SUMMER INSTITUTE ON SET THEORETIC TOPOLOGY

(Summary of Lectures and Seminars)

Summary of Lectures and Seminars

SUMMER INSTITUTE ON SET THEORETIC TOPOLOGY

Madison, Wisconsin 1955

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FOREWORD

The third summer institute sponsored by the American Mathematical Society with financial support from the National Science Foundation was devoted to set theoretic topology. Meetings were held on the campus of the University of Wisconsin from July 24 to August 20, 1955.

At its organizational meeting, the Institute elected R. H. Bing as chairman. It selected six of its members to give hour addresses; also it set up numerous seminars to consider various phases of set theoretic topology. Each seminar chose its own leaders and organized itself. A summary of the six lectures and the deliberations of the seminars is included in this pamphlet.

It was thought that it would be valuable to have this summary come out without delay. Hence, there was essentially no editing of this report. In most cases, handwritten reports were turned into the chairman during the last week of the Institute. He turned these over to the typist and as quickly as the typing was done the duplicating process was begun. Perhaps a list of corrections will follow. It is hoped that the advantage of getting the report promptly will outweigh the disadvantage of culling out possible errors.

The reports of the seminars contain abstracts of talks and questions raised. Perhaps the answers to some of these questions are known but it is presumed that at the time a question was raised, no one present in the seminar knew the answer. Perhaps many of the questions are not of great moment but no effort was made to cull them.

It is hoped that this summary will prove useful both to those who attended the Institute and those who did not attend.

We mention again that this is a "crash" summary, run off without delay. Much credit for getting the typing done rapidly is due to Mrs. Beatrice Holmburg, Secretary of the Mathematics Department, University of Wisconsin.

While the supply lasts, these summaries may be obtained free from Professor R. H. Bing, 301B North Hall, University of Wisconsin, Madison 6, Wisconsin.

TRANSFORMATIONS

by

D. Montgomery

Let G be a compact group which acts as a topological transformation group of a manifold M; assume that the action is effective, that is that every element except the identity moves at least one point of M. Then questions may be raised on roughly three levels of generality as follows:

1. If G is not assumed to be a Lie group, do the above conditions nevertheless force it to be a Lie group? Equivalently must G have a neighborhood of the identity which contains no non-trivial subgroup? In particular if G is compact and zero-dimensional, must it be finite?

In this direction it has been shown (Montgomery and Zippin) that if $M = E^3$ and G is connected, then G is a Lie group, and furthermore, that G is either the circle group or the orthogonal group on 3 variables and that in either case coordinates can be chosen so that G acts in the usual way.

2. Assume G to be a Lie group. It has been proved that if r is the highest dimension of any orbit then points on orbits of dimension $< r$ form a closed set of dimension $\leq n-2$ (Montgomery, Samelson, Zippin). This is a generalization of a theorem of Newman.

If $M = E^n$ the following questions are suggested: a) must some point be fixed under all elements of G? b) can two orbits be linked?

3. Assume G to be a Lie group and that M is a differentiable manifold on which G acts in a differentiable manner. The two questions raised above are unanswered in this more special case as well.

Now let G_x be the subgroup of G consisting of all elements leaving x fixed. Among the points on orbits of highest dimension there may be some where G_x is discontinuous. Let these be denoted by E. It has been shown (Montgomery, Samelson, Yang) that $\dim E \leq n-2$.

For $M = S^n$, what can be said about the set of fixed points of G, and must they be a sphere of lower dimension or at least resemble one in the sense of homology. A number of related questions about fixed points suggest themselves in both 2) and 3).

SET THEORETIC TOPOLOGY - PRESENT AND FUTURE

by

G. T. Whyburn

Remarks were made concerning the nature of set theoretic topology as of today and its role in the current and possible future development of the whole of mathematics was discussed. By way of example, some recent contributions of topology to the classical theory of functions of a complex variable were sketched. Indications were given as to how properties of such functions which are topological in character are obtained by topological type arguments from the fundamental properties of lightness and openness belonging to the class of all mappings generated by analytic functions. Attention was called to a still missing topological proof of the fact that a mapping generated in this way can be a local homeomorphism at a point z of the complex plane only if the derivative of the generating function does not vanish at z. The problem of establishing a relation between the continuity properties of the decomposition of the z-plane into sets on which an analytic function is constant and the exceptional and asymptotic values of the function was discussed in some detail. Lack of "upper semi-continuity" in a certain sense of the decomposition at an element is a phenomenon analogous to that of an exceptional value for the function on this set. The problem is to pin down this relationship precisely and determine to what extent these phenomena are interdependent or simultaneous.

In conclusion some possible improvements in the performance of set theoretic topologists and their contribution to the overall mathematical scene were mentioned. Better writing, simpler statement of results and more readable proofs were stressed. It was suggested that nearly every important theorem which survives permanently in mathematics eventually becomes stateable in simple terms. Also its proof becomes reasonably comprehensible through one of two occurrencies, namely, either by direct simplification or by a whole new theory being developed in mathematics involving new concepts and results which eventually engulfs the theorem in question and offers it in palatable form in its proper setting.

<u>Key references</u>: See the bibliography at the end of the author's paper "Open mappings on locally compact sets", A.M.S., Memoirs, No. 1.

DEVELOPMENT OF A THEORY FOR MANIFOLDS

by

R. L. Wilder

The first ambitious program that could be called topological was carried out by Poincaré. By subdividing a surface into a finite number of pieces called cells the determination of the "Betti numbers" was reduced to a finite algebraic procedure—a simplication which rendered proof of topological invariance quite difficult, however. His study of manifolds and discovery of the duality theorem named after him, as well as of the fundamental group, profoundly influenced later work.

Figure 1 is a simplified diagram indicating lines of influence and later developments. The "generalized manifolds" of Veblen, van Kampen, etc., were still composed of cells, while those of Čech and Lefschetz (1933) took advantage of the extensions (Vietoris, Čech) of homology theory to general spaces in order to set up spaces which could be called "manifolds" in the sense that they satisfied the Poincaré duality, etc. However, they were strictly "homology manifolds", defined entirely in terms of homology (and set-theoretic) properties that failed to eliminate spaces that were not 1 - L C, for instance.

Figure 2 sketches an independent line of development (I do not recall a single mention of Poincaré's name in the work of Schoenflies mentioned below, although he was familiar with and made use of, Riemann's work on "connectivity"). As the figure is perhaps self-explanatory, I will point out only two general matters which the details of the figure may not bring out. Firstly, note that the arrow from Moore's characterization of E^2 over to the column headed "Decomposition spaces" leads not into the first item of the latter ("prime point"), but to upper semi-continuous decompositions of E^2. So far as I know, Hahn's work was independent of Moore's. But as I recall it from my personal association with Moore at the time, his interest in upper semi-continuous collections of continua in the plane stemmed from the fact that he recognized, as a good "Axiomatiker", that the elements of such collections when considered as points, with suitable definition of certain sub-collections as regions ("point" and "region" were the undefined elements in his axioms), satisfied all his axioms for the plane (so long as no continuum in the collection separated the plane). As is well-known, we may consider this as the definition

POINCARÉ'S COMBINATORIAL TOPOLOGY

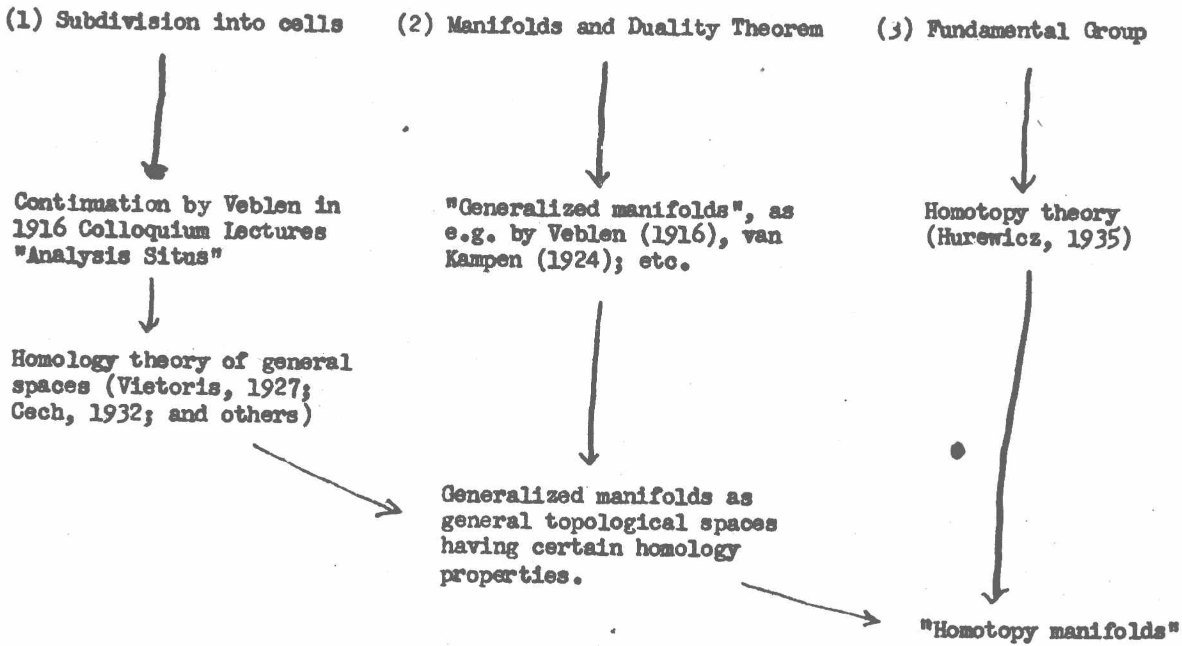

Figure 1

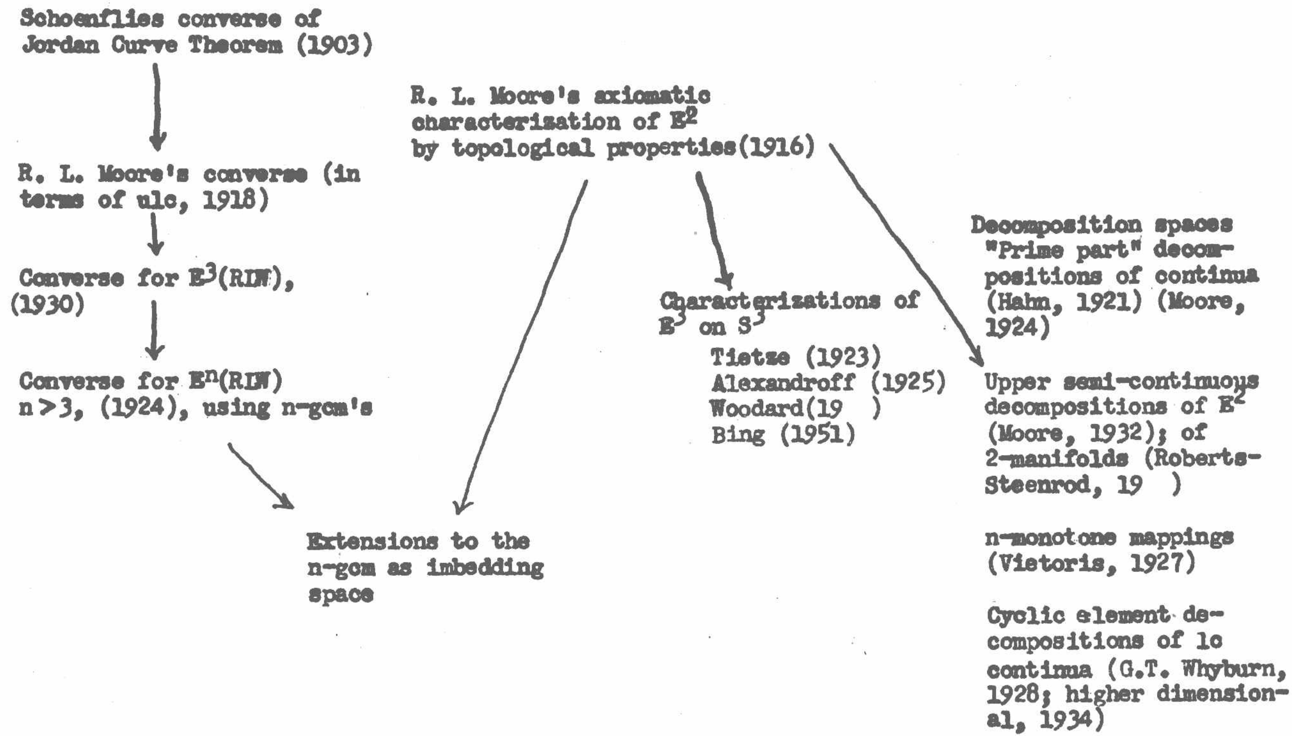
Schoenflies converse of Jordan Curve Theorem (1903)
R. L. Moore's converse (in terms of ulc, 1918)
Converse for E^3(RLW), (1930)
Converse for E^n(RLW) n > 3, (1924), using n-gcm's
R. L. Moore's axiomatic characterization of E^2 by topological properties(1916)
Extensions to the n-gcm as imbedding space
Characterizations of E^3 on S^3
Tietze (1923)
Alexandroff (1925)
Woodard(19)
Bing (1951)
Decomposition spaces
"Prime part" decompositions of continua (Hahn, 1921) (Moore, 1924)
Upper semi-continuous decompositions of E^2 (Moore, 1932); of 2-manifolds (Roberts-Steenrod, 19)
n-monotone mappings (Vietoris, 1927)
Cyclic element decompositions of lc continua (G.T. Whyburn, 1928; higher dimensional, 1934)

Figure 2

of a continuous mapping of E^2 onto E^2, but this was not the point of view that was of interest. Later developments emphasize both these points of view, of course.

Secondly, note that the bottom item on the left ("Extension to the n-gcm ...) was motivated from two directions. My "Converse for E^n" with $n > 3$, contained a defect due to the lack of a topological characterization of the n-manifold for $n > 2$, which resulted in my being able only to assert that the converse in E^n, for $n > 3$, yielded what I called an "(n - 1)-dimensional generalized closed manifold" ... (n - 1)-gcm for short. Conscious of this, as well as of the underlying cause (lack of a topological characterization of E^n), led me to decide to use the n-gcm as the imbedding space itself (in place of E^n), and see how much of the program for the set-theoretic investigation of E^n could be carried through; from another point of view, then, "setting" for a set of axioms for the imbedding space which, while not completely characterizing E^n as Moore's axioms did for E^2, might at least go far in this direction. It turned out that not only did such theorems as the Jordan-Brouwer theorem and the converse go over to the new spaces, but almost without exception all known theorems of plane topology become special cases of theorems concerning n-gcms.

Incidentally, the metric cases of the 1-gcm and 2-gcm reduce to the S^1 and the classical closed 2-manifolds. Also, the axioms which I used can be shown to be equivalent to the independently devised axioms of Čech-Lefschetz; they have been subsequently reduced by Begle and myself to very simple form.

Example 1. In order that a space S should be topologically a euclidean plane, it is necessary and sufficient that it be 2-dimensional, connected, locally compact, separable metric and such that $p^1(S) = 0$, and for every $X \in S$, $p^1(S,X) = 0$ and $p^2(S,X) = 1$.

[$p^r(S,X)$ is the Alexandroff-Čech local Betti number of S at the pt X].

Example 2. If S is an orientable n-gcm such that $p^1(S) = 0$, D is a ulc^{n-2} (= uniform local r-connected for $r = 0, 1, \ldots, n - 2$) domain of S whose boundary B is a non-n-dimensional continuum, then B is an orientable n-gcm.

[For $n = 2$, and S metric, this becomes Moore's converse of 1918 mentioned as second item, column 1 of Figure 2.]

Turning to Figure 3, we consider a line of development stemming directly from a problem arising in 19th century Analysis: Can a plane curve $x = f(t)$, $y = g(t)$, where the

Space-filling Curve Problem

Schoenflies characterization of continuous curves in E^2 (1908)

Extension to E^n (including recognition of lc^k sets by positional properties (RLW, 1941)

Complete solution by intrinsic topological properties (Hahn, Mazurkiewicz, 1914)

Study of continuous curves (Peano continua) (1920-1930); other characterizations—Sierpinski's "Property S".

Moore's modification of Property S and application to recognition of domains in E^2 whose boundaries are lc.

Extensions to E^n (RLW) (1938-1946)

Global properties ("$(P, Q)^r$").

Figure 3

real parameter t varies continuously from 0 to 1, fill up a square and its interior? While the <u>curves</u> defined by Peano, Hilbert and E. H. Moore did this, they contributed little to the next obvious question. What types of configuration can constitute a set of points which a plane curve passes through? Schoenflies gave a complete solution of the latter question. Six years later Hahn and Mazurkiewicz independently, influenced by the new concept of "topological space", developed in such works as those of Frechet and Hausdorff, gave intrinsic characterizations of those topological spaces that can be continuous images of the real closed interval—spaces that are metric continua and are 0-lc—a result of beautiful simplicity and generality.

Each of these results gave rise to new problems: The Schoenflies result, phrased in terms of the "situation" of the configuration in E^2, gave rise to the problem—how to recognize the "continuous curves" by their situation in higher dimension, i.e., by positional properties in E^n, $n > 2$? The Hahn-Mazurkiewicz theorem gave impetus to the search for other characteristic properties of "continuous curve spaces", as well as to an intensive structural analysis (the Whyburn cyclic element theory is an excellent example of the latter). I should like to select one of these characteristic properties, due to Sierpinski and subsequently slightly modified by Moore and called "Property S", for purposes of exemplification.

Sierpinski showed that in order that a compact metric space M should be the continuous image of a real interval, it is necessary and sufficient that it have the following property: For any $\varepsilon > 0$, M is the union of a finite number of closed and connected sets of diameter $< \varepsilon$. This property, with "closed" omitted, Moore called Property S. Notice that it is a global property, in contrast to the Hahn-Mazurkiewicz <u>local connectedness</u>. Now suppose S is a locally compact (not necessarily metric) space, M a subset of S, and G any group of compact cycles of S. Then we say that M has Property $(P,Q;G)^r$ if for arbitrary open sets P and Q such that Q is compact and $P \supset \overline{Q}$, at most a finite number of cycles of $M \cap Q$ are linearly independent relative to homology (= lirh) in $P \cap M$. For G the group of <u>all</u> compact r-cycles of S, we simply write $(P,Q)^r$; for G the group of <u>bounding</u> compact r-cycles of S, we write $(P,Q;\sim)^r$.

Now in locally compact spaces, Properties S and $(P,Q)^0$ are equivalent, so that $(P,Q)^n$ may be considered a generalization to n dimensions of Property S. And analogous to the relationship between Property S and 0-lc, we have that for a compact space to be lc^n, it is necessary and sufficient that it have Property $(P,Q)^n_0$ —that is, locally connected in all dimensions

0 to n is equivalent to having $(P,Q)^m$ in all dimensions 0 to n. However, many properties of lc^n spaces can be generalized to spaces having $(P,Q)^r$ only over a limited range $m \leqq r \leqq n$ where $m > 0$. And among the most interesting properties that the (P,Q)-properties have are the dualities they satisfy when we introduce the corresponding properties for cohomology.

Example 3. From such dualities, for instance, one easily derives that if M is a closed subset of an orientable n-gcm S, and $r < n - 1$, then for M to have $(P,Q;\sim)^r$ it is necessary and sufficient that $S - M$ have $(P,Q;\sim)^{n-r-2}$. Since for a continuum $(P,Q;\sim)^0 = (P,Q)^0$, the special case $r = 0$ and S the 2-sphere S^2 gives at once that a nasc that a subcontinuum M of S^2 be 0-lc is that its complement have property $(P,Q;\sim)^0$—i.e., that each complementary domain have property S and the diameters of these domains form a null sequence (the Schoenflies-Moore theorem).

Example 4. One of the well known theorems of plane topology was the Torhorst Theorem: If M is a continuous curve in S^2, and D a complementary domain of M, then the boundary F(D) of D is a continuous curve. For generalized manifolds we have: If M is a 0-lc closed subset of an orientable n-gcm S, which has property $(P,Q;\sim)^{n-2}$, and D is a complementary domain of M, then F(D) is 0-lc. (Note that for the case where $S = S^2$ and M is connected, the 0-lc condition on M is equivalent to the $(P,Q,\sim)^0$ condition so that in this case the two conditions merge.) Another generalization is: If M is an lc^{n-2} closed subset of an orientable n-gcm S and D a component of S-M, then F(D) is 0-lc.

Unsolved problems. As observed above, the separable metric cases of the gm's merge with the classical types for 1 and 2 dimensions. Examples show that this is not true for 3 dimensions. It would be interesting to know if every 3-gcm in E^4 which is spherelike in both homology and homotopy is an S^3. If the manifold is spherelike only in the homology sense, the answer seems to be negative, since there evidently exist surfaces of "Poincaré space" type in E^4 that are not 3-spheres [Cf. the minutes of the seminar on generalized manifolds, 7th session]. A related question is: Is every 3-gcm in E^4 a 3-manifold in the classical sense?[1] If not, is every 3-gcm in the homotopy sense (see below) in E^4 a 3-manifold in the classical sense?

A basic question concerning n-gms, $n > 2$, that has remained unsolved is: Is every n-gm locally orientable? (An n-gm is locally orientable if each point has a neighborhood which is an orientable n-gm.)

Twenty years ago Alexandroff raised the question whether a gm which is perfectly normal is necessarily metric? This remains unanswered.

Due to their generality as topological spaces, little is known of the dimension theory characteristics of gm's. For example, it is not even known whether an n-dimensional continuum in an n-gcm S must contain interior points of S; not even whether a common boundary of two domains in an n-gcm is at most (n - 1)-dimensional. (Cf. Example 2 above.)

There are many other unsolved problems, as for instance those concerning mappings of n-gcm's and various types of decompositions thereof. And the theory of "homotopy manifolds" is a virtually untouched field.

[1] [Using certain results reported on in the Manifolds seminar, the answer is evidently negative.]

CERTAIN CLASSICAL PROBLEMS

OF EUCLIDEAN TOPOLOGY

By E. E. Moise

Perhaps I should begin by explaining that there will be very little factual content in what I have to say: I will be discussing various unsolved problems. Moreover, these problems are not new: all of them can be considered classical, and the most recent of them was proposed in print fifteen years ago. Nevertheless, I believe that they deserve discussion now.

In the first place, while all of them are of a high order of difficulty, I am by no means convinced that they are so hopeless, in the present state of knowledge, as to be best forgotten or ignored. In the second place, the nature of their interest has in some cases shifted rather drastically during the past twenty years or so: some of the classical problems of combinatorial topology are now much more deserving of the attention of set-theoretic topologists, who until recently have paid rather little attention to them.

This is true, in particular, of the Triangulation Problem. There was a time when a solution of this problem was urgently needed, in order to show that the use of triangulations, in the algebraic topology of manifolds, represented a method, rather than an ad hoc hypothesis. But invariantly defined homology theories are now long familiar; and the Wilder theory constitutes an extended proof that the homology theory of manifolds is autonomous, in the sense that the deep homology properties of such spaces are deducible from their trivial homology properties, with no strong use of their Euclidean structure, and in particular, without use of triangulations. The categoricity of the Eilenberg-Steenrod axioms has been established not only for triangulable spaces, but also for absolute neighborhood retracts. Examples could be multiplied to show that in algebraic topology the Triangulation Problem has been bypassed with great success.

To the set-theorist, however, the problem appears in a different light. Its primary importance is as a symptom. It seems fair to say that the problem has not been solved because the foundations of set-theoretic Euclidean topology are not rightly understood; and it seems reasonable to predict that if the triangulation theorem is proved, in any direct sort of way then the methodology of the proof would be of much broader applicability. Some have expressed the hope that a proof can be found, using the elegant and powerful methods which have lately been developed in the theory of groups. This may be so. But from the point of view that I am taking here, it would be more desirable to find a "direct proof," which instead of bypassing the essential geometric difficulties, would come to grips with these difficulties and break them. Perhaps the best criterion for such a "direct" proof is that it should lead also to a proof of the so-called Hauptvermutung. This asserts in effect that a manifold can be triangulated in essentially only one way; that is to say, any two triangulations of the same manifold must be combinatorially equivalent.

The essential geometric difficulties are substantial; some of them represent major problems in their own right. For example, consider the star St(v) of a vertex v of a triangulated n-manifold. For n greater than 3, it is not known whether St(v) must be combinatorially equivalent, or even topologically equivalent, to an n-cell. This is the so-called Sphere Problem. If St(v) can be thrown into n-space by a piecewise linear homeomorphism, then the answer to both questions can easily be shown to be affirmative. But our hypothesis, taken on its face, asserts merely that St(v) can be thrown into n-space by a homeomorphism. It is plausible to conjecture that the latter condition implies the former, not only for star-shaped sets, but for any n-manifold with boundary. (This has been proved for dimension 3; it represents the case epsilon equals infinity of an approximation theorem. See Ann. of Math. vol. 55 (1952) pp. 215-222.) Thus, if there are forms of combinatorial pathology, refuting the Sphere Conjecture, then there are forms of set-theoretic topology, of an at least equally implausible kind.

We observe that the Triangulation Problem, the Hauptvermutung, and the Sphere Conjecture have at least one property in common: each of them calls for the construction of a homeomorphism. (In fact, all of the problems which I shall be discussing have this property.) In dimension 3, the usual way of doing this is to define the desired homeomorphism first over a 2-dimensional set, and then extend it. For example, if C is a 3-cell with boundary B, then any piecewise linear homeomorphism of B into 3-space can be extended to give a piecewise linear homeomorphism of C into 3-space. This depends on a strengthened combinatorial form of a theorem of Alexander, to the effect that every polyhedral 2-sphere in 3-space is the boundary of a 3-cell. To extend the 3-dimensional methods to higher dimensions, we would need to know, at least, that every polyhedral (n-1)-sphere in n-space is the boundary of an n-cell. This remains an un-

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