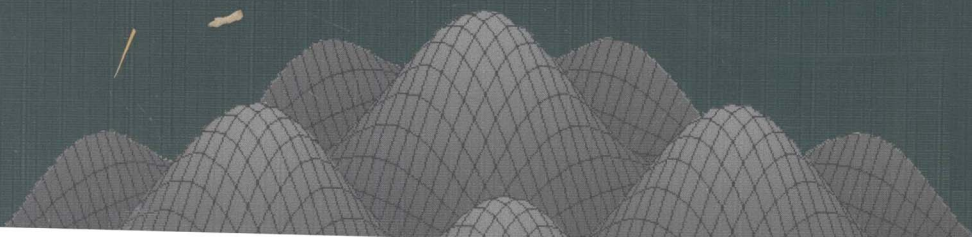


Holomorphic Functions in the Plane and n-dimensional Space

Klaus Gürlebeck
Klaus Habetha
Wolfgang Sprößig



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附光盘 壹张

Klaus Gürlebeck
Institut für Angewandte Mathematik
Bauhaus-Universität Weimar
Coudraystr. 13
99423 Weimar
Germany
e-mail: klaus.guerlebeck@bauing.uni-weimar.de

Klaus Habetha
Lehrstuhl II für Mathematik
RWTH Aachen
Templergraben 55
52062 Aachen
Germany
e-mail: habetha@math2.rwth-aachen.de

Wolfgang Sprößig
Institut für Angewandte Analysis
Fakultät für Mathematik
TU Bergakademie Freiberg
Prüferstr. 9
09596 Freiberg
Germany
e-mail: sproessig@math.tu-freiberg.de

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Preface to the German Edition

Complex analysis has produced a large number of deep recumbent and aesthetic results in its more than 200 year-old history. In the classical context complex analysis is the theory of complex differentiable functions of a complex variable, or also the theory of *holomorphic functions*. These are the solutions of a (2×2) -system of partial differential equations, which usually are called *Cauchy–Riemann differential equations (CRD)*.

Indeed, the algebra of the real quaternions of W.R. Hamilton has been available since 1843 and the real Clifford algebra of W.K. Clifford since 1878, but until the 1930s the prevailing view was that complex analysis is a purely two-dimensional theory. Only the group around the Swiss mathematician R. Fueter and the Romanian mathematicians G.C. Moisil and N. Teodorescu around 1930 started to develop a hypercomplex analysis in the algebra of real quaternions and in real Clifford algebras. In the late 1960s a group of Belgian mathematicians around R. Delanghe in Ghent created a rich higher dimensional analogy to complex analysis, called Clifford analysis. Since 1990 the number of relevant articles and books has increased significantly. Today intensive research is going on in Clifford analysis to which the more than 9000 entries in our database on the relevant literature testify. The database can be found on the CD attached to our book.

The purpose of the present textbook is to collect the essentials of classical complex analysis and to present its higher dimensional generalizations at a level suitable for university instruction. The typical users we have in mind are, first of all, students of mathematics and physics, but also students of any discipline requiring some sophisticated mathematics from the second year onward. The material to be covered is extensive, and we have attempted to make it as self-sufficient as possible within the limits of a modest size book. We have covered not only analytical and geometrical aspects but numerical procedures as well. Historical references outline the development of the field and present some of the personal characteristics of the most important personalities who have contributed to that history.

In the first chapter complex numbers, quaternions, and the Clifford numbers are introduced. We have emphasized the parallelism of our presentation. Quaternions and Clifford numbers take up more space than complex numbers. Besides the algebraic and geometrical properties we treat in particular also rotations and representations.

In Section 4 we illustrate the topological and analytical basic facts for the treatment of functions up to Riemann spheres. This section is deliberately kept short in view of its relationship to classical analysis. Section 5 treats some of the possible definitions of holomorphic functions. We keep this name also in higher dimensions, because the definitions are almost independent of dimension. The standard literature uses here mostly the concept of Weierstrass monogenic functions. However, it seems to us at least debatable whether this best describes the meaning

of the definition (cf. end of Section 5). Also, the notion of holomorphic functions fits conceptually better than that of meromorphic functions. Since the articles by H. Malonek the concept of holomorphic functions can be introduced also via local approximation by suitable linear functions, so even in that context the analogy holds in all dimensions. Section 6 is devoted to “simple” functions, namely powers and Möbius transforms. The polynomials named after R. Fueter are suitable as power functions in higher dimensions since they have many nice qualities. Unfortunately, the reduction of Fueter polynomials to the planar case leads to powers $(-iz)^n$ and not to z^n ; however, the parallelism is still given. In particular L.V. Ahlfors has studied Möbius transforms in higher dimensions. Here too the clear comparability of all dimensions can be recognized.

We have put together the necessary aids for integration in Appendix 2, and a short introduction to the theory of differential forms in Appendix 1. We believe that this can be helpful, because in lectures to beginners these areas are often treated only very briefly, if at all. Indeed, we do not include the proof of Stokes’ integral theorem as this would lead too far away from the subject. Then in Chapter III Cauchy’s integral theorem and the Borel–Pompeiu formula are easy consequences of Stokes’ theorem. However, we also consider the boundary value formulae of Plemelj–Sokhotzki. Conclusions on Cauchy’s integral formula follow. Moreover, the Teodorescu transform is examined and the Hodge decomposition of quaternionic Hilbert space is treated. The needed functional spaces are briefly introduced in Appendix 3.

Chapter IV is devoted to different areas of hypercomplex analysis. We firstly treat Taylor and Laurent series. The effort is clearly larger in higher dimensions than in the plane, but the similarity helps. Unfortunately, the Taylor series in dimensions greater than 2 are not orthogonal expansions. For quaternions orthogonal expansions are introduced, which are especially adequate for numerical purposes.

The elementary functions in the plane have no special difficulties. They are given more or less canonically. For all generalizations to higher dimensions, a royal way does not exist symptomatically. Different generalizations of the exponential function are pointed out, and one generalization given by the method of separation of variables is developed. This exponential function has some nice properties and is an appropriate kernel for the Fourier transform of quaternion-valued functions.

Section 12, which explores the local structure of holomorphic functions, shows that in higher dimensions this is still an active field of research. The pleasant qualities of zeros and isolated singularities in the plane at first sight get lost in space. There is still no suitable structure in which to understand all the relevant phenomena. At least the residue theorem can be transferred, and also first attempts for an argument principle were found.

Section 13 deals with special functions. The Gamma function and the Riemann Zeta function are treated, followed by considerations about automorphic functions and forms in $\mathcal{Cl}(n)$ which offer an insight into the latest research in this field.

Problems at the end of every section should help the reader toward a better understanding of the corresponding area. The use of the skewfield structure of the real quaternions allows one to formulate some statements more precisely and in a more readable form than in general Clifford algebras. Since applications in $\mathbb{R}^3$ and $\mathbb{R}^4$ are of special interest, we have sometimes waived the more general case of real Clifford algebras $Cl(n)$.

Results of many colleagues working in the area of Clifford analysis are used in the presentation of higher dimensional results. We thank especially our colleagues Professor Krüger (Kaiserslautern/Germany), Professor Malonek (Aveiro/Portugal), and Professor Kraußhar (Leuven/Belgium), who helped us to write some of the sections. We discussed details with Professor Sommen (Ghent/Belgium) and Professor Shapiro (Mexico-City/Mexico). We thank Professor Jank (Aachen/Germany) and the editor Dr. Hempfling (Birkhäuser) for improving the typescript. Critical remarks by the referees of a first version of the book were of great value for us. M. Sprößig and T. Lahmer helped us by very carefully long work for this book.

Weimar, Aachen and Freiberg, August 2005

Klaus Gürlebeck, Klaus Habetha and Wolfgang Sprößig

Preface to the English Edition

We thank the publisher for the opportunity to translate this book into English. Of course we have corrected all mistakes we found in the German edition, other changes have been made only rarely.

We thank in particular Professor E. Venturino (Torino/Italy) who translated one chapter of the book, but also improved the translation of the rest of the book. Some sections have been translated by M. Schneider and A. Schlichting who are students in Freiberg/Germany.

Weimar, Aachen, Freiberg, May 2007

Klaus Gürlebeck, Klaus Habetha, Wolfgang Sprößig

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Chapter I

Numbers

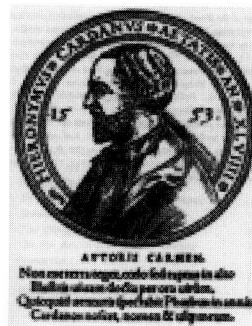
1 Complex numbers

1.1 The History of Their Discovery

The European Renaissance as rebirth of humanistic thought began in the middle of the fourteenth century in Italy and had its maximal development in the sixteenth century, in which with some delay the ideas of renewal of natural sciences and mathematics were accepted. In particular, algebra experienced a push forward, but also important results came from mechanics, astronomy, trigonometry and geometry. A few learned men began to build on the mathematical results of the ancient scientists. Thus the works of ARCHIMEDES, APOLLONIUS, EUCLID and HERON OF ALEXANDRIA became of remarkable interest. It was discovered that in the year 50 A.D. already HERON in his book *Stereometria* had described roots of negative numbers.

The main influence on the algebraically oriented mathematics of this time came from the Arabic world. Rules for finding roots of negative numbers were described for the first time in 499 in the principal work *Āryabhaṭīyā* of ĀRYABHAṬA THE OLDER. Through the method of “completion of the square”, the Babylonians were able to solve quadratic equations with positive coefficients.

The algebraists of the Renaissance had two principal tasks: to extend the number system in order to understand the meaning of negative and complex numbers, and to develop an efficient mathematical symbolism.



Girolamo Cardano

With the exception of DIOPHANTUS OF ALEXANDRIA, the ancient Greek and Arabic mathematicians were accustomed to describing mathematical situations by rhetorical means. With his principal work, printed in 1545 in Nürnberg *Artis magnae sive de regulis algebraicis* or simply *Ars magna*, GIROLAMO CARDANO (1501–1576) established the foundations of modern mathematics. In his book he not only showed how to solve cubic equations, but using roots of negative numbers he gave also the formulas for solving algebraic equations of fourth order. CARDANO himself is seen as personification of the Renaissance. In a singular way he was able to bridge the Middle Age and modern perspectives. On the one hand he was an adept and active practitioner of occultism and of natural magic. He authored general treatises on palmistry and the meaning of dreams, and he wrote about spirits, angels, and deamons. On the other hand, his researches are completely free from mystical and supernatural influxes. The solution methods that he made public were based on firmly established results of his predecessors.

These methods date back to SCIPIONE DAL FERRO (1465–1526) and NICCOLÒ FONTANA (“TARTAGLIA”) (1500–1557). Although he swore to TARTAGLIA, who gave him the “secret knowledge” for solving algebraic equations of third and fourth order, not to publish these new methods without TARTAGLIA’s consent, CARDANO considered it his duty to make them known to the mathematical world of that time, indicating their discoverer. The story of the *Ars magna* is indeed a fascinating historical episode.



Niccolò Fontana

A new stage began with RENÉ DESCARTES (1596–1650). He introduced in 1637 the terms *real* and *imaginary* and with his works in general tried to popularize these new concepts. ALBERT GIRARD (1595–1632) gave geometrical interpretations of roots of negative numbers. Also references to imaginary numbers are found in the writings of the German universal scientist GOTTFRIED WILHELM LEIBNIZ (1646–1716). He writes in 1675 that an imaginary number is “a wonderful creature of an ideal world, of amphibic nature between the things which are and those which are not.” Later, general contributions were made by JOHN WALLIS (1616–1703), ROGER COTES (1682–1716) and above all by LEONHARD EULER (1707–1783), who in 1777 introduced also the symbol $i = \sqrt{-1}$, shortly after having discovered the relation

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

and from it the astonishing result:

$$e^{i\pi} + 1 = 0.$$

The concept *complex number* was introduced only in 1832 by CARL FRIEDRICH GAUSS. Since then, for a complex number z the notation $z = x + iy$ became standard. The introduction of complex numbers as pairs of real numbers dates back to WILLIAM ROWAN HAMILTON (1837).

1.2 Definition and Properties

The complex numbers can be defined in several different ways: we try to remain consistent through the following chapters and begin by defining an extended field of the real numbers $\mathbb{R}$. We have then to establish a set and corresponding relationships.

Definition 1.1 (Field of the complex numbers). Let the set $\mathbb{C} := \{z : z = (x, y), x, y \in \mathbb{R}\}$ be the set of ordered pairs of real numbers. The numbers x and y are called the *coordinates* of z .

Componentwise *Addition* is defined as

$$z_1 + z_2 := (x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

for

$$z_i = (x_i, y_i) \in \mathbb{C},$$

and multiplication with a real number a as

$$a(x, y) := (ax, ay).$$

Multiplication is defined as the linear continuation of the multiplication of the basis elements $\mathbf{1}$ and i , where $\mathbf{1} := (1, 0)$ represents the multiplicative unit element and $i := (0, 1)$ satisfies the rule $i^2 = -1$.

Some remarks: multiplication with real numbers allows the usual representation for the *complex numbers*:

$$z = (x, y) = 1x + iy = x + iy.$$

Thus in place of $x + i0$ we can simply write x , and in place of $0 + iy$ simply iy , in particular in place of $\mathbf{1}$ only 1 (or nothing at all). The continuation of multiplication of the basis elements is carried out through formal expansion of the product

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2),$$

where we apply $i^2 = -1$.

The pair $(0, 0)$ is the neutral element of addition. To prove that addition and multiplication in $\mathbb{C}$ are associative and distributive is a somewhat tiresome exercise, while commutativity follows immediately from the definition. All nonzero elements have multiplicative inverses. Thus $\mathbb{C}$ is a field, the *field of complex numbers*.

If we identify the complex numbers $x + i0$ with the real numbers, then addition and multiplication in $\mathbb{C}$ correspond to those in $\mathbb{R}$, so that $\mathbb{C}$ is an extension field of $\mathbb{R}$. Also the above defined multiplication by a real number, which leads to a vector space structure, is encompassed by the multiplication in $\mathbb{C}$.

The complex numbers $x + iy$ clearly correspond in a one-to-one manner to the vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ of the vector space $\mathbb{R}^2$, however vectors cannot be multiplied as complex numbers, so that the structure of $\mathbb{C}$ is richer.

In higher dimensions the basis elements are frequently denoted by e_0, e_1 and so on: in the plane this is not yet necessary, as here the use of summation signs does not introduce any simplification.

Definition 1.2. $x := \operatorname{Re} z$ is called the *real part* and $y := \operatorname{Im} z$ the *imaginary part* of the complex number $z = (x, y)$. The number $\bar{z} := x - iy$ is called the *complex conjugate* of z . The expression $|z| := \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ will be denoted as *modulus* or *absolute value* of z .

The modulus of a complex number is obviously equal to the Euclidean norm of the corresponding two-dimensional vector.

Proposition 1.3. *Let $z, z_1, z_2 \in \mathbb{C}$ be given. Then the following relations hold:*

- (i) $\operatorname{Re} z = \frac{z + \bar{z}}{2},$ (ii) $\operatorname{Im} z = \frac{z - \bar{z}}{2i},$
- (iii) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}, z \neq 0,$ (iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,$
- (v) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2,$ (vi) $\bar{\bar{z}} = z,$
- (vii) $|z_1 z_2| = |z_1| |z_2|,$ in particular for all $n \in \mathbb{Z}$ (viii) $|z^n| = |z|^n,$
- (ix) $|\bar{z}| = |-z| = |z|.$

Proof. The relations (i)–(vi) and (ix) require only simple calculations. For (vii) write $|z_1 z_2|^2 = \bar{z}_1 \bar{z}_2 z_1 z_2 = \bar{z}_1 z_1 \bar{z}_2 z_2 = |z_1|^2 |z_2|^2$. $\square$

This proposition shows that conjugation is exchangeable with addition and multiplication in $\mathbb{C}$: (iv), (v) and (vi) mean that conjugation is an *involution*. Thus in physics and operator theory frequently the notation z^* is used in place of $\bar{z}$. The reader should convince him/herself that conjugation is the only automorphism in the algebra $\mathbb{C}$ apart from the identity. An example follows:

Example 1.4. If we want to split a fraction of complex numbers in its real and imaginary parts, we have to expand it with the complex conjugate part of the denominator as follows:

$$\frac{i+3}{2i-4} = \left(\frac{i+3}{2i-4} \right) \left(\frac{-2i-4}{-2i-4} \right) = \frac{-10-10i}{20} = -\frac{1}{2} - \frac{i}{2}.$$

Remark 1.5. Proposition 1.3 contains the following statements: a sum of two squares can be written as the product of linear expressions, obviously using the complex unit i :

$$x^2 + y^2 = (x + iy)(x - iy).$$

Moreover, the *two-squares-theorem* holds:

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2,$$

which says that a product of two sums of two squares can be written again as the sum of squares. In higher dimensions an extensive theory has been constructed around this theorem (see [66]).

Now a few inequalities concerning the modulus of a complex number follow:

Proposition 1.6. *For the modulus of a complex number we have*

- (i) $|\operatorname{Re} z| \leq |z|, \quad |\operatorname{Im} z| \leq |z|,$