Progress in Mathematics

Arithmetic and Geometry Around Hypergeometric Functions

Lecture Notes of a CIMPA Summer School held at Galatasaray University, Istanbul, 2005

Rolf-Peter Holzapfel
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Progress in Mathematics

Volume 260

Preface

This volume comprises the Lecture Notes of the CIMPA Summer School *Arithmetic and Geometry around Hypergeometric Functions* held at Galatasaray University, Istanbul during June 13-25, 2005. In the Summer School there were fifteen lectures forming an impressive group of mathematicians covering a wide range of topics related to hypergeometric functions. The full schedule of talks from the workshop appears on the next page. In addition to the lecture notes submitted by its lecturers, this volume contains several research articles.

A group of fourty graduate students and young researchers attended the school. Among the participants there were 2 Algerian, 3 American, 1 Armenian, 1 Bulgarian, 1 Canadian, 3 Dutch, 2 Georgian, 7 German, 1 Indian, 2 Iraqi, 1 Iranian, 1 Italian, 1 Russian, 5 Japanese, 23 Turkish and 1 Ukrainian mathematicians, including the lecturers.

We would like to thank the *Centre des Mathématiques Pures et Appliquées*, for their financial support and Professor Michel Jambu for organizational help. We could support participants from across the region thanks to the generous financial help provided by the *International Center for Theoretical Physics* (ICTP) and the *International Mathematical Union* (IMU). The local participants has been supported by the *Scientific and Technological Research Councel of Turkey* (TÜBİTAK).

This summer school has been realized not only by financial support from its sponsors but also thanks to the generousity of its lecturers, who all agreed to finance their travel from their own personal grants. Some of them did so also for the accomodation.

The proposal for the AGAHF Summer School was submitted to CIMPA in February 2004. During the long preparatory process and during the summer school, Ayşegül Ulus, Özgür Ceyhan, and Özgür Kişisel contributed at various levels to the organization. We are grateful to them.

Sabine Buchmann is a French artist living in Istanbul, who likes to draw Ottoman-style miniatures of the boats serving across the bosphorus; these boats are an inseparable part of the city panorama. When asked, she liked the idea of a boat full of mathematicians and drew it for the conference poster — with the names of all the lecturers hidden inside, written in minute letters. Her miniature helped us much in attracting the audience of the summer school.

We are thankful to the student team hired by the university comprising Anet İzmitli, Egemen Kırant, Günce Orman, Haris Saybaşılı and Eylem Şentürk for turning this summer school into a pleasant experience.

Finally we would like to thank warmly Prof. Dr. Duygun Yarsuvat, the rector of the Galatasaray University for offering us the great location and financial support of the university.

The second named editor was supported by TÜBİTAK grant Kariyer 103T136 during the summer school and during the preperation of this volume.

Rolf-Peter Holzapfel, A. Muhammed Uludağ and Masaaki Yoshida, Editors

PROGRAM

Daniel Allcock: Real hyperbolic geometry in moduli problems

Igor Dolgachev: Moduli spaces as ball quotients (followed by Kondo's lectures)

Rolf Peter Holzapfel: Orbital Varieties and Invariants

Michel Jambu: Arrangements of Hyperplanes

A. Kochubei: Hypergeometric functions and Carlitz differential equations over function fields

Shigeyuki Kondo: Complex ball uniformizations of the moduli spaces of del Pezzo surfaces

Edward Looijenga: (first week) Introduction to Deligne-Mostow theory

Edward Looijenga: (second week) Hypergeometric functions associated to arrangements

Keiji Matsumoto: Invariant functions with respect to the Whitehead link

Hironori Shiga: Hypergeometric functions and arithmetic geometric means (followed by Wolfart's lectures)

Jan Stienstra: Gel'fand-Kapranov-Zelevinsky hypergeometric systems and their role in mirror symmetry and in string theory

Toshiaki Terada: Hypergeometric representation of the group of pure braids.

A. Muhammed Uludağ: Geometry of Complex Orbifolds

Alexander Varchenko: Special functions, KZ type equations, and representation theory

Jürgen Wolfart: Arithmetic of Schwarz maps (preceded by Shiga's lectures)

Masaaki Yoshida: Schwarz maps (general introduction)

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Hyperbolic Geometry and the Moduli Space of Real Binary Sextics

Daniel Allcock, James A. Carlson and Domingo Toledo

Abstract. The moduli space of real 6-tuples in $\mathbb{C}P^1$ is modeled on a quotient of hyperbolic 3-space by a nonarithmetic lattice in $\operatorname{Isom} H^3$. This is partly an expository note; the first part of it is an introduction to orbifolds and hyperbolic reflection groups.

Keywords. Complex hyperbolic geometry, hyperbolic reflection groups, orbifolds, moduli spaces, ball quotients.

These notes are an exposition of the key ideas behind our result that the moduli space $\mathcal{M}_s$ of stable real binary sextics is the quotient of real hyperbolic 3-space H^3 by a certain Coxeter group (together with its diagram automorphism). We hope they can serve as an aid in understanding our work [3] on moduli of real cubic surfaces, since exactly the same ideas are used, but the computations are easier and the results can be visualized.

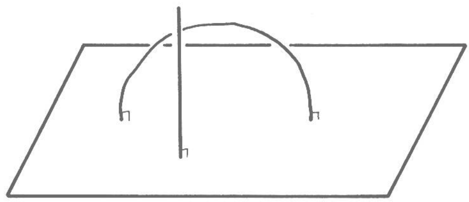
These notes derive from the first author's lectures at the summer school "Algebra and Geometry around Hypergeometric Functions", held at Galatasary University in Istanbul in July 2005. He is grateful to the organizers, fellow speakers and students for making the workshop very rewarding. To keep the flavor of lecture notes, not much has been added beyond the original content of the lectures; some additional material appears in an appendix. The pictures are hand-drawn to encourage readers to draw their own.

Lecture 1

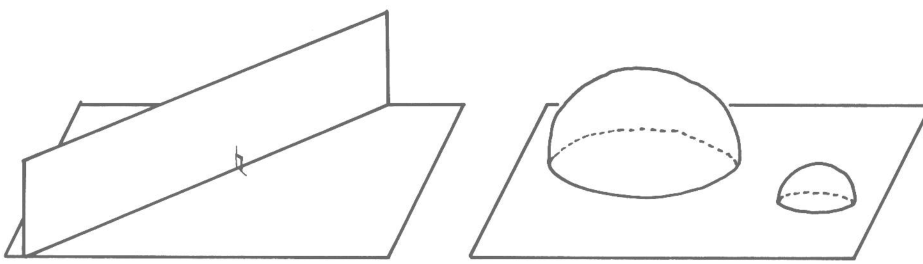
Hyperbolic space H^3 is a Riemannian manifold for which one can write down an explicit metric, but for us the following model will be more useful; it is called the upper half-space model. Its underlying set is the set of points in $\mathbb{R}^3$ with

First author partly supported by NSF grant DMS 0231585. Second and third authors partly supported by NSF grants DMS 9900543 and DMS 0200877.

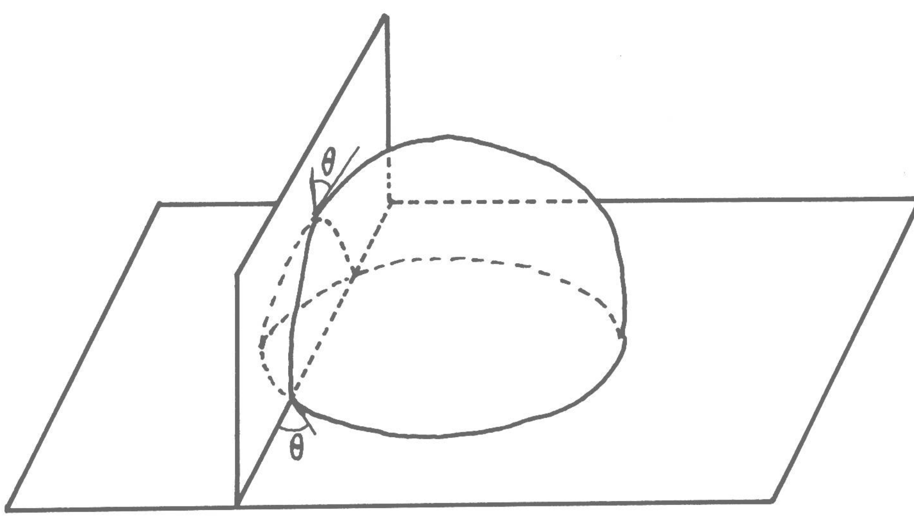
positive vertical coordinate, and geodesics appear either as vertical half-lines, or as semicircles with both ends resting on the bounding $\mathbb{R}^2$:



Note that the 'endpoints' of these geodesics lie in the boundary of H^3, not in H^3 itself. Planes appear either as vertical half-planes, or as hemispheres resting on $\mathbb{R}^2$:



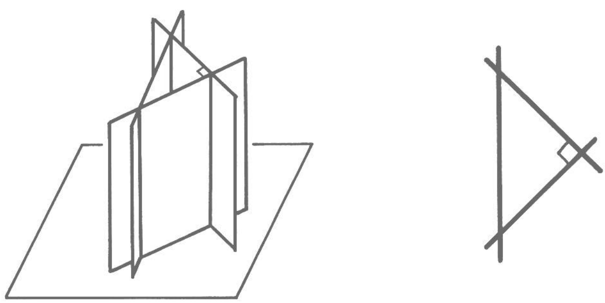
If two planes meet then their intersection is a geodesic. The most important property of the upper half-space model is that it is conformal, meaning that an angle between planes under the hyperbolic metric equals the Euclidean angle between the half-planes and/or hemispheres. For example, the following angle θ looks like a $\pi/4$ angle, so it *is* a $\pi/4$ angle:



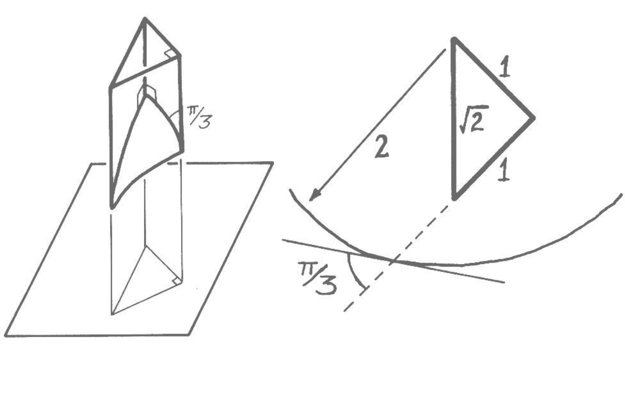
This lets us build hyperbolic polyhedra with specified angles by pushing planes around. For example, the diagram

P_0 (1)

describes a polyhedron P_0 with four walls, corresponding to the nodes, with the interior angle between two walls being $\pi/2$, $\pi/3$ or $\pi/4$ according to whether the nodes are joined by no edge, a single edge or a double edge. For now, ignore the colors of the nodes; they play no role until Theorem 2. We can build a concrete model of P_0 by observing that the first three nodes describe a Euclidean $(\pi/2, \pi/4, \pi/4)$ triangle, so the first three walls should be arranged to appear as vertical halfplanes. Sometimes pictures like this can be easier to understand if you also draw the view down from vertical infinity; here are both pictures:



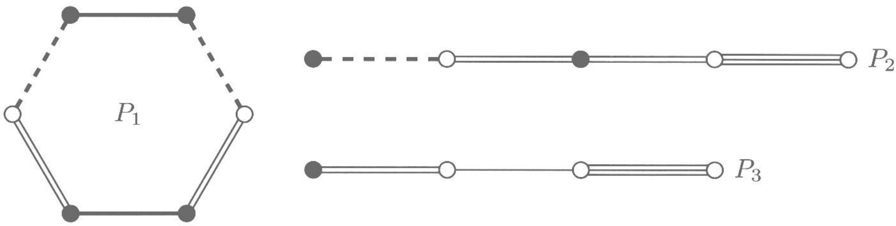
How to fit in the fourth plane? After playing with it one discovers that it cannot appear as a vertical halfplane, so we look for a suitable hemisphere. It must be orthogonal to two of our three walls, so it is centered at the foot of one of the halflines of intersection. The radius of the hemisphere is forced to be 2 because of the angle it makes with the remaining wall (namely $\pi/3$). We have drawn the picture so that the hemisphere is centered at the foot of the back edge. The figure should continue to vertical infinity, but we cut it off because seeing the cross-section makes the polyhedron easier to understand. We've also drawn the view from above; the boundary circle of the hemisphere strictly contains the triangle, corresponding to the fact that P_0 does not descend all the way to the boundary $\mathbb{R}^2$.



(2)

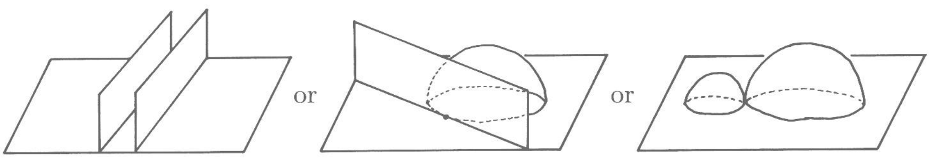
We think of P_0 as an infinitely tall triangular chimney with its bottom bitten off by a hemisphere. The dimensions we have drawn on the overhead view refer to Euclidean distances, not hyperbolic ones. The "radius" of a hemisphere has no intrinsic meaning in hyperbolic geometry; indeed, the isometry group of H^3 acts transitively on planes.

Readers may enjoy trying their hands at this by drawing polyhedra for the diagrams

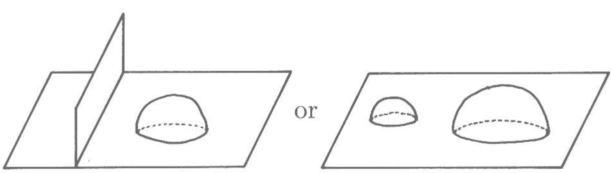


(3)

where the absent, single and double bonds mean the same as before, a triple bond indicates a $\pi/6$ angle, a heavy bond means parallel walls and a dashed bond means ultraparallel walls. In the last two cases we describe the meaning by pictures: Parallelism means



and ultraparallelism means

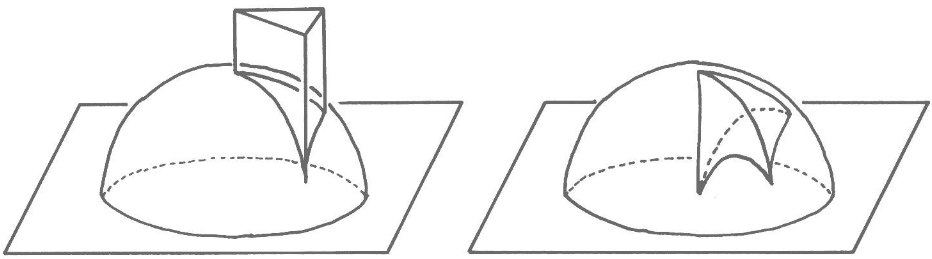


That is, when two planes do not meet in H^3, we call them parallel if they meet at the boundary of H^3, and ultraparallel if they do not meet even there.

Diagrams like (1) and (3) are called Coxeter diagrams after H. S. M. Coxeter, who introduced them to classify the finite groups generated by reflections. Given a random diagram, there is no guarantee that one can find a hyperbolic polyhedron with those angles, but if there is one then it describes a discrete group acting on H^3:

Theorem 1 (Poincaré Polyhedron Theorem). *Suppose $P \subseteq H^3$ is a polyhedron (i.e., the intersection of a finite number of closed half-spaces) with every dihedral angle of the form* $\pi/$(an integer). *Let Γ be the group generated by the reflections across the walls of P. Then Γ is discrete in* Isom H^3 *and P is a fundamental* *domain for* Γ *in the strong sense: every point of* H^3 *is* Γ*-equivalent to exactly one point of* P.

The proof is a very pretty covering space argument; see [5] for this and for a nice introduction to Coxeter groups in general. A reflection across a plane means the unique isometry of H^3 that fixes the plane pointwise and exchanges the components of its complement. A reflection across a vertical half-plane looks like an ordinary Euclidean reflection, and a reflection across a hemisphere means an inversion in it; here are before-and-after pictures of an inversion.



An inversion exchanges vertical infinity with the point of $\mathbb{R}^2$ "at the center" of the hemisphere.

The data of a group Γ acting discretely on H^3 is encoded by an object called an orbifold. As a topological space it is H^3/Γ, but the orbifold has more structure. An orbifold chart on a topological space X is a continuous map from an open subset U of $\mathbb{R}^n$ to X, that factors as

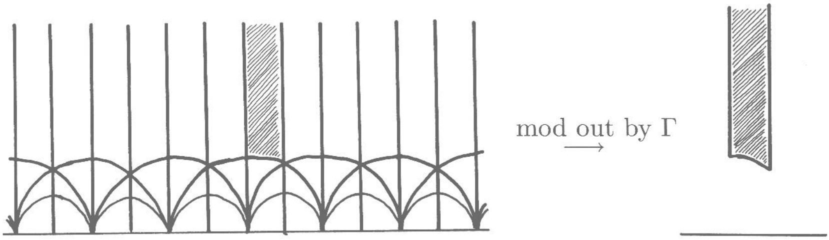
$$U \to U/\Gamma_U \to X,$$

where Γ_U is a finite group acting on U and the second map is a homeomorphism onto its image. Our H^3/Γ has lots of such charts, because if $x \in H^3$ has stabilizer Γ_x and U is a sufficiently small open ball around x, then

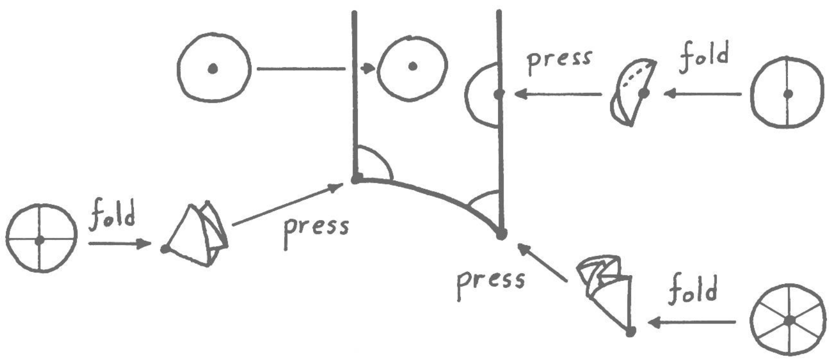
$$U \to U/\Gamma_x \to H^3/\Gamma$$

is an orbifold chart. An orbifold is a space locally modeled on a manifold modulo finite groups. Formally, an orbifold X is a hausdorff space covered by such charts, with the compatibility condition that if $x \in X$ lies in the image of charts $U \to U/\Gamma_U \to X$ and $U' \to U'/\Gamma_{U'} \to X$ then there are preimages v and v' of x in U and U' with neighborhoods V and V' preserved by $\Gamma_{U,v}$ and $\Gamma_{U',v'}$, an isomorphism $\Gamma_{U,v} \cong \Gamma_{U',v'}$ and an equivariant isomorphism $\tau_{V,V'}$ between V and V' identifying v with v'. The group $\Gamma_{U,v}$ is called the local group at x, and the nature of the isomorphisms $\tau_{V,V'}$ determines the nature of the orbifold. That is, if all the $\tau_{V,V'}$ are homeomorphisms then X is a topological orbifold, if all are real-analytic diffeomorphisms then X is a real-analytic orbifold, if all are hyperbolic isometries then X is a hyperbolic orbifold, and so on. So H^3/Γ is a hyperbolic orbifold. There is a notion of orbifold universal cover which allows one to reconstruct H^3 and its Γ-action from the orbifold H^3/Γ.

Only in two dimensions is it easy to draw pictures of orbifold charts; here they are for the quotient of the upper half-plane H^2 by the group Γ generated by reflections across the edges of the famous $(\pi/2, \pi/3, \pi/\infty)$ triangle.



Here are local orbifold charts around various points of H^3/Γ:



For three-dimensional Coxeter groups essentially the same thing happens: the local chart at a generic point of a wall is the quotient of a 3-ball by a reflection, and along an edge it is the quotient of a 3-ball by a dihedral group. One needs to understand the finite Coxeter groups in dimension 3 in order to understand the folding at the vertices, but this is not necessary here.

We care about hyperbolic orbifolds because it turns out that moduli spaces arising in algebraic geometry are usually orbifolds, and it happens sometimes that such a moduli space happens to coincide with a quotient of hyperbolic space (or complex hyperbolic space or one of the other symmetric spaces). So we can sometimes gain insight into the algebraic geometry by manipulating simple objects like tilings of hyperbolic space.

Suppose a Lie group G acts properly on a smooth manifold X, with finite stabilizers. (Properly means that each compact set K in X meets only "compactly many" of its translates—that is, there exists a compact set in G such that if $g \in G$ lies outside it, then $K \cap gK = \emptyset$. This is needed for the quotient space to be Hausdorff.) Because G acts on the left, we write $G\backslash X$ for the quotient, which is an orbifold by the following construction. For $x \in X$ one can find a small transversal T to the orbit $G.x$, which is preserved by the stabilizer G_x. Then $T \to G_x\backslash T \to G\backslash X$ gives an orbifold chart. In particular, the local group at the image of x in $G\backslash X$

is G_x. If X is real-analytic and G acts real-analytically then $G\backslash X$ is a real-analytic orbifold.

Now we come to the case which concerns us. Let $\mathcal{C}$ be the set of binary sextics, i.e., nonzero 2-variable homogeneous complex polynomials of degree 6, modulo scalars, so $\mathcal{C} = \mathbb{C}P^6$. Let $\mathcal{C}^{\mathbb{R}}$ be the subset given by those with real coefficients, $\mathcal{C}_0$ the smooth sextics (those with 6 distinct roots), and $\mathcal{C}_0^{\mathbb{R}}$ the intersection. Then $G = \mathrm{PGL}_2\mathbb{C}$ acts on $\mathcal{C}$ and $\mathcal{C}_0$ and $G^{\mathbb{R}} = \mathrm{PGL}_2\mathbb{R}$ acts on $\mathcal{C}^{\mathbb{R}}$ and $\mathcal{C}_0^{\mathbb{R}}$. The moduli space $\mathcal{M}_0$ of smooth binary sextics is $G\backslash\mathcal{C}_0$, of 3 complex dimensions. The real moduli space $\mathcal{M}_0^{\mathbb{R}} = G^{\mathbb{R}}\backslash\mathcal{C}_0^{\mathbb{R}}$ is *not* the moduli space of 6-tuples in $\mathbb{R}P^1$; rather it is the moduli space of nonsingular 6-tuples in $\mathbb{C}P^1$ which are preserved by complex conjugation. This set has 4 components, $\mathcal{M}_{0,j}^{\mathbb{R}}$ being $G^{\mathbb{R}}\backslash\mathcal{C}_{0,j}^{\mathbb{R}}$, where j indicates the number of pairs of conjugate roots. It turns out that G acts properly on $\mathcal{C}_0$, and since the point stabilizers are compact algebraic subgroups of G they are finite; therefore $\mathcal{M}_0$ is a complex-analytic orbifold and the $\mathcal{M}_{0,j}^{\mathbb{R}}$ are real-analytic orbifolds. The relation with hyperbolic geometry begins with the following theorem:

Theorem 2. *Let Γ_j be the group generated by the Coxeter group of P_j from* (1) *or* (3)*, together with the diagram automorphism when $j = 1$. Then $\mathcal{M}_{0,j}^{\mathbb{R}}$ is the orbifold H^3/Γ_j, minus the image therein of the walls corresponding to the blackened nodes and the edges corresponding to triple bonds. Here, 'is' means an isomorphism of real-analytic orbifolds.*

In the second lecture we will see that the faces of the P_j corresponding to blackened nodes and triple bonds are very interesting; we will glue the P_j together to obtain a real-hyperbolic description of the entire moduli space.

References. The canonical references for hyperbolic geometry and an introduction to orbifolds are Thurston's notes [15] and book [16]. The book is a highly polished treatment of a subset of the material in the notes, which inspired a great deal of supplementary material, e.g., [4]. For other applications of hyperbolic geometry to real algebraic geometry, see Nikulin's papers [12] and [13], which among other things describe moduli spaces of various sorts of K3 surfaces as quotients of H^n.

Lecture 2

We will not really provide a proof of Theorem 2; instead we will develop the ideas behind it just enough to motivate the main construction leading to Theorem 4 below. Although Theorem 2 concerns smooth sextics, it turns out to be better to consider mildly singular sextics as well. Namely, let $\mathcal{C}_s$ be the set of binary sextics with no point of multiplicity 3 or higher, and let $\Delta \subseteq \mathcal{C}_s$ be the discriminant, so $\mathcal{C}_0 = \mathcal{C}_s - \Delta$. (For those who have seen geometric invariant theory, $\mathcal{C}_s$ is the set of stable sextics, hence the subscript s.) It is easy to see that Δ is a normal crossing divisor in $\mathcal{C}_s$. (In the space of *ordered* 6-tuples in $\mathbb{C}P^1$ this is clear; to get the picture in $\mathcal{C}_s$ one mods out by permutations.) Now let $\mathcal{F}_s$ be the universal branched cover of $\mathcal{C}_s$, with ramification of order 6 along each component of the preimage of Δ. $\mathcal{F}_s$ turns out to be smooth and the preimage of Δ a normal crossing divisor. More precisely, in a neighborhood of a point of $\mathcal{F}_s$ describing a sextic with k double points, the map to $\mathcal{C}_s$ is given locally by

$$(z_1, \ldots, z_6) \mapsto (z_1^6, \ldots, z_k^6, z_{k+1}, \ldots, z_6),$$

where the branch locus is the union of the hypersurfaces $z_1 = 0, \ldots, z_k = 0$. Let $\mathcal{F}_0$ be the preimage of $\mathcal{C}_0$ and let Γ be the deck group of $\mathcal{F}_s$ over $\mathcal{C}_s$. We call an element of $\mathcal{F}_s$ (resp. $\mathcal{F}_0$) a framed stable (resp. smooth) binary sextic. Geometric invariant theory implies that G acts properly on $\mathcal{C}_s$, and one can show that this G-action lifts to one on $\mathcal{F}_s$ which is not only proper but free, so $G\backslash\mathcal{F}_s$ is a complex manifold. The reason we use 6-fold branching rather than some other sort of branching is that in this case $G\backslash\mathcal{F}_s$ has a nice description, given by the following theorem. See the appendix for a sketch of the Hodge theory involved in the proof.

Theorem 3 (Deligne–Mostow [6]**).** *There is a properly discontinuous action of* Γ *on complex hyperbolic* 3*-space* $\mathbb{C}H^3$ *and a* Γ*-equivariant complex-manifold diffeomorphism* $g : G\backslash\mathcal{F}_s \to \mathbb{C}H^3$*, identifying* $G\backslash\mathcal{F}_0$ *with the complement of a hyperplane arrangement* $\mathcal{H}$ *in* $\mathbb{C}H^3$*.*

Complex hyperbolic space is like ordinary hyperbolic space except that it has 3 complex dimensions, and hyperplanes have complex codimension 1. There is an upper-half space model analogous to the real case, but the most common model for it is the (open) complex ball. This is analogous to the Poincaré ball model for real hyperbolic space; we don't need the ball model except to see that complex conjugation of $\mathbb{C}H^3$, thought of as the complex 3-ball, has fixed-point set the real 3-ball, which is H^3.

Given a framed stable sextic $\tilde{S}$, Theorem 3 gives us a point $g(\tilde{S})$ of $\mathbb{C}H^3$. If $\tilde{S}$ lies in $\mathcal{F}_0^{\mathbb{R}}$ (the preimage of $\mathcal{C}_0^{\mathbb{R}}$), say over $S \in \mathcal{C}_0^{\mathbb{R}}$, then we can do better, obtaining not just a point of $\mathbb{C}H^3$ but also a copy of H^3 containing it. The idea is that complex conjugation κ of $\mathcal{C}_0$ preserves S and lifts to an antiholomorphic involution (briefly, an anti-involution) $\tilde{\kappa}$ of $\mathcal{F}_0$ that fixes $\tilde{S}$. This uses the facts that $\mathcal{F}_0 \to \mathcal{C}_0$ is a covering space and that $\pi_1(\mathcal{F}_0) \subseteq \pi_1(\mathcal{C}_0)$ is preserved by κ. Riemann extension extends $\tilde{\kappa}$ to an anti-involution of $\mathcal{F}_s$. Since κ normalizes G's action on $\mathcal{C}_s$, $\tilde{\kappa}$ normalizes G's action on $\mathcal{F}_s$, so $\tilde{\kappa}$ descends to an anti-involution κ' of $\mathbb{C}H^3 = G\backslash\mathcal{F}_s$. Each anti-involution of $\mathbb{C}H^3$ has a copy of H^3 as its fixed-point set, so we have defined a map $g^{\mathbb{R}}$ from $\mathcal{F}_0^{\mathbb{R}}$ to the set of pairs

$$(x \in \mathbb{C}H^3, \text{ a copy of } H^3 \text{ containing } x). \qquad (4)$$

Note that $\tilde{\kappa}$ fixes every point of $\mathcal{F}_0^{\mathbb{R}}$ sufficiently near $\tilde{S}$, so all nearby framed real sextics determine the same anti-involution κ' of $\mathbb{C}H^3$. Together with the G-invariance of g, this proves that $g^{\mathbb{R}}$ is invariant under the identity component of $G^{\mathbb{R}}$. A closer study of $g^{\mathbb{R}}$ shows that it is actually invariant under all of $G^{\mathbb{R}}$. We write K_0 for the set of pairs (4) in the image $g^{\mathbb{R}}(\mathcal{F}_0^{\mathbb{R}})$. An argument relating points of $\mathcal{C}_s$ preserved by anti-involutions in $G \rtimes (\mathbb{Z}/2)$ to points of $\mathbb{C}H^3$ preserved by anti-involutions in $\Gamma \rtimes (\mathbb{Z}/2)$ shows that if $x \in \mathcal{F}_0^{\mathbb{R}}$ has image $(g(x), H)$, then every pair $(y \in H - \mathcal{H}, H)$

also lies in K_0. That is, K_0 is the disjoint union of a bunch of H^3's, minus their intersections with $\mathcal{H}$. The theoretical content of Theorem 2 is that $g^{\mathbb{R}} : G^{\mathbb{R}}\backslash\mathcal{F}_0^{\mathbb{R}} \to K_0$ is a diffeomorphism.

The computational part of Theorem 2 is the explicit description of K_0, in enough detail to understand $\mathcal{M}_0 = G\backslash\mathcal{F}_0^{\mathbb{R}}/\Gamma = K_0/\Gamma$ concretely. It turns out that Γ, $\mathcal{H}$ and the anti-involutions can all be described cleanly in terms of a certain lattice Λ over the Eisenstein integers $\mathcal{E} = \mathbb{Z}[\omega{=}e^{2\pi i/3}]$. Namely, Λ is a rank 4 free $\mathcal{E}$-module with Hermitian form

$$\langle a|a\rangle = a_0\bar{a}_0 - a_1\bar{a}_1 - a_2\bar{a}_2 - a_3\bar{a}_3 \,. \tag{5}$$

The set of positive lines in $P(\mathbb{C}^{1,3} = \Lambda \otimes_{\mathcal{E}} \mathbb{C})$ is a complex 3-ball (i.e., $\mathbb{C}H^3$), $\Gamma = P\mathrm{Aut}\,\Lambda$, $\mathcal{H}$ is the union of the hyperplanes orthogonal to norm -1 elements of Λ, and the anti-involutions of $\mathbb{C}H^3$ corresponding to the elements of K_0 are exactly

$$\begin{aligned}
\kappa_0 &: (x_0,x_1,x_2,x_3) \mapsto (\bar{x}_0, \quad \bar{x}_1, \quad \bar{x}_2, \quad \bar{x}_3)\\
\kappa_1 &: (x_0,x_1,x_2,x_3) \mapsto (\bar{x}_0, \quad \bar{x}_1, \quad \bar{x}_2, -\bar{x}_3)\\
\kappa_2 &: (x_0,x_1,x_2,x_3) \mapsto (\bar{x}_0, \quad \bar{x}_1, -\bar{x}_2, -\bar{x}_3)\\
\kappa_3 &: (x_0,x_1,x_2,x_3) \mapsto (\bar{x}_0, -\bar{x}_1, -\bar{x}_2, -\bar{x}_3)
\end{aligned} \tag{6}$$

and their conjugates by Γ. We write H_j^3 for the fixed-point set of κ_j.

Since $H_0^3, \ldots, H_3^3$ form a complete set of representatives for the H^3's comprising K_0, we have

$$\mathcal{M}_0^{\mathbb{R}} = K_0/\Gamma = \coprod_{j=0}^{3} \left(H_j^3 - \mathcal{H}\right)\Big/(\text{its stabilizer } \Gamma_j \text{ in } \Gamma)$$

Understanding the stabilizers Γ_j required a little luck. Vinberg devised an algorithm for searching for a fundamental domain for a discrete group acting on H^n that is generated by reflections [18]. It is not guaranteed to terminate, but if it does then it gives a fundamental domain. We were lucky and it did terminate; the reflection subgroup of Γ_j turns out to be the Coxeter group of the polyhedron P_j.

One can obtain our polyhedra by applying his algorithm to the $\mathbb{Z}$-sublattices of Λ fixed by each κ_j. For example, an element of the κ_2-invariant part of Λ has the form $(a_0, a_1, a_2\sqrt{-3}, a_3\sqrt{-3})$ with $a_0, \ldots, a_3 \in \mathbb{Z}$, of norm $a_0^2 - a_1^2 - 3a_2^2 - 3a_3^2$. Similar analysis leads to the norm forms

$$\begin{aligned}
\langle a|a\rangle &= a_0^2 - a_1^2 - a_2^2 - a_3^2\\
\langle a|a\rangle &= a_0^2 - a_1^2 - a_2^2 - 3a_3^2\\
\langle a|a\rangle &= a_0^2 - a_1^2 - 3a_2^2 - 3a_3^2\\
\langle a|a\rangle &= a_0^2 - 3a_1^2 - 3a_2^2 - 3a_3^2
\end{aligned}$$

in the four cases of (6). Now, Γ_j lies between its reflection subgroup and the semidirect product of this subgroup by its diagram automorphisms. After checking