Topics in Graph Automorphisms and Reconstruction

Josef Lauri and Raffaele Scapellato

London Mathematical Society
Student Texts **54**

Topics in Graph Automorphisms and Reconstruction

JOSEF LAURI
University of Malta

RAFFAELE SCAPELLATO
Politecnico di Milano

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2003

Printed in the United States of America

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

Lauri, Josef, 1955–
Topics in graph automorphisms and reconstruction / Josef Lauri, Raffaele Scapellato.
p. cm. – (London Mathematical Society student texts)
Includes bibliographical references and index.
ISBN 0-521-82151-7 – ISBN 0-521-52903-4 (pbk.)
1. Graph theory. 2. Automorphisms. 3. Reconstruction (Graph theory) I. Scapellato, Raffaele, 1955– . II. Title. III. Series

QA166 .L39 2002
511′.5–dc21 2002073691

ISBN 0 521 82151 7 hardback
ISBN 0 521 52903 4 paperback

LONDON MATHEMATICAL SOCIETY STUDENT TEXTS

Managing editor: Professor W. Bruce, Department of Mathematics
University of Liverpool, United Kingdom

3 *Local fields,* J. W. S. CASSELS
4 *An introduction to twister theory, second edition,* S. A. HUGGET & K. P. TOD
5 *Introduction to general relativity,* L. P. HUGHSTON & K. P. TOD
7 *The theory of evolution and dynamical systems,* J. HOFBAUER & K. SIGMUND
8 *Summing and nuclear norms in Banach space theory,* G. J. O. JAMESON
9 *Automorphisms of surfaces after Nielson and Thurston,* A. CASSON & S. BLEILER
11 *Spacetime and singularities,* G. NABER
12 *Undergraduate algebraic geometry,* M. REID
13 *An introduction to Hankel operators,* J. R. PARTINGTON
15 *Presentations of groups, second edition,* D. L. JOHNSON
17 *Aspects of quantum field theory in curved spacetime,* S. A. FULLING
18 *Braids and coverings: Selected topics,* V. LUNDSGAARD HANSEN
19 Steps in commutative algebra, R. Y. SHARP
20 *Communication theory,* C. M. GOLDIE & R. G. E. PINCH
21 *Representations of finite groups of Lie type,* F. DIGNE & J. MICHEL
22 *Designs, graphs, codes, and their links,* P. J. CAMERON & J. H. VAN LINT
23 *Complex algebraic curves,* F. KIRWAN
24 *Lectures on elliptic curves,* J. W. S. CASSELS
25 *Hyperbolic geometry,* B. IVERSEN
26 *An introduction to the theory of L-functions and Eisenstein series,* H. HIDA
27 *Hilbert space: Compact operators and the trace theorem,* J. R. RETHERFORD
28 *Potential theory in the complex lane,* T. RANSFORD
29 *Undergraduate cummutative algebra,* M. REID
31 *The Laplacian on a Riemannian manifold,* S. ROSENBERG
32 *Lectures on Lie groups and Lie algebras,* R. CARTER, G. SEGAL & I. MACDONALD
33 *A primer of algebraic* D-*modules,* S. C. COUTINHO
34 *Complex algebraic surfaces,* A. BEAUVILLE
35 *Young tableaux,* W. FULTON
37 *A mathematical introduction to wavelets,* P. WOJTASZCZYK
38 *Harmonic maps, loop groups and integrable systems,* M. GUEST
39 *Set theory for the working mathematician,* K. CIESIELSKI
40 *Ergodic theory and dynamical systems,* M. POLLICOTT & M. YURI
41 *The algorithmic resolution of diophantine equations,* N. P. SMART
42 *Equilibrium states in ergodic theory,* G. KELLER
43 *Fourier analysis on finite groups and applications,* A. TERRAS
44 *Classical invariant theory,* P. J. OLVER
45 *Permutation groups,* P. J. CAMERON
46 *Riemann surfaces: A primer,* A. BEARDON
47 *Introductory lectures on rings and modules,* J. BEACHY
48 *Set theory,* A. HAJNÁL & P. HAMBURGER
49 *An introduction to* K-*theory for* C*-*algebras,* M. RØRDAM, F. LARSEN & N. LAUSTSEN
50 *A brief guide to algebraic number theory,* H. P. F. SWINNERTON-DYER
51 *Steps in commutative algebra,* R. Y. SHARP
52 *Finite Markov chains and algorithmic applications,* O. HÄGGSTRÖM
53 *The prime number theorem,* G. J. O. JAMESON
54 *Topics in graph automorphisms and reconstruction,* J. LAURI & R. SCAPELLATO

Lil
Mary Anne,
Christina, Beppe u Sandrina

A
Fiorella

Preface

This book arose out of lectures given by the first author to Masters students at the University of Malta and by the second author at the Università Cattolica di Brescia.

This book is not intended to be an exhaustive coverage of graph theory. There are many excellent texts that do this, some of which are mentioned in the Bibliography. Rather, the intention is to provide the reader with a more in-depth coverage of some particular areas of graph theory. The choice of these areas has been largely governed by the research interests of the authors, and the flavour of the topics covered is predominantly algebraic, with emphasis on symmetry properties of graphs. Thus, standard topics such as the automorphism group of a graph, Frucht's Theorem, Cayley graphs and coset graphs, and orbital graphs are presented early on because they provide the background for most of the work presented in later chapters. Here, more specialised topics are tackled, such as graphical regular representations, pseudosimilarity, graph products, Hamiltonicity of Cayley graphs and special types of vertex-transitive graphs, including a brief treatment of the difficult topic of classifying vertex-transitive graphs. The last four chapters are devoted to the Reconstruction Problem, and even here greater emphasis is given to those results that are of a more algebraic character and involve the symmetry of graphs. A special chapter is devoted to graph products. Such operations are often used to provide new examples from existing ones but are seldom studied for their intrinsic value.

Throughout we have tried to present results and proofs, many of which are not usually found in textbooks but have to be looked for in journal papers. Also, we have tried, where possible, to give a treatment of some of these topics that is different from the standard published material (for example, the chapter on graph products and much of the work on reconstruction).

Although the prerequisites for reading this book are quite modest (exposure to a first course in graph theory and some discrete mathematics, and elementary knowledge about permutation groups and some linear algebra) it was our intention when preparing the book that a student who has mastered its contents would be in a good position to understand the current state of research in most of the specialised topics covered, would be able to read with profit journal papers in these areas, and would hopefully have his or her interest sufficiently aroused to consider carrying out research in one of these areas of graph theory.

We would finally like to thank Professor Caroline Series for showing an interest in the book when it was still in an early draft form and the staff at Cambridge University Press for their help and encouragement, especially Roger Astley, Senior Editor, Mathematical Sciences, and, for technical help with LaTeX, Alison Woollatt who, with a short style file, solved problems that would have baffled us for ages. Thanks are also due to Elise Oranges who edited the book thoroughly and pointed out several corrections.

The first author would also like to thank the Academic Work Resources Fund Committee and the Computing Services Centre of the University of Malta, the first for some financial help while writing this book and the second for technical assistance. He also thanks his M.Sc. students at the University of Malta who worked through draft chapters of the book and whose comments and criticism helped to improve the final product.

Josef Lauri
Raffaele Scapellato

Contents

1

Graphs and Groups: Preliminaries

1.1 Graphs and digraphs

In these chapters a *graph* $G = (V(G), E(G))$ will consist of two disjoint sets: a nonempty set $V = V(G)$ whose elements will be called *vertices* and a set $E = E(G)$ whose elements, called *edges*, will be unordered pairs of distinct elements of V. Unless explicitly stated otherwise, the set of vertices will always be finite. An edge $\{u, v\}, u, v \in V$, is also denoted by uv. Sometimes E is allowed to be a multiset, that is, the same edge can be repeated more than once in E. Such edges are called *multiple edges*. Also, edges uu consisting of a pair of repeated vertices are sometimes allowed; such edges are called *loops*. But unless otherwise stated, it will always be assumed that a graph does not have loops or multiple edges. The *complement* of the graph G, denoted by $\overline{G}$, has the same vertex-set as G, but two distinct vertices are adjacent in the complement if and only if they are not adjacent in G. In the chapter on graph products we shall need a modified version of this definition: here, the complement of G, denoted by G^c, also contains a loop $\{v, v\}$ whenever $\{v, v\}$ is not a loop in G.

The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges in $E(G)$ to which v belongs. A vertex of degree k is sometimes said to be a *k-vertex*. Two vertices belonging to the same edge are said to be *adjacent*, while a vertex and an edge to which it belongs are said to be *incident*. A loop incident to a vertex v contributes a value of 2 to $\deg(v)$. A graph is said to be *regular* if all of its vertices have the same degree. A regular graph with degree equal to 3 is sometimes called *cubic*. The minimum and maximum degrees of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

If S is a set of vertices of a graph G, then $G - S$ will denote that graph obtained by removing S from $V(G)$ and removing from $E(G)$ all edges incident to some vertex in S. If F is a set of edges of G, then $G - F$ will denote that graph whose vertex-set is $V(G)$ and whose edge-set is $E(G) - F$. If $S = \{u\}$ and $F = \{e\}$, we shall, for short, denote $G - S$ and $G - F$ by $G - u$ and $G - e$, respectively.

If S is a subset of the vertices of G, then $G[S]$ will denote the subgraph of G *induced* by S, that is, the subgraph consisting of the vertices in S and all of the edges joining pairs of vertices from S.

In general, given any two sets A, B, then $A - B$ will denote the set consisting of all of the elements that are in A but not in B. Also, a set containing k elements is often said to be a *k-set.*

An important modification of the above definition of a graph gives what is called a *directed graph*, or *digraph* for short. In a digraph $D = (V(D), A(D))$ the set $A = A(D)$ consists of ordered pairs of vertices from $V = V(D)$ and its elements are called *arcs.* Again, an arc (u, v) is sometimes denoted by uv when it is clear from the context whether we are referring to an arc or an edge. The arc uv is said to be *incident to* v and *incident from* u; the vertex u is said to be *adjacent to* v whereas v is *adjacent from* u. The number of arcs incident from a vertex v is called its *out-degree*, denoted by $\deg_{\text{out}}(v)$, while the number of arcs incident to v is called its *in-degree* and is denoted by $\deg_{\text{in}}(v)$. A digraph is said to be *regular* if all of its vertices have the same out-degree or, equivalently, the same in-degree.

The number of vertices of a graph G or digraph D is called its *order* and is generally denoted by $n = n(G)$ or $n(D)$, while the number of edges or arcs is called its *size* and is denoted by $m = m(G)$ or $m(D)$.

A sequence of distinct vertices $v_1, v_2, \ldots, v_{k+1}$ and edges $e_1, e_2, \ldots, e_k$ such that each edge $e_i = v_iv_{i+1}$ is called a *path.* If we allow v_1 and v_{k+1} to be the same vertex, then we get what is called a *cycle.*

The *length* of a path or a cycle in G is the number of edges in the path or cycle. A path of length k is denoted by P_k while a cycle of length k is denoted by C_k. The *distance* between two vertices u, v in a connected graph G, denoted by $d(u, v)$, is the length of a shortest path joining u and v. The *diameter* of G is the length of a longest path in G and the *girth* is the length of a shortest cycle.

In these definitions, if we are dealing with a digraph and the $e_i = v_iv_{i+1}$ are arcs, then the path or cycle is called a *directed path* or *directed cycle*, respectively.

Given a digraph D, the *underlying graph* of D is the graph obtained from D by considering each pair in $A(D)$ to be an unordered pair. Given a graph G, the digraph $\overleftrightarrow{G}$ is obtained from G by replacing each edge in $E(G)$ by a pair of oppositely directed arcs.

We adopt the usual convention of representing graphs and digraphs by diagrams in which each vertex is shown by a dot, each edge by a curve joining the corresponding pair of dots and each arc (u, v) by a curve with an arrowhead pointing in the direction from u to v.

A number of definitions on graphs and digraphs will be given as they are required. However, several standard graph theoretic terms will be used but not defined in these chapters; these can be found in any of the references [194] or [196].

1.2 Groups

A *permutation group* will be a pair (Γ, Y) where Y is a finite set and Γ is a subgroup of the symmetric group S_Y, that is, the group of all permutations of Y. The stabiliser of an element $y \in Y$ under the action of Γ is denoted by Γ_y while the orbit of y is denoted by $\Gamma(y)$. The *Orbit-Stabiliser Theorem* states that, for any element $y \in Y$,

$$|\Gamma| = |\Gamma(y)| \cdot |\Gamma_y|.$$

If the elements of Y are all in one orbit, then (Γ, Y) is said to be a *transitive permutation group* and Γ is said to act *transitively* on Y. The permutation group Γ is said to act *regularly* on Y if it acts transitively and the stabiliser of any element of Y is trivial. By the Orbit-Stabiliser Theorem, this is equivalent to saying that Γ acts transitively on Y and $|\Gamma| = |Y|$. Also, Γ acts regularly on Y is equivalent to saying that, for any $y_1, y_2 \in Y$, there exists exactly one $\alpha \in \Gamma$ such that $\alpha(y_1) = y_2$.

One important regular action of a permutation group arises as follows. Let Γ be any group, let $Y = \Gamma$ and, for any $\alpha \in \Gamma$, let λ_α be the permutation of Y defined by $\lambda_\alpha(\beta) = \alpha\beta$. Let $L(\Gamma)$ be the set of all permutations λ_α for all $\alpha \in \Gamma$. Then $(L(\Gamma), Y)$ defines a permutation group acting regularly on Y. This is called the *left regular representation* of the group Γ on itself. One can similarly consider the *right regular representation* of the group Γ on itself, and this is denoted by $(R(\Gamma), Y)$.

The following is an important generalisation of the previous definitions. If Γ is a group and $\mathcal{H} \leq \Gamma$, let $Y = \Gamma/\mathcal{H}$ be the set of left cosets of $\mathcal{H} \in \Gamma$. For any $\alpha \in \Gamma$, let $\lambda_\alpha^{\mathcal{H}}$ be a permutation on Y defined by $\lambda_\alpha^{\mathcal{H}}(\beta\mathcal{H}) = \alpha\beta\mathcal{H}$. Let $L^{\mathcal{H}}(\Gamma)$ be the set of all $\lambda_\alpha^{\mathcal{H}}$ for all $\alpha \in \Gamma$. Then $(L^{\mathcal{H}}(\Gamma), Y)$ defines a permutation group that reduces to the left regular representation of Γ if $\mathcal{H} = \{1\}$.

Two permutation groups $(\Gamma_1, Y_1), (\Gamma_2, Y_2)$ are said to be *equivalent*, denoted by $(\Gamma_1, Y_1) \equiv (\Gamma_2, Y_2)$, if there exists a bijective isomorphism $\phi : \Gamma_1 \to \Gamma_2$ and a bijection $f : Y_1 \to Y_2$ such that, for all $y \in Y_1$ and for all $\alpha \in \Gamma_1$,

$$f(\alpha(x)) = \phi(\alpha)(f(x)).$$

In this case we also say that the action of Γ_1 on Y_1 is equivalent to the action of Γ_2 on Y_2, and sometimes we denote this simply by $\Gamma_1 = \Gamma_2$, when the two sets on which the groups are acting is clear from the context.

Note in particular that, if $(\Gamma_1, Y_1) \equiv (\Gamma_2, Y_2)$, then $\Gamma_1 \simeq \Gamma_2$ as abstract groups, $|Y_1| = |Y_2|$ and the cycle structure of the permutations of Γ_1 on Y_1 must be the same as those of Γ_2 on Y_2. However, the converse is not true; that is, Γ_1 and Γ_2 could be isomorphic and the cycle structures of their respective actions could be the same, but (Γ_1, Y_1) might not be equivalent to (Γ_2, Y_2) (see Exercise 7).

Figure 1.1 shows a simple example of two graphs whose automorphism groups are isomorphic as abstract groups but not equivalent as permutation groups.

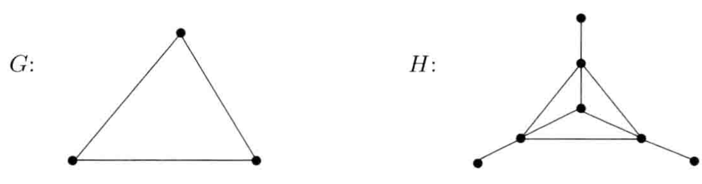

Fig. 1.1. Aut(G), Aut(H) are isomorphic but not equivalent

If (Γ, Y) is a permutation group acting on Y and Y' is a union of orbits of Y, then we can talk about the action of Γ *restricted* to Y', that is, the permutation group (Γ, Y') where, for $\alpha \in \Gamma$ and $y' \in Y'$, $\alpha(y')$ is the same as in (Γ, Y). When Y' is a union of orbits we also say that it is *invariant* under the action of Γ, because in this case $\alpha(y') \in Y'$ for all $\alpha \in \Gamma$ and $y' \in Y'$. Also, (Γ', Y') is said to be a subpermutation group of (Γ, Y) if $\Gamma' \leq \Gamma$ and Y' is a union of orbits of Γ' acting on Y.

The following is a useful well-known result on permutation groups whose proof is not difficult and is left as an exercise.

Theorem 1.1 *Let (Γ, Y) be a permutation group acting transitively on*

Y. Let $y \in Y$, let $\mathcal{H} = \Gamma_y$ be the stabiliser of y and let W be $\Gamma/\mathcal{H}$, the set of left cosets of $\mathcal{H}$ in Γ. Then (Γ, Y) is equivalent to $(L^{\mathcal{H}}(\Gamma), W)$.

If (Γ, Y) is not transitive, and $\mathcal{O}$ is the orbit containing y, then $(L^{\mathcal{H}}(\Gamma), W)$ is equivalent to the action of Γ on Y restricted to $\mathcal{O}$.

In the context of groups and graphs we shall need the very important idea of a group acting on pairs of elements of a set. Thus, let (Γ, Y) be a permutation group acting on the set Y. By $(\Gamma, Y \times Y)$ we shall mean the action on ordered pairs of Y induced by Γ as follows: If $\alpha \in \Gamma$ and $x, y \in Y$, then

$$\alpha((x, y)) = (\alpha(x), \alpha(y)).$$

Similarly, by $(\Gamma, \binom{Y}{2})$ we shall mean the action on unordered pairs of distinct elements of Y induced by

$$\alpha(\{x, y\}) = \{\alpha(x), \alpha(y)\}.$$

These ideas will be developed further in a later chapter.

In later chapters we shall also need the notions of k-transitivity and primitivity of a permutation group. A permutation group (Γ, Y) is said to be *k-transitive* if, given any two k-tuples $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ of distinct elements of Y, then there is an $\alpha \in \Gamma$ such that

$$(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_k)) = (y_1, y_2, \ldots, y_k).$$

Thus, a transitive permutation group is 1-transitive. Also, (Γ, Y) is said to be *k-homogeneous* if, for any two k-subsets A, B of Y, there is an $\alpha \in \Gamma$ such that $\alpha(A) = B$, where $\alpha(A) = \{\alpha(a) : a \in A\}$. Finally, let (Γ, Y) be transitive and suppose that $\mathcal{R}$ is an equivalence relation on Y, and let the equivalence classes of Y under $\mathcal{R}$ be $Y_1, Y_2, \ldots, Y_r$. Then (Γ, Y) is said to be *compatible* with $\mathcal{R}$ if, for any $\alpha \in \Gamma$ and any equivalence class Y_i, the set $\alpha(Y_i)$ is also an equivalence class.

Any permutation group is clearly compatible with the trivial equivalence relations on Y, namely, those in which either all of Y is an equivalence class or when each singleton set is an equivalence class. If these are the only equivalence relations with which (Γ, Y) is compatible, then the permutation group is said to be *primitive*. Otherwise it is *imprimitive*.

If (Γ, Y) is imprimitive and $\mathcal{R}$ is a nontrivial equivalence relation on Y with which the permutation group is compatible, then the equivalence classes of $\mathcal{R}$ are called *imprimitivity blocks* and their set Y/R is an *imprimitivity block system* for the permutation group (Γ, Y).

It is an easy exercise to show that a 2-transitive permutation group is primitive.

We shall also need some elementary ideas on the presentation of a group in terms of of generators and relations.

Let Γ be a group and let $X \subseteq \Gamma$. A *word* in X is a product of a finite number of terms, each of which is an element of X or an inverse of an element of X. The set X is said to *generate* Γ if every element in Γ can be written as a word in X; in this case the elements of X are said to be *generators* of Γ. A *relation* in X is an equality between two words in X. By taking inverses, any relation can be written in the form $w = 1$, where w is some word in X.

If X generates Γ and every relation in Γ can be deduced from one of the relations $w_1 = 1, w_2 = 1, \ldots$ in X, then we write

$$\Gamma = \langle X | w_1 = 1, w_2 = 1, \ldots \rangle.$$

This is called a *presentation* of Γ in terms of generators and relators. The group Γ is said to be *finitely generated* (*finitely related*) if $|X|$ (the number of relations) is finite; it is called *finitely presented*, or we say that it has a *finite presentation*, if it is both finitely generated and finitely related.

It is clear that every finite group has a finite presentation (although the converse is false). Simply take $X = \Gamma$ and, as relations, take all expressions of the form $\alpha_i \alpha_j = \alpha_k$ for all $\alpha_i, \alpha_j \in \Gamma$. In other words, the multiplication table of Γ serves as the defining relations.

It is well to point out that removing relations from a presentation of a group in general gives a larger group, the extreme case being that of the *free group* which has only generators and no relations.

The simplest free group is the infinite cyclic group that has the presentation

$$\langle \alpha \rangle$$

with just one generator and no defining relation, whereas the cyclic group of order n has the presentation

$$\langle \alpha | \alpha^n = 1 \rangle;$$

this group is denoted by $\mathbb{Z}_n$.

The group with presentation

$$\langle \alpha, \beta \rangle$$

is the infinite free group on two elements. The dihedral group of degree

n is denoted by D_n. It has order $2n$ and also has a presentation with two generators:

$$\langle \alpha, \beta | \alpha^2 = 1, \beta^n = 1, \alpha^{-1}\beta\alpha = \beta^{-1} \rangle.$$

The reader is referred to [111, 166] for any terms and concepts on group theory that are used but not defined in these chapters and, in particular, to [35, 46] for more information on permutation groups.

1.3 Graphs and groups

Let G, G' be two graphs. A bijection $\alpha : V(G) \to V(G')$ is called an *isomorphism* if

$$\{u, v\} \in E(G) \Leftrightarrow \{\alpha(u), \alpha(v)\} \in E(G').$$

The graphs G, G' are, in this case, said to be *isomorphic*, and this is denoted by $G \simeq G'$. Similarly, if D, D' are digraphs, then a bijection $\alpha : V(D) \to V(D')$ is called an *isomorphism* if

$$(u, v) \in A(D) \Leftrightarrow (\alpha(u), \alpha(v)) \in A(D'),$$

and in this case the digraphs D, D' are also said to be *isomorphic*, and again this is denoted by $D \simeq D'$.

If the two graphs, or digraphs, in the above definition are equal, then α is said to be an *automorphism* of G or of D. The set of automorphisms of a graph or a digraph is a group under composition of functions, and it is denoted by $\mathrm{Aut}(G)$ or $\mathrm{Aut}(D)$.

Note that an automorphism α of G is an element of $S_{V(G)}$, although it is its induced action on $E(G)$ that determines whether α is an automorphism. This fact, although clear from the definition of automorphism, is worth emphasising especially because of its importance in work that will be done in later chapters.

For example, for the graph in Figure 1.2, the permutation of edges given by $(12 \quad 23 \quad 34)$ is not induced by any permutation of the vertex-set $\{1, 2, 3, 4\}$. The only automorphisms for this graph are the identity and the permutation $(14)(23)$, which induces the permutation $(12 \;\, 34)(23)$ of the edges in the graph.

The question of edge permutations not induced by vertex permutations will be considered in some more detail later in this chapter.

The process of obtaining a permutation group from a digraph can be reversed in a very natural manner. Suppose that (Γ, Y) is a group of