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THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

Theory of Ordinary Differential Equations

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PREFACE

This book has developed from courses given by the authors and probably contains more material than will ordinarily be covered in a one-year course. The selection of material is partly conditioned by the interests of the authors.

It is hoped that the book will be a useful text in the application of differential equations as well as for the pure mathematician. Prerequisite for this book is a knowledge of matrices and of the essentials of functions of a complex variable. The notion of the Lebesgue integral is used in Chaps. 2, 7, 9, and 10. However, Chap. 2 is needed only for certain parts of Chap. 15, which, so far as applications go, are adequately covered by Chap. 13. The Lebesgue integral can easily be avoided in Chap. 7, as is indicated there. However, a rigorous study of Chaps. 9 and 10 requires a mathematical sophistication that would certainly include the ability to understand the statements of the theorems required from integration theory. An alternative approach is to apply the theory of Chaps. 9 and 10 to a restricted class of functions as is done in the proof of Theorem 3.1 of Chap. 9. This approach requires a knowledge of the Riemann-Stieltjes integral only.

Chapters 3 through 12 are on linear equations. For linear theory, it is not necessary to cover the existence theory of Chap. 1. For Chap. 3, the necessary theorem is sketched in Prob. 1 at the end of that chapter. The discussion in Sec. 7 of Chap. 3 suffices for Chaps. 4 and 5. For Chaps. 7 through 12, Prob. 7 of Chap. 1 provides the additional existence theory needed.

Chapters 4, 5, and 6 are not needed for any later chapters. Chapter 8 is not required for any later chapter, nor are Chaps. 9 and 10. Chapter 8 does not depend on Chap. 7.

Chapter 12 requires only Chap. 7 and, for Sec. 5, also Chap. 11.

Chapters 1 and 3 only are required for Chaps. 13 and 14. Chapter 1 will suffice for most of Chap. 15 and for Chaps. 16 and 17.

No attempt has been made to give the historical origin of the theory, and only a limited number of references are given at the end of the book. In keeping with this approach, the authors make no mention in the text where they present new results.

The problems, in some cases, give additional material not considered in the text.

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CONTENTS

CHAPTER 1

EXISTENCE AND UNIQUENESS OF SOLUTIONS

1. Existence of Solutions

Let I denote an open interval on the real line $-\infty < t < \infty$, that is, the set of all real t satisfying $a < t < b$ for some real constants a and b. The set of all complex-valued functions having k continuous derivatives on I is denoted by $C^k(I)$. If f is a member of this set, one writes $f \varepsilon C^k(I)$, or $f \varepsilon C^k$ on I. The symbol ε is to be read "is a member of" or "belongs to." It is convenient to extend the definition of C^k to intervals I which may not be open. The real intervals $a < t < b$, $a \leqq t \leqq b$, $a \leqq t < b$, and $a < t \leqq b$ will be denoted by (a,b), $[a,b]$, $[a,b)$, and $(a,b]$, respectively. If $f \varepsilon C^k$ on (a,b), and in addition the right-hand kth derivative of f exists at a and is continuous from the right at a, then f is said to be of class C^k on $[a,b)$. Similarly, if the kth derivative is continuous from the left at b, then $f \varepsilon C^k$ on $(a,b]$. If both these conditions hold, one says $f \varepsilon C^k$ on $[a,b]$.

If D is a *domain*, that is, an open connected set, in the real (t,x) plane, the set of all complex-valued functions f defined on D such that all kth-order partial derivatives $\partial^k f/\partial t^p \partial x^q$ $(p + q = k)$ exist and are continuous on D is denoted by $C^k(D)$, and one writes $f \varepsilon C^k(D)$, or $f \varepsilon C^k$ on D.

The sets $C^0(I)$ and $C^0(D)$, the continuous functions on I and D, will be denoted by $C(I)$ and $C(D)$, respectively.

Let D be a domain in the (t,x) plane and suppose f is a real-valued function such that $f \varepsilon C(D)$. Then the central problem of this chapter may be phrased as follows:

Problem. *To find a differentiable function φ defined on a real t interval I such that*

(i) $$(t,\varphi(t)) \varepsilon D \qquad (t \varepsilon I)$$

(ii) $$\varphi'(t) = f(t,\varphi(t)) \qquad \left(t \varepsilon I,\ ' = \frac{d}{dt}\right)$$

This problem is called an *ordinary differential equation of the first order*, and is denoted by

(E) $$x' = f(t,x) \qquad \left(' = \frac{d}{dt}\right)$$

If such an interval I and function φ exist, then φ is called a *solution of the differential equation* (E) *on* I. Clearly if φ is a solution of (E) on I, then $\varphi \in C^1$ on I, on account of (ii).

In geometrical language, (E) prescribes a slope $f(t,x)$ at each point of D. A solution φ on I is a function whose graph [the set of all points $(t,\varphi(t))$, $t \in I$] has the slope $f(t,\varphi(t))$ for each $t \in I$.

The problem (E) may have many solutions on an interval I. For example, the simple equation

$$x' = 1$$

has, for any given real constant c, the solution φ_c given by

$$\varphi_c(t) = t + c$$

on any t interval I. However, there exists only one solution passing through the point (1,1), say, and existing on an interval I containing $t = 1$, namely, φ_0. Therefore, in order to be able to talk about uniqueness of solutions of (E), one is led to the problem of finding a solution passing through a given point in the (t,x) plane.

Suppose (τ,ξ) is a given point in D. Then an *initial-value problem* associated with (E) and this point is defined in the following way:

Initial-value Problem. *To find an interval I containing τ and a solution φ of* (E) *on I satisfying*

$$\varphi(\tau) = \xi$$

This problem is denoted by

$$x' = f(t,x) \qquad x(\tau) = \xi$$

Suppose φ is such a solution which exists on an interval I. Then by integrating (ii) one obtains immediately the integral equation

$$\varphi(t) = \xi + \int_\tau^t f(s,\varphi(s))\,ds \qquad (t \in I)$$

Conversely, suppose $\varphi \in C$ is a function satisfying the above integral equation on I. Then clearly $\varphi(\tau) = \xi$, and by differentiating the equation it follows that φ is a solution of (E) on I. In other words, there is a correspondence between solutions φ of (E) on I satisfying $\varphi(\tau) = \xi$, and continuous functions φ satisfying the above integral relation on I. Thus the initial-value problem for (E) and (τ,ξ) on I is completely equivalent to the finding of all continuous φ on I satisfying the integral equation.

Given a continuous function f on a domain D as above, the first question to be answered is whether there exists a solution of the equation (E). The answer is yes, if I is properly prescribed. An indication of the limitation of any general existence theorem can be seen by considering the simple example

$$x' = x^2$$

It is clear that a solution of this equation which passes through the point $(1,-1)$ is given by $\varphi(t) = -t^{-1}$. However, this solution does not exist at $t = 0$, although $f(t,x) = x^2$ is continuous there. This shows that any general existence theorem will necessarily have to be of a *local* nature, and existence in the large can only be asserted under additional conditions on f.

The local existence proof proceeds by two stages. First, it is shown by an actual construction that there exists an "approximate" solution to (E), in a sense to be made precise below. Then one proves that there exists a sequence of these approximate solutions which tend to a solution of (E).

Let f be a real-valued continuous function on a domain D in the (t,x) plane. An *ϵ-approximate solution* of (E) on a t interval I is a function $\varphi \varepsilon C$ on I such that

(i) $(t,\varphi(t)) \varepsilon D \qquad (t \varepsilon I)$

(ii) $\varphi \varepsilon C^1$ on I, except possibly for a finite set of points S on I, where φ' may have simple discontinuities†

(iii) $|\varphi'(t) - f(t,\varphi(t))| \leqq \epsilon \qquad (t \varepsilon I - S)$

Any function $\varphi \varepsilon C$ satisfying property (ii) on I is said to have a *piecewise continuous derivative* on I, and this is denoted by $\varphi \varepsilon C_p^1(I)$.

If $f \varepsilon C$ on the rectangle

$$R: \qquad |t - \tau| \leqq a \qquad |x - \xi| \leqq b \qquad (a,b > 0)$$

about the point (τ,ξ), it is bounded there. Let

$$M = \max |f(t,x)| \qquad ((t,x) \varepsilon R)$$

and

$$\alpha = \min\left(a, \frac{b}{M}\right)$$

Theorem 1.1. *Let $f \varepsilon C$ on the rectangle R. Given any $\epsilon > 0$, there exists an ϵ-approximate solution φ of* (E) *on $|t - \tau| \leqq \alpha$ such that $\varphi(\tau) = \xi$.*

Proof. Let $\epsilon > 0$ be given. An ϵ-approximate solution will be constructed for the interval $[\tau, \tau + \alpha]$; a similar construction will define it for $[\tau - \alpha, \tau]$. This approximate solution will consist of a polygonal path starting at (τ,ξ), that is, a finite number of straight-line segments joined end to end.

† A function g is said to have a simple discontinuity at a point c if the right and left limits of g at c exist but are not equal. In case $\epsilon = 0$, it will be understood that the set S is empty.

Since $f \varepsilon C$ on R, it is uniformly continuous there, and hence, for the given ϵ, there exists a $\delta_\epsilon > 0$ such that

$$|f(t,x) - f(\bar{t},\bar{x})| \leq \epsilon \tag{1.1}$$

if

$$(t,x) \varepsilon R, \ (\bar{t},\bar{x}) \varepsilon R, \text{ and } |t - \bar{t}| \leq \delta_\epsilon \qquad |x - \bar{x}| \leq \delta_\epsilon$$

Now divide the interval $[\tau, \tau + \alpha]$ into n parts

$$\tau = t_0 < t_1 < \cdots < t_n = \tau + \alpha$$

in such a way that

$$\max |t_k - t_{k-1}| \leq \min \left(\delta_\epsilon, \frac{\delta_\epsilon}{M} \right) \tag{1.2}$$

From (τ,ξ), construct a straight-line segment with slope $f(\tau,\xi)$ proceeding to the right of τ until it intersects the line $t = t_1$ at some point (t_1,x_1).

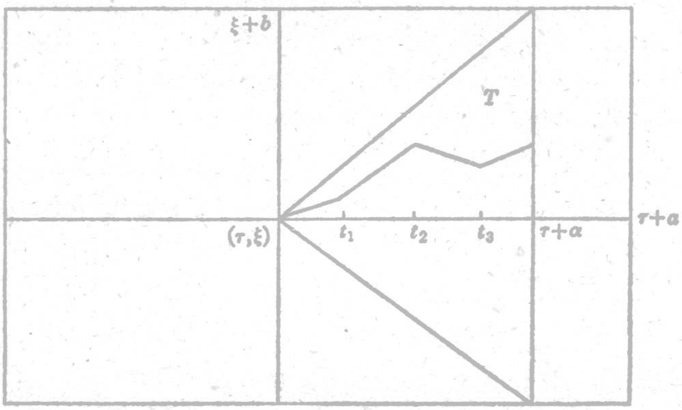

FIG. 1

This segment must lie inside the triangular region T bounded by the lines issuing from (τ,ξ) with slope M and $-M$, and the line $t = \tau + \alpha$ (see Fig. 1, where α is shown as b/M). This follows immediately from the definition of α and the fact that $|f(t,x)| \leq M$. In particular, the constructed segment actually meets the line $t = t_1$ in T. At the point (t_1,x_1) construct to the right of t_1 a straight-line segment with slope $f(t_1,x_1)$ up to the intersection with $t = t_2$, say at (t_2,x_2). Continuing in this fashion, in a finite number of steps the resultant path φ will meet the line $t = \tau + \alpha$. Further, the path will lie completely within T.

This φ is the required ϵ-approximate solution. Analytically it may be expressed as

$$\begin{gathered} \varphi(\tau) = \xi \\ \varphi(t) = \varphi(t_{k-1}) + f(t_{k-1},\varphi(t_{k-1}))(t - t_{k-1}) \\ t_{k-1} < t \leq t_k \qquad k = 1, \ldots, n \end{gathered} \tag{1.3}$$

From the construction of φ it is clear that $\varphi \in C_p^1$ on $[\tau, \tau + \alpha]$, and that

$$|\varphi(t) - \varphi(\bar{t})| \leq M|t - \bar{t}| \qquad (t, \bar{t} \text{ in } [\tau, \tau + \alpha]) \tag{1.4}$$

If t is such that $t_{k-1} < t < t_k$, then (1.4) together with (1.2) imply that $|\varphi(t) - \varphi(t_{k-1})| \leq \delta_\epsilon$. But from (1.3) and (1.1),

$$|\varphi'(t) - f(t,\varphi(t))| = |f(t_{k-1},\varphi(t_{k-1})) - f(t,\varphi(t))| \leq \epsilon$$

This shows that φ is an ϵ-approximate solution, as desired.

The construction of Theorem 1.1 is sometimes used as a practical means for finding an approximate solution. In fact, what has been found is really a set of points $(t_k,\varphi(t_k))$ and these are joined by line segments. The points, by (1.3), satisfy the difference equation

$$x_k - x_{k-1} = (t_k - t_{k-1})f(t_{k-1},x_{k-1})$$

This is a formulation that might be used on a digital computing machine, for example.

The existence of a solution of (E) will now be deduced. For the reader mainly interested in the applications, other existence proofs, under more restricted assumptions on f, are given in Theorems 2.3 and 3.1; the rest of this section can be omitted.

In order to prove the existence of a sequence of approximate solutions tending to a solution of (E), where the only hypothesis is $f \in C$ on R, the notion of an equicontinuous set of functions is required. A set of functions $F = \{f\}$ defined on a real interval I is said to be *equicontinuous* on I if, given any $\epsilon > 0$, there exists a $\delta_\epsilon > 0$, independent of $f \in F$ and also $t, \bar{t} \in I$ such that

$$|f(t) - f(\bar{t})| < \epsilon \qquad \text{whenever } |t - \bar{t}| < \delta_\epsilon$$

The fundamental property of such sets of functions needed here is given in the following lemma:

Lemma (Ascoli). *On a bounded interval I, let $F = \{f\}$ be an infinite, uniformly bounded, equicontinuous set of functions. Then F contains a sequence $\{f_n\}$, $n = 1, 2, \ldots$, which is uniformly convergent on I.*

Proof. Let $\{r_k\}$, $k = 1, 2, \ldots$, be the rational numbers in I enumerated in some order. The set of numbers $\{f(r_1)\}$, $f \in F$, is bounded, and hence there exists a sequence of distinct functions $\{f_{n1}\}$, $f_{n1} \in F$, such that the sequence $\{f_{n1}(r_1)\}$ is convergent. Similarly, the set of numbers $\{f_{n1}(r_2)\}$ has a convergent subsequence $\{f_{n2}(r_2)\}$. Continuing in this way, an infinite set of functions $f_{nk} \in F$, $n, k = 1, 2, \ldots$, is obtained which have the property that $\{f_{nk}\}$ converges at $r_1, \ldots, r_k$. Define f_n to be the function f_{nn}. Then $\{f_n\}$ is the required sequence which is uniformly convergent on I.

Clearly $\{f_n\}$ converges at each of the rationals on I. Thus, given any

$\epsilon > 0$ and rational number $r_k \varepsilon I$, there exists an integer $N_\epsilon(r_k)$ such that

$$|f_n(r_k) - f_m(r_k)| < \epsilon \qquad (n,m > N_\epsilon(r_k))$$

For the given ϵ there exists a δ_ϵ, independent of $t,\bar{t}$ and $f \varepsilon F$ such that

$$|f(t) - f(\bar{t})| < \epsilon \qquad |t - \bar{t}| < \delta_\epsilon$$

Divide the interval I into a finite number of subintervals $I_1, \ldots, I_p$ such that the length of the largest subinterval is less than δ_ϵ. For each I_k choose a rational number $\tilde{r}_k \varepsilon I_k$. If $t \varepsilon I$, then t is in some I_k, and hence

$$|f_n(t) - f_m(t)| \leqq |f_n(t) - f_n(\tilde{r}_k)| + |f_n(\tilde{r}_k) - f_m(\tilde{r}_k)| + |f_m(\tilde{r}_k) - f_m(t)| < 3\epsilon$$

provided that $n,m > \max (N_\epsilon(\tilde{r}_1), \ldots, N_\epsilon(\tilde{r}_p))$. This proves the uniform convergence of the sequence $\{f_n\}$ on I.

Theorem 1.2 (Cauchy-Peano Existence Theorem). *If $f \varepsilon C$ on the rectangle R, then there exists a solution $\varphi \varepsilon C^1$ of* (E) *on $|t - \tau| \leqq \alpha$ for which $\varphi(\tau) = \xi$.*

Proof. Let $\{\epsilon_n\}$, $n = 1, 2, \ldots$, be a monotone decreasing sequence of positive real numbers tending to zero as $n \to \infty$. By Theorem 1.1, for each ϵ_n there exists an ϵ_n-approximate solution, φ_n, of (E) on $|t - \tau| \leqq \alpha$ such that $\varphi_n(\tau) = \xi$. Choose one such solution φ_n for each ϵ_n. From (1.4) it follows that

$$|\varphi_n(t) - \varphi_n(\bar{t})| \leqq M|t - \bar{t}| \tag{1.5}$$

Applying (1.5) to $\bar{t} = \tau$, one readily sees, since $|t - \tau| \leqq b/M$, that the sequence $\{\varphi_n\}$ is uniformly bounded by $|\xi| + b$. Moreover, (1.5) implies that $\{\varphi_n\}$ is an equicontinuous set. By the Ascoli lemma, there exists a subsequence $\{\varphi_{n_k}\}$, $k = 1, 2, \ldots$, of $\{\varphi_n\}$, converging uniformly on $[\tau - \alpha, \tau + \alpha]$ to a limit function φ, which must be continuous since each φ_n is continuous. [Indeed, it follows from (1.5) that $|\varphi(t) - \varphi(\bar{t})| \leqq M|t - \bar{t}|$.]

This limit function φ is a solution of (E) which meets the required specifications. To see this, one writes the relation defining φ_n as an ϵ_n-approximate solution in an integral form, as follows:

$$\varphi_n(t) = \xi + \int_\tau^t (f(s,\varphi_n(s)) + \Delta_n(s))\, ds \tag{1.6}$$

where $\Delta_n(t) = \varphi_n'(t) - f(t,\varphi_n(t))$ at those points where φ_n' exists, and $\Delta_n(t) = 0$ otherwise. Because φ_n is an ϵ_n-approximate solution, $|\Delta_n(t)| \leqq \epsilon_n$. Since f is uniformly continuous on R, and $\varphi_{n_k} \to \varphi$ uniformly on

$[\tau - \alpha, \tau + \alpha]$, as $k \to \infty$, it follows that $f(t,\varphi_{n_k}(t)) \to f(t,\varphi(t))$ uniformly on $[\tau - \alpha, \tau + \alpha]$, as $k \to \infty$. Replacing n by n_k in (1.6) one obtains, in letting $k \to \infty$,

$$\varphi(t) = \xi + \int_\tau^t f(s,\varphi(s))\, ds \tag{1.7}$$

But from (1.7), $\varphi(\tau) = \xi$, and, upon differentiation, $\varphi'(t) = f(t,\varphi(t))$, for $f(t,\varphi(t))$ is a continuous function. It is clear from this that φ is a solution of (E) on $|t - \tau| \leqq \alpha$ of class C^1.

In general, the choice of a subsequence of $\{\varphi_n\}$ in the above proof is necessary, for there exist polygonal paths $\{\varphi_n\}$ which diverge everywhere on a whole interval about $t = \tau$ as $\epsilon_n \to 0$; see Prob. 12.

If it is assumed that a solution of (E) through (τ,ξ) (if it exists) is unique, then *every* sequence of polygonal paths $\{\varphi_n\}$ for which $\epsilon_n \to 0$ must converge on $|t - \tau| \leqq \alpha$, and hence uniformly, to a solution, for $\{\varphi_n\}$ is an equicontinuous set on $|t - \tau| \leqq \alpha$. Suppose this were false. Then there would exist a sequence of polygonal paths $\{\varphi_n\}$ divergent at some point on $|t - \tau| \leqq \alpha$. This implies the existence of at least two subsequences of $\{\varphi_n\}$ tending to different limit functions. Both will be solutions, and this gives a contradiction. Therefore, if uniqueness is assured, the choice of a subsequence in Theorem 1.2 is unnecessary.

It can happen that the choice of a subsequence is unnecessary even though uniqueness is not satisfied. The example

$$x' = x^{1/3} \tag{1.8}$$

illustrates this. There are an *infinite* number of solutions starting at (0,0) which exist on [0,1]. For any c, $0 \leqq c \leqq 1$, the function φ_c defined by

$$\begin{aligned} \varphi_c(t) &= 0 \qquad (0 \leqq t \leqq c) \\ \varphi_c(t) &= \left(\frac{2(t - c)}{3}\right)^{3/2} \qquad (c < t \leqq 1) \end{aligned} \tag{1.9}$$

is a solution of (1.8) on [0,1]. If the construction of Theorem 1.1 is applied to Eq. (1.8), one finds that the only polygonal path starting at the point (0,0) is φ_1. This shows that this method cannot, in general, give *all* solutions of (E).

Theorem 1.3. *Let $f \in C$ on a domain D in the (t,x) plane, and suppose (τ,ξ) is any point in D. Then there exists a solution φ of* (E) *on some t interval containing τ in its interior.*

Proof. Since D is open, there exists an $r > 0$ such that all points, whose distance from (τ,ξ) is less than r, are contained in D. Let R be any closed rectangle containing (τ,ξ), and contained in this open circle of radius r. Then Theorem 1.2 applied to (E) on R gives the required result.