{A^2} \leqq \cdots$ we have either the sign $<$ throughout or we have this sign for k ($\geqq 0$) terms and thereafter equality. Now suppose that $\mathfrak{N}A = \mathfrak{N}$ and $\mathfrak{Z}_A \neq 0$. Then $\mathfrak{Z}_{A^2} > \mathfrak{Z}_A$. For, each z in $\mathfrak{Z}_A$ has the form xA for a suitable x and so $zA = xA^2 = 0$. Hence if $\mathfrak{Z}_{A^2} = \mathfrak{Z}_A$, $xA = 0$, i.e. every $z = 0$. Similarly we see that $0 < \mathfrak{Z}_A < \mathfrak{Z}_{A^2} < \cdots$. Hence

THEOREM 7. *If $\mathfrak{N}$ satisfies the ascending chain condition and if A is an endomorphism such that $\mathfrak{N}A = \mathfrak{N}$, then $\mathfrak{Z}_A = 0$.*

If A is a normal endomorphism, the chain $\mathfrak{N} \geqq \mathfrak{N}A \geqq \mathfrak{N}A^2 \geqq \cdots$ is a normal chain. We have either $\mathfrak{N} > \mathfrak{N}A > \cdots$ or $\mathfrak{N} > \mathfrak{N}A > \cdots > \mathfrak{N}A^k = \mathfrak{N}A^{k+1} = \cdots$. The first of these alternatives certainly holds if $\mathfrak{Z}_A = 0$ and

$\mathfrak{N} > \mathfrak{N}A$. For if $\mathfrak{N}A^k = \mathfrak{N}A^{k+1}$, $xA^{k+1} = yA^k$ for any x and a suitable y. Hence $(xA^k - yA^{k-1})A = 0$ and $xA^k = yA^{k-1}$, i.e. $\mathfrak{N}A^{k-1} = \mathfrak{N}A^k$. Thus we have

THEOREM 8. *If $\mathfrak{N}$ satisfies the descending chain condition and if A is a normal Ω-endomorphism such that $\mathfrak{Z}_A = 0$, then $\mathfrak{N} = \mathfrak{N}A$.*

If we combine the two preceding theorems, we obtain

THEOREM 9. *If $\mathfrak{N}$ satisfies both chain conditions and if A is a normal Ω-endomorphism, then either A is an automorphism or $\mathfrak{N}A < \mathfrak{N}$ and $\mathfrak{Z}_A \neq 0$.*

Assume again the ascending chain condition. Then $0 < \mathfrak{Z}_A < \cdots < \mathfrak{Z}_{A^k} = \mathfrak{Z}_{A^{k+1}} = \cdots$ for a finite k. It follows that $\mathfrak{Z}_{A^k} \wedge \mathfrak{N}A^k = 0$. For if w is in this intersection, $w = xA^k$ and $wA^k = 0$. Hence $xA^{2k} = 0$ and since $\mathfrak{Z}_{A^k} = \mathfrak{Z}_{A^{2k}}$, $xA^k = w = 0$. Since $\mathfrak{N}A^{k+1} \leqq \mathfrak{N}A^k$, A induces an Ω-endomorphism in $\mathfrak{P} = \mathfrak{N}A^k$ and since there are no elements z in $\mathfrak{P}$ other than 0 such that $zA = 0$, A is an isomorphism between $\mathfrak{P}$ and $\mathfrak{P}A$. Hence if D is any transformation in $\mathfrak{P}$ such that $DA = 0$, then $D = 0$. Evidently A induces a nilpotent endomorphism ($A^k = 0$) in $\mathfrak{Z}_{A^k}$.

If A is normal and $\mathfrak{N}$ satisfies the descending chain condition, we have $\mathfrak{N} > \cdots > \mathfrak{N}A^l = \mathfrak{N}A^{l+1} = \cdots$. If x is any element of $\mathfrak{N}$, $xA^l = yA^{2l}$ for a suitable y and so $x = yA^l + (-yA^l + x) = (x - yA^l) + yA^l \in \mathfrak{N}A^l + \mathfrak{Z}_{A^l} = \mathfrak{Z}_{A^l} + \mathfrak{N}A^l$. The transformation induced by A in $\mathfrak{Z}_{A^l}$ is nilpotent. If D is any transformation in $\mathfrak{N}A^l$ such that $AD = 0$, where A is the induced endomorphism in $\mathfrak{N}A^l$, then $D = 0$.

If both chain conditions hold, the integers k and l of the last two paragraphs are equal. For $\mathfrak{N}A^k \wedge \mathfrak{Z}_{A^k} = 0$ and hence the only element of $\mathfrak{N}A^k$ mapped into 0 by A is 0. It follows that $\mathfrak{N}A^k = \mathfrak{N}A^{k+1}$ so that $l \leqq k$. On the other hand, $\mathfrak{N}A^l = (\mathfrak{N}A^l)A$ implies that $\mathfrak{N}A^l \wedge \mathfrak{Z}_A = 0$. Thus if $yA^{l+1} = (yA^l)A = 0$, $yA^l = 0$; hence $\mathfrak{Z}_{A^{l+1}} = \mathfrak{Z}_{A^l}$ and $k \leqq l$. Hence we have proved the important

LEMMA (Fitting). *Suppose that both chain conditions hold for $\mathfrak{N}$ and that A is a normal Ω-endomorphism. Then for a suitable k we have $\mathfrak{N} = \mathfrak{N}A^k + \mathfrak{Z}_{A^k}$, $\mathfrak{N}A^k \wedge \mathfrak{Z}_{A^k} = 0$ where A is nilpotent in $\mathfrak{Z}_{A^k}$ and an automorphism in $\mathfrak{N}A^k$.*

Remark. We need not suppose that A is an Ω-endomorphism in the above discussion. Instead let Ω contain the inner automorphisms and let A satisfy the condition that $A\Omega = \Omega A$, i.e. for each α in Ω there is an α' and an α'' in Ω such that $A\alpha = \alpha' A$, $\alpha A = A\alpha''$. Since Ω contains the inner automorphisms, Ω-subgroups are invariant. The groups $\mathfrak{N}A$ and $\mathfrak{Z}_A$ are Ω-subgroups and one may carry over the above arguments without change. However, we shall sketch a more direct proof of the final result. Consider the chains $\mathfrak{N} \geq \mathfrak{N}A \geqq \cdots$ and $0 \leqq \mathfrak{Z}_A \leqq \cdots$. The terms of these chains are Ω-subgroups and so by the chain conditions there is an integer m such that $\mathfrak{N}A^m = \mathfrak{N}A^{m+1} = \cdots$ and $\mathfrak{Z}_{A^m} = \mathfrak{Z}_{A^{m+1}} = \cdots$. Set $A^m = B$. Then $\mathfrak{N}B = \mathfrak{N}B^2$, $\mathfrak{Z}_B = \mathfrak{Z}_{B^2}$ and hence by the chain conditions $\mathfrak{N}B \wedge \mathfrak{Z}_B = 0$. If x is any element of $\mathfrak{N}$, $xB^2 = yB$ for a suitable y and so $x = yB + (-yB + x) \in \mathfrak{N}B + \mathfrak{Z}_B$.

6. Direct sums. In the remainder of this chapter we consider an Ω-group $\mathfrak{N}$ for which the endomorphisms in Ω induce all of the inner automorphisms of $\mathfrak{N}$.

We shall also suppose that $\mathfrak{R}$ satisfies both chain conditions. As we have seen, the first assumption implies that every Ω-subgroup is invariant and that Ω-endomorphisms are normal. The ascending chain condition may be stated in the simpler form: Every ascending chain $0 < \mathfrak{R}_1 < \mathfrak{R}_2 \cdots$ terminates after a finite number of terms.

We say that $\mathfrak{R}$ is a *direct sum* of the Ω-subgroups $\mathfrak{R}_i$, $i = 1, \cdots, h$ if

$$\mathfrak{R} = \mathfrak{R}_1 + \cdots + \mathfrak{R}_h$$

and

$$\mathfrak{R}_i \wedge (\mathfrak{R}_1 + \cdots + \mathfrak{R}_{i-1} + \mathfrak{R}_{i+1} + \cdots + \mathfrak{R}_h) = 0$$

for all i. The decomposition is *proper* if all $\mathfrak{R}_i \neq 0$. If no proper decomposition exists other than $\mathfrak{R} = \mathfrak{R}$, $\mathfrak{R}$ is *indecomposable.* We use the notation $\mathfrak{R} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_h$ for a direct sum. Since the $\mathfrak{R}_i$ are invariant, $\mathfrak{R}_i + \mathfrak{R}_j = \mathfrak{R}_j + \mathfrak{R}_i$ and we may equally well write $\mathfrak{R} = \mathfrak{R}_{1'} \oplus \cdots \oplus \mathfrak{R}_{h'}$ for any permutation $1', \cdots, h'$ of $1, \cdots, h$. If $a \in \mathfrak{R}_i$ and $b \in \mathfrak{R}_j$, $j \neq i$, then the commutator $-a - b + a + b \in \mathfrak{R}_i \wedge \mathfrak{R}_j = 0$. Hence $a + b = b + a$ and any element of $\mathfrak{R}_i$ commutes with any in $\mathfrak{R}_j$.

A necessary and sufficient condition that $\mathfrak{R} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_h$, where the $\mathfrak{R}_i$ are Ω-subgroups, is that every x in $\mathfrak{R}$ be expressible in one and only one way in the form $x_1 + \cdots + x_h$, x_i in $\mathfrak{R}_i$. This implies directly that if $\mathfrak{R} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_h$, then $\mathfrak{R}_1' = \mathfrak{R}_1 + \cdots + \mathfrak{R}_{k_1} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_{k_1}$ and if $\mathfrak{R}_2' = \mathfrak{R}_{k_1+1} + \cdots + \mathfrak{R}_{k_1+k_2}, \cdots, \mathfrak{R}_l' = \mathfrak{R}_{k_1+\cdots+k_{l-1}+1} + \cdots + \mathfrak{R}_{k_1+\cdots+k_l}$, then $\mathfrak{R} = \mathfrak{R}_1' \oplus \cdots \oplus \mathfrak{R}_l'$. Conversely, if $\mathfrak{R} = \mathfrak{R}_1' \oplus \cdots \oplus \mathfrak{R}_l'$ and $\mathfrak{R}_1' = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_{k_1}, \cdots, \mathfrak{R}_l' = \mathfrak{R}_{k_1+\cdots+k_{l-1}+1} \oplus \cdots \oplus \mathfrak{R}_{k_1+\cdots+k_l}$, then $\mathfrak{R} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_h$, $h = k_1 + \cdots + k_l$.

If $\mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2$, the Second Isomorphism Theorem implies that $\mathfrak{R}_2$ is Ω-isomorphic to $\mathfrak{R} - \mathfrak{R}_1$. Evidently the length of $\mathfrak{R}$ = length $\mathfrak{R}_1$ + length $\mathfrak{R}_2$. If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are Ω-subgroups of $\mathfrak{R}$ such that $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2$, and $\mathfrak{R}_3 = \mathfrak{R}_1 \wedge \mathfrak{R}_2$, then $\mathfrak{R} - \mathfrak{R}_3 = (\mathfrak{R}_1 - \mathfrak{R}_3) \oplus (\mathfrak{R}_2 - \mathfrak{R}_3)$. It follows that

$$\text{length } \mathfrak{R} + \text{length } (\mathfrak{R}_1 \wedge \mathfrak{R}_2) = \text{length } \mathfrak{R}_1 + \text{length } \mathfrak{R}_2 .$$

We may, of course, replace $\mathfrak{R}$ by $\mathfrak{R}_1 + \mathfrak{R}_2$ and obtain this relation for arbitrary Ω-subgroups of $\mathfrak{R}$.

If $\mathfrak{R} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_h$ so that we have, for every x, $x = x_1 + \cdots + x_h$, x_i in $\mathfrak{R}_i$, then we define the mapping E_i by $xE_i = x_i$. Since the expression for x is unique, E_i is single valued. If $y = y_1 + \cdots + y_h$, $x + y = (x_1 + y_1) + \cdots + (x_h + y_h)$. Hence $(x + y)E_i = xE_i + yE_i$. If $\alpha \in \Omega$, $x\alpha = x_1\alpha + \cdots + x_n\alpha$ so that $\alpha E_i = E_i\alpha$. The E_i are therefore Ω-endomorphisms. Evidently the following relations hold:

$$(1) \qquad E_i^2 = E_i, \qquad E_iE_j = 0 \text{ if } i \neq j, \qquad E_1 + \cdots + E_h = 1.$$

We note also that $E_i + E_j = E_j + E_i$ and that any partial sum $E_{i_1} + \cdots + E_{i_m}$, i_k distinct, is an endomorphism.

An Ω-endomorphism E that is idempotent ($E^2 = E$) will be called a *projection.*

The E_i determined by the direct decomposition are of this type. Now suppose, conversely, that the E_i are arbitrary projections that satisfy (1). Then $\mathfrak{N}E_i \equiv \mathfrak{N}_i$ are Ω-subgroups such that $\mathfrak{N} = \mathfrak{N}_1 \oplus \cdots \oplus \mathfrak{N}_h$ and the E_i are the projections determined by this decomposition. Furthermore if E is any projection and $\mathfrak{Z}_E$ is the set of elements z such that $zE = 0$, then by Fitting's lemma, or directly, we have $\mathfrak{N} = \mathfrak{N}E \oplus \mathfrak{Z}_E$. Hence there is a projection E' such that $E + E' = E' + E = 1$, $EE' = E'E = 0$. We shall call an idempotent element E of any ring *primitive* if it is impossible to write $E = E' + E''$ where E' and E'' are idempotent elements $\neq 0$ of the ring and $E'E'' = E''E' = 0$. Thus $\mathfrak{N}$ is indecomposable if and only if 1 is a primitive projection.

By Fitting's lemma we have

THEOREM 10. *Let $\mathfrak{N}$ be an Ω-group for which Ω contains all the inner automorphisms of $\mathfrak{N}$ and both chain conditions hold. If $\mathfrak{N}$ is indecomposable, then any Ω-endomorphism is either nilpotent or an automorphism.*

7. The Krull-Schmidt theorems. Suppose that $\mathfrak{N}$ is decomposable so that $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$, $\mathfrak{N}_i \neq 0$. If $\mathfrak{N}_1$ is decomposable, $\mathfrak{N}_1 = \mathfrak{N}_{11} \oplus \mathfrak{N}_{12}$ and $\mathfrak{N} = \mathfrak{N}_{11} \oplus \mathfrak{N}_{12} \oplus \mathfrak{N}_2$. Thus $\mathfrak{N} > \mathfrak{N}_1 > \mathfrak{N}_{11} \neq 0$ and continuing in this way, we obtain an indecomposable $\mathfrak{N}_{1\ldots1}$ such that $\mathfrak{N} = \mathfrak{N}_{1\ldots1} \oplus \mathfrak{N}_1'$. We simplify the notation and write $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_1'$ where $\mathfrak{N}_1$ is indecomposable and $\neq 0$. If $\mathfrak{N}_1'$ is decomposable, we have $\mathfrak{N}_1' = \mathfrak{N}_2 \oplus \mathfrak{N}_2'$ where $\mathfrak{N}_2$ is indecomposable and $\neq 0$. Then $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2 \oplus \mathfrak{N}_2'$. This process yields a descending chain $\mathfrak{N}_1' > \mathfrak{N}_2' > \cdots$. Hence it breaks off and we obtain $\mathfrak{N} = \mathfrak{N}_1 \oplus \cdots \oplus \mathfrak{N}_h$ where the $\mathfrak{N}_i$ are indecomposable and $\neq 0$.

Now suppose that $\mathfrak{N} = \mathfrak{P}_1 \oplus \cdots \oplus \mathfrak{P}_k$ is a second decomposition where the Ω-subgroups $\mathfrak{P}_j$ are indecomposable and $\neq 0$. Let E_i and F_j be the projections determined by the two decompositions. Since any sum $E_{i_1} + \cdots + E_{i_n}$, i_m distinct, is an endomorphism, this is true also for $AE_{i_1} + \cdots + AE_{i_n} = A(E_{i_1} + \cdots + E_{i_n})$ and $E_{i_1}A + \cdots + E_{i_n}A = (E_{i_1} + \cdots + E_{i_n})A$ for any endomorphism A. If we apply the endomorphism F_jE_1 to $\mathfrak{N}_1$, we obtain an endomorphism in this group and we have $F_1E_1 + \cdots + F_kE_1 = E_1$ as the identity in $\mathfrak{N}_1$. We wish to show that at least one of the F_jE_1 is an automorphism in $\mathfrak{N}_1$. This will follow from the following lemma.

LEMMA. *Let $\mathfrak{N}$ be an Ω-group for which Ω contains all the inner automorphisms of $\mathfrak{N}$ and both chain conditions hold. If $\mathfrak{N}$ is indecomposable and A and B are Ω-endomorphisms such that $A + B = 1$, then either A or B is an automorphism.*

Since $A + B = 1$ and A and B are endomorphisms, $A^2 + AB = A^2 + BA$ and hence $AB = BA$. If neither A nor B is an automorphism, both are nilpotent. Then $1 = (A + B)^m$ is a sum of terms of the type A^rB^s where $r + s = m$. If m is sufficiently large, we have either $A^r = 0$ or $B^s = 0$, and so we obtain the contradiction $1 = 0$.

We apply this to $F_1E_1 = A$ and $F_2E_1 + \cdots + F_kE_1 = B$ acting in $\mathfrak{N}_1$. If F_1E_1 is not an automorphism, then B is and hence B^{-1} exists. It follows that $F_2E_1B^{-1} + \cdots + F_kE_1B^{-1} = 1$. Either $F_2E_1B^{-1}$ is an automorphism or

$F_3E_1B^{-1} + \cdots + F_kE_1B^{-1}$ is. If we continue in this way, we obtain the result that for some j, $F_jE_1B^{-1}C^{-1} \cdots G^{-1}$ is an automorphism where B^{-1}, C^{-1}, $\cdots$ are automorphisms. It follows that F_jE_1 is an automorphism in $\mathfrak{N}_1$. For simplicity we write $j = 1$.

Consider the Ω-homomorphism F_1 between $\mathfrak{N}_1$ and $\mathfrak{N}_1F_1 \leqq \mathfrak{P}_1$. Since F_1E_1 is an automorphism, F_1 is an isomorphism. Now $\mathfrak{N}_1F_1$ is an Ω-subgroup of $\mathfrak{P}_1$, as is also $\bar{\mathfrak{P}}_1$, the subset of $\mathfrak{P}_1$ of elements z such that $zE_1 = 0$. If y is any element of $\mathfrak{P}_1$, $yE_1 = wF_1E_1$ for some w in $\mathfrak{N}_1$. Hence $y = (y - wF_1) + wF_1$ where $y - wF_1$ is in $\bar{\mathfrak{P}}_1$. Since $\bar{\mathfrak{P}}_1 \wedge \mathfrak{N}_1F_1 = 0$, this contradicts the indecomposability of $\mathfrak{P}_1$ unless $\bar{\mathfrak{P}}_1 = 0$ and $\mathfrak{N}_1F_1 = \mathfrak{P}_1$. Thus $\mathfrak{N}_1F_1 = \mathfrak{P}_1$ and hence F_1 is an isomorphism between $\mathfrak{N}_1$ and $\mathfrak{P}_1$, and E_1 is an isomorphism between $\mathfrak{P}_1$ and $\mathfrak{N}_1$. We assert that $H_1 = E_1F_1 + E_2 + \cdots + E_h$ is an Ω-endomorphism. This is a consequence of the following general remark: Suppose that $\mathfrak{N} = \mathfrak{N}_1 \oplus \cdots \oplus \mathfrak{N}_h$ and that $\mathfrak{N}' = \mathfrak{N}_1' \oplus \cdots \oplus \mathfrak{N}_h'$ is an Ω-subgroup of $\mathfrak{N}$. If A_i is an Ω-homomorphism between $\mathfrak{N}_i$ and $\mathfrak{N}_i'$, then $E_1A_1 + \cdots + E_hA_h$ is an Ω-endomorphism in $\mathfrak{N}$. Our result follows by noting that $\mathfrak{P}_1 \wedge (\mathfrak{N}_2 + \cdots + \mathfrak{N}_h) = 0$ so that $\mathfrak{N}' = \mathfrak{P}_1 + \mathfrak{N}_2 + \cdots + \mathfrak{N}_h = \mathfrak{P}_1 \oplus \mathfrak{N}_2 \oplus \cdots \oplus \mathfrak{N}_h$. Since $zH_1 = 0$ implies that $z = 0$, H_1 is an automorphism, i.e. $\mathfrak{N}' = \mathfrak{N}$.

Now suppose that we have already obtained a pairing between $\mathfrak{P}_i$ and $\mathfrak{N}_i$ for $i = 1, \cdots, r$ such that E_i is an Ω-isomorphism between $\mathfrak{P}_i$ and $\mathfrak{N}_i$ and F_i is one between $\mathfrak{N}_i$ and $\mathfrak{P}_i$. Suppose also that $\mathfrak{N} = \mathfrak{P}_1 \oplus \cdots \oplus \mathfrak{P}_r \oplus \mathfrak{N}_{r+1} \oplus \cdots \oplus \mathfrak{N}_h$, and $H_r = E_1F_1 + \cdots + E_rF_r + E_{r+1} + \cdots + E_h$ is an automorphism. Since the inner automorphisms of a difference group are induced by inner automorphism of the group, $\bar{\mathfrak{N}} = \mathfrak{N} - (\mathfrak{P}_1 + \cdots + \mathfrak{P}_r)$ satisfies our conditions. We have

$$\bar{\mathfrak{N}} = \bar{\mathfrak{N}}_{r+1} \oplus \cdots \oplus \bar{\mathfrak{N}}_h = \bar{\mathfrak{P}}_{r+1} \oplus \cdots \oplus \bar{\mathfrak{P}}_k$$

where $\bar{\mathfrak{N}}_l = (\mathfrak{P}_1 + \cdots + \mathfrak{P}_r + \mathfrak{N}_l) - (\mathfrak{P}_1 + \cdots + \mathfrak{P}_l)$, $\bar{\mathfrak{P}}_j = (\mathfrak{P}_1 + \cdots + \mathfrak{P}_r + \mathfrak{P}_j) - (\mathfrak{P}_1 + \cdots + \mathfrak{P}_r)$ are Ω-isomorphic to $\mathfrak{N}_l$ and $\mathfrak{P}_j$ respectively. By the above discussion we may pair $\bar{\mathfrak{P}}_{r+1}$ with, say, $\bar{\mathfrak{N}}_{r+1}$ so that the corresponding projections $\bar{E}_{r+1}$, $\bar{F}_{r+1}$ are isomorphisms between $\bar{\mathfrak{P}}_{r+1}$ and $\bar{\mathfrak{N}}_{r+1}$. We also have the equation $\bar{\mathfrak{N}} = \bar{\mathfrak{P}}_{r+1} \oplus \bar{\mathfrak{N}}_{r+2} \oplus \cdots \oplus \bar{\mathfrak{N}}_h$. If $x \in (\mathfrak{P}_1 + \cdots + \mathfrak{P}_{r+1}) \wedge (\mathfrak{N}_{r+2} + \cdots + \mathfrak{N}_h)$, the coset $\bar{x} = x + (\mathfrak{P}_1 + \cdots + \mathfrak{P}_r) \in \bar{\mathfrak{P}}_{r+1} \wedge (\bar{\mathfrak{N}}_{r+2} + \cdots + \bar{\mathfrak{N}}_h)$. Hence $\bar{x} = 0$ and $x \in \mathfrak{P}_1 + \cdots + \mathfrak{P}_r$. Since $(\mathfrak{P}_1 + \cdots + \mathfrak{P}_r) \wedge (\mathfrak{N}_{r+1} + \cdots + \mathfrak{N}_h) = 0$, $x = 0$. Thus

$$\mathfrak{P}_1 + \cdots + \mathfrak{P}_{r+1} + \mathfrak{N}_{r+1} + \cdots + \mathfrak{N}_h = \mathfrak{P}_1 \oplus \cdots \oplus \mathfrak{P}_{r+1} \oplus \mathfrak{N}_{r+2} \oplus \cdots \oplus \mathfrak{N}_h .$$

Hence $H_{r+1} = E_1F_1 + \cdots + E_{r+1}F_{r+1} + E_{r+2} + \cdots + E_h$ is an endomorphism. Since $\bar{F}_{r+1}$ is an isomorphism between $\bar{\mathfrak{N}}_{r+1}$ and $\bar{\mathfrak{P}}_{r+1}$, $z_{r+1}F_{r+1} \neq 0$ if $z_{r+1} \neq 0$ is in $\mathfrak{N}_{r+1}$. Hence $zH_{r+1} = 0$ only if $z = 0$; H_{r+1} is an automorphism and $\mathfrak{N} = \mathfrak{P}_1 \oplus \cdots \oplus \mathfrak{P}_{r+1} \oplus \mathfrak{N}_{r+2} \oplus \cdots \oplus \mathfrak{N}_h$. This proves the following theorems.

THEOREM 11 (Krull-Schmidt). *Let $\mathfrak{N}$ be an Ω-group such that Ω contains all the inner automorphisms and both chain conditions hold. Suppose that $\mathfrak{N} = \mathfrak{N}_1 \oplus \cdots \oplus \mathfrak{N}_h = \mathfrak{P}_1 \oplus \cdots \oplus \mathfrak{P}_k$ are two decompositions of $\mathfrak{N}$ as direct sums of in-*