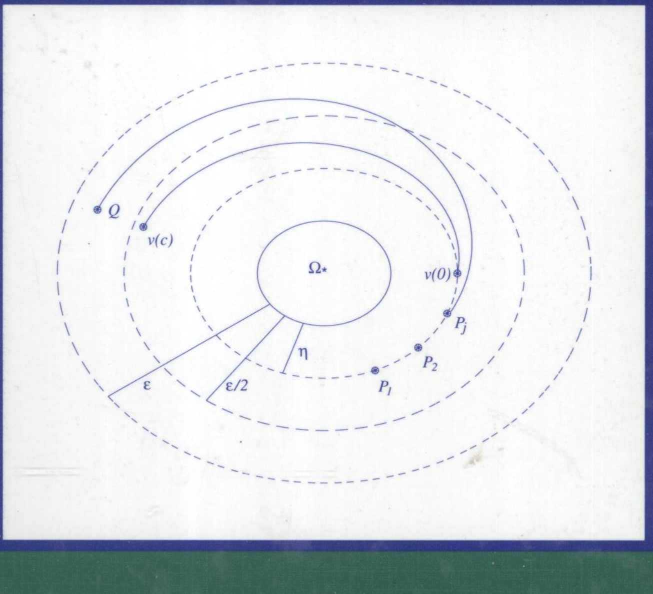
Progress in Nonlinear Differential Equations and Their Applications

Victor A. Galaktionov
Juan Luis Vázquez

A Stability Technique for Evolution Partial Differential Equations

A Dynamical Systems Approach



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Progress in Nonlinear Differential Equations and Their Applications
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This book is dedicated to our wives, Olga and Mariluz, and to our children, Oleg, Isabel, and Miguel.

Introduction: Stability Approach and Nonlinear Models

The S-Theorem

This book contains the description and application of a method of asymptotic analysis, a new stability theorem that we call the *S-Theorem*, originated in the study of the large-time behaviour of a class of partial differential equations known generally as *nonlinear reaction-diffusion equations*. These equations are among the best-known equations of mathematical physics. But, as shown in the text, the method has a more general scope in the study of evolution problems which can be posed in an abstract setting as infinite-dimensional dynamical systems. This is why we often refer to it as a *Dynamical Systems Approach*.

The study of asymptotic behaviour of solutions of evolution equations is a classical subject of mechanics and dynamical systems, and a number of quite effective methods have been developed, such as Lyapunov techniques, stable and centre manifold analysis, scaling and renormalization group arguments, etc. These methods can be used quite successfully to understand the asymptotic properties of many quasilinear reaction-diffusion equations, also known as nonlinear heat equations, in particular, when they admit global-in-time solutions, so that no essential singularities occur in the large-time evolution. In principle, we will not deal with such problems with known global behaviour, and will be concerned with problems that exhibit a complicated structure of asymptotic patterns that makes our analysis necessary or convenient.

The method presented here is suitable for application to different evolution problems described by nonlinear partial differential equations (PDEs) of parabolic or hyperbolic type, involving first-order, second-order or higher-order operators, many of them admitting free boundaries, or for other types of equations or systems. The common feature is that these evolution problems can be formulated as *asymptotically small perturbations* of certain dynamical systems with better-known behaviour. Now, it usually happens that the perturbation is small in a very weak sense, hence the difficulty (or impossibility) of applying more classical techniques.

Though the method originated with the analysis of critical behaviour for evolution PDEs, in its abstract formulation it deals with a nonautonomous abstract differential equation (NDE)

$$u_t = \mathbf{A}(u) + \mathbf{C}(u, t), \quad t > 0, \tag{1}$$

where u has values in a Banach space, like an L^p space, $\mathbf{A}$ is an autonomous (time-independent) operator and $\mathbf{C}$ is an asymptotically small perturbation, so that $\mathbf{C}(u(t), t) \to 0$ as $t \to \infty$ along orbits $\{u(t)\}$ of the evolution in a sense to be made precise, which in practice can be quite weak. We work in a situation in which the autonomous (limit) differential equation (ADE)

$$u_t = \mathbf{A}(u) \tag{2}$$

has a well-known asymptotic behaviour, and we want to prove that for large times the orbits of the original evolution problem converge to a certain class of limits of the autonomous equation.

More precisely, we want to prove that the orbits of (NDE) are attracted by a certain limit set Ω_* of (ADE), which may consist of equilibria of the autonomous equation, or it can be a more complicated object. A set of *three basic requirements* allows this conclusion, the main one being the Lyapunov stability of the closed set Ω_*, and this is the contents of the S-Theorem. It is typical of standard methods that such stability assumptions have to be imposed on the original equation (NDE). An important feature of our method is that it places the stability assumption on the limit equation (ADE). Note also that the convergence result *does not* depend on the knowledge of any rate of decay for the perturbation $\mathbf{C}(u, t)$ as t grows.

In Chapter 1 we state our main stability theorem (S-Theorem, in short). We establish that under three hypotheses (H1)–(H3), the omega-limit set of a perturbed dynamical system is stable under arbitrary asymptotically small perturbation. This result will be used throughout the book. The problem has been formulated above for convenience in the language of differential equations, but actually the S-Theorem is of a more general character, and applies to abstract dynamical systems posed in a complete metric space.

Asymptotics of nonlinear evolution PDEs

The rest of the book is devoted to the study of a selection of nonlinear asymptotic phenomena which occur for classes of equations involving different nonlinear operators. Indeed, the second goal of the book is to contribute a number of techniques and results to the wide field of *asymptotics of nonlinear evolution PDEs*.

The concrete examples of application have been chosen because they are relevant asymptotic problems that attracted the interest of the authors, were not covered by existing theories, and motivated the development of this theory. We present nine main examples, starting with classical reaction-diffusion-convection theory, and go on to cover subjects in blow-up, fluid flows (Navier–Stokes), Hamilton–Jacobi and fully nonlinear equations. We contribute to the theory of such equations, describe some general nonlinear effects and present a classification of the involved singularities.

Indeed, a first motivation of the theory has been the study of typical models of nonlinear diffusion. We devote Chapter 2 to presenting the main equations along with the concepts, tools and typical results on existence, uniqueness and differential properties of weak solutions, that might be useful in setting the context, as a technical preliminary for subsequent chapters. We will in particular examine the known *asymptotic properties* as $t \to \infty$. We demonstrate basic mathematical tools developed in the second half of the twentieth century on a benchmark equation, the *Porous Medium Equation* (PME, in short)

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{3}$$

where $m > 1$ is a fixed exponent. For $m = 1$ it is just the classical *Heat Equation*.

In subsequent chapters, our text contributes to the general theory by supplying a further analysis tool that has allowed the authors to perform a complete asymptotic study in a number of open cases, many of them involving critical situations and striking phenomena of singularity formation. Especially, we will be interested in *blow-up properties*, when solutions become unbounded (in L^∞ or in another natural norm) after a finite time.

Before we proceed with the outline of the applications, let us try to understand in a few words why the study of nonlinear evolution equations or reaction-diffusion type leads to the consideration of *small asymptotic perturbations* of better-known autonomous dynamical systems.

Consider the case of critical diffusion-absorption treated in Chapter 4. It is well known that the solutions of the heat equation $u_t = \Delta u$ and the PME (3) posed in the whole space $\mathbb{R}^N$ with integrable initial data $u_0 \in L^1(\mathbb{R}^N)$, decay as $t \to \infty$ like $O(t^{-\alpha})$ for an exponent α that is shown to be $\alpha = N/[N(m-1)+2]$.

When we want to be more precise we rescale (i.e., we zoom) the variable u into a new variable θ that equals u times the decay factor t^α, hence it has size $O(1)$ for large t. But if we want θ to be a solution of a nice equation we have to also re-scale space in the form $x = \xi\, t^{\alpha/N}$. We are also interested for the same reason in using logarithmic time $\tau = \ln t$. This is all well known using dimensional analysis and exploits the property of scale invariance of the equation, and leads to the rescaled PME for $\theta(\xi, \tau) = t^\alpha\, u(x, t)$:

$$\theta_\tau = \mathbf{A}(\theta) \equiv \Delta\theta^m + \tfrac{\alpha}{N}\, \xi \cdot \nabla\theta + \alpha\, \theta. \tag{4}$$

It is an autonomous equation and its solutions tend to a nontrivial equilibrium, namely, the Gaussian kernel if $m = 1$, and the ZKB profile if $m > 1$. The asymptotic profile of the original problem is now read as the transformation of that equilibrium in terms of u.

Suppose now that you consider the more complicated model equation

$$u_t = \Delta u^m - u^\beta, \tag{5}$$

with $\beta, m \geq 1$. This is a model of nonlinear diffusion in an absorptive medium, well known in the literature. The absorption term is not an asymptotically small perturbation in principle. Now, we happen to know that the decay rate for this equation is the same as before when $\beta > \beta_* = m + 2/N$. If this is so we perform the same type of re-scaling to find

$$\theta_\tau = \mathbf{A}(\theta) + \mathbf{C}(\theta, \tau), \qquad \mathbf{C}(\theta, \tau) = -e^{-\sigma\tau}\theta^\beta, \qquad \sigma = (\beta - \beta_*)\alpha. \tag{6}$$

In this form we arrive at an asymptotically small perturbation of the rescaled PME (4) and the problem falls into the scope of the text. The appearance of the small exponential factor reminds us that we have lost the scale invariance in the original equation (5). Curiously, the most difficult analysis occurs for the critical case $\beta = \beta_*$, where we will concentrate the attention, and is naturally done with the S-Theorem.

Description of the applications

In Chapter 3 we perform a first application of the S-Theorem to study the asymptotic behaviour of nonnegative solutions for the equation of *superslow diffusion* which in N-dimensional geometry takes the form

$$u_t = \Delta(e^{-1/u}). \tag{7}$$

It can be treated as a formal limit case of the PME with $m = \infty$. We separately consider the initial-value problem for $t > 0$ in a bounded domain $\Omega \subset \mathbb{R}^N$ and the Cauchy problem in $\mathbb{R} \times \mathbb{R}_+$. Interesting transformations are needed to present those problems as small asymptotic perturbations of some well-known equation, and this is an important aspect of the theory. It turns out that in these two problems the asymptotic patterns look similar, but the rescaled variables and perturbed equations differ essentially. In the case of the bounded domain the rescaled equation with small asymptotic perturbations is rather involved and is given by

$$\theta_\tau = \mathbf{A}(\theta) + \frac{4\ln\tau}{\tau}\theta\,\Delta\theta + \frac{2}{\tau}\,(\theta - \theta\ln\theta\,\Delta\theta) + \frac{4\ln^2\tau}{\tau^2}\theta\,\Delta\theta - \frac{4\ln\tau}{\tau^2}\theta\ln\theta\,\Delta\theta + \tfrac{1}{\tau^2}\theta\,(\ln\theta)^2\,\Delta\theta,$$

with $\mathbf{A}(\theta) = \theta\,\Delta\theta + \theta$.

In Chapter 4 we describe the asymptotic behaviour of a PME with absorption in the case of a *critical* exponent,

$$u_t = \Delta u^m - u^\beta \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \beta = \beta_* = m + 2/N. \tag{8}$$

The exponent β_* (often called *critical Fujita exponent* for equations with source term $+u^\beta$) has been chosen because it is precisely the case when more standard methods of asymptotic analysis fail. Briefly explained, the difficulty stems from the fact that the two operators on the right-hand side have effects of the same order of magnitude, as can be easily shown by dimensional analysis or scaling. Thus, in the rescaling calculations done above for $\beta > \beta_*$, we see that the perturbation is not small when we pass to the limit $\beta \to \beta_*$. Consequently, the problem exhibits a typical *critical* situation, which is called a *resonance* in physical parlance. One of the main consequences is that the decay rate is modified to include extra *logarithmic factors* (a typical feature of resonance in dynamical systems).

The authors used the S-Theorem in 1991 to prove that all weak, space-integrable solutions behave for $t \to \infty$ as a unique orbit of the PME *without absorption*, and the resonance is felt as a rescaling in u and x by slow-growth unbounded factors, logarithmic functions of time. This is an example of a *transitional* behaviour between two different asymptotic structures for $\beta < \beta_*$ and $\beta > \beta_*$. The behaviour for the critical exponent $\beta = \beta_*$ then inherits certain features of both the subcritical and the supercritical ranges. This kind of transitional behaviour has a quite general nature and occurs for other equations; we will present some other instances of the phenomenon. The paper [169] was the first instance of an application of the "dynamical systems approach with asymptotically small perturbations" developed in this book.

Chapter 5 deals with the asymptotics of a problem involving extinction. *Extinction in finite time* is the term which denotes the phenomenon whereby a positive solution of an evolution process becomes identically zero after a finite time T, $u(\cdot, T) = 0$. The phenomenon is also called complete *quenching*. It is well known that this is not possible for the standard problems associated to the heat equation and other parabolic evolution operators with good coefficients. The phenomenon arises in nonlinear equations due to the presence of terms that either degenerate or are singular at $u = 0$. The extinction of a solution is usually associated with the formation of a singularity for the solution at the level of some derivative. Therefore, it can be understood as blow-up for the derivatives of the solution, with the advantage that the L^∞ norm of the solution itself remains bounded. In this chapter we still consider the PME with absorption, but the presence of a strong absorption term produces extinction. We concentrate on the equation with another *critical* exponent

$$u_t = \Delta u^m - u^p, \quad m > 1, \;\; p = p_* = 2 - m < 1. \tag{9}$$

In this case the singular behaviour close to the extinction time, $t \to T < \infty$, is governed by the ODE without diffusion:

$$u_t = -u^{2-m}.$$

This is the first time that we face the case of *singular perturbation*: the limit equation is of lower order than the original PME with absorption. As is well known from the theory of singular perturbations, the passage to the limit becomes a hard problem. In order to apply the S-Theorem, we need to prove several estimates on rescaled orbits in a metric space C_ρ with a singular weight.

We follow with two chapters where the S-Theorem is used in combination with the technique of *Matched Asymptotic Expansions*. This is a very important tool of asymptotic analysis that is needed to reflect the multiple behaviour of many problems arising in several applied fields, hence our interest in the study that combines both machineries. Chapter 6 is devoted to the study of the fast diffusion equation with *critical* parameter

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad m = m_* = (N-2)/N, \quad N \geq 3. \tag{10}$$

We establish that $m = m_*$ corresponds to the transition between two different types of self-similar asymptotic behaviour in a neighbourhood of the critical value for $m > m_*$ (self-similarity of the *first kind* given by the ZKB solution), and $0 < m < m_*$ (self-similarity of the *second kind*). As a consequence, we describe two different asymptotic domains, the outer and the inner ones, with quite different asymptotic scalings. The leading part of the asymptotics in the outer domain is governed by a radial solution of the first-order equation (the conservation law)

$$v_t + N(v^{(N-2)/N})_s = 0, \quad \text{where } s = \ln|x|,$$

to which the stability theory applies. The inner one has a simple "flat" shape and some parabolic properties are necessary to match both the asymptotics.

Chapter 7 is devoted to the PME in exterior domains. We need to use expansions in the inner and outer regions and a matching procedure (the approach is different from that in Chapter 6). The main feature of the topic is the role played by singular solutions as asymptotic limits in the S-Theorem. We address here the critical situation that appears in dimension two and produces a typical $\ln(t)$ factor in the delicate matching process.

We cover next some topics from *fluid mechanics*. In Chapter 8 we turn to a classical problem and study a singularly perturbed dynamical system which describes some special blow-up patterns of the Navier–Stokes equations in $\mathbb{R}^2$,

$$\begin{cases} u_t + uu_x + vu_y = -p_x/\rho + \nu\Delta u, \\ v_t + uv_x + vv_y = -p_y/\rho + \nu\Delta v, \\ u_x + v_y = 0, \end{cases} \tag{11}$$

where (u, v) is the velocity field, p is the pressure, $\rho > 0$ is the constant density and $\nu > 0$ is the constant kinematic viscosity. We are interested in the particular solutions similar to the famous stationary *von Kármán solution* of the form

$$u = \int_0^x f(z,t)dz, \quad v = -yf(x,t), \quad p = h(x,t).$$

They describe a *plane jet* with a stagnation point at (0, 0) and free boundaries. Then the function f solves a *semilinear nonlocal heat equation*

$$f_t + \left(\int_0^x f(z,t)dz\right) f_x - f^2 = \nu f_{xx}$$

with free boundary conditions. We study the first stable blow-up pattern which gives the asymptotic structure of the plane jet for the Navier–Stokes equations. In particular, we prove that asymptotically this generic blow-up pattern is described by a nonlocal semilinear first-order Hamilton–Jacobi equation

$$f_t + \left(\int_0^x f(z,t)dz\right) f_x - f^2 = 0,$$

so that this asymptotic analysis falls in the scope of a singular perturbation theory.

In Chapter 9 we study a problem of *blow-up*, i.e., the solutions become unbounded in a finite time, and the profile that is formed at this time is under investigation. Blow-up is a major area of research in nonlinear evolution equations, cf. [32, 180, 286]. We consider the semilinear equation with "almost linear" reaction term

$$u_t = u_{xx} + (1+u)\ln^2(1+u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+. \tag{12}$$

The study presents an important aspect, i.e., the asymptotic *degeneracy* of the parabolic equations near blow-up. More concretely, we prove that for bounded bell-shaped initial data $u_0(x) \geq 0$, the asymptotic behaviour as $t \to T$ is described by the nonlinear quadratic *Hamilton–Jacobi equation*

$$u_t = \frac{(u_x)^2}{1+u} + (1+u)\ln^2(1+u),$$

and the S-Theorem makes it possible to pass to the limit in a singularly perturbed dynamical system. Finally we prove that this equation exhibits *regional* blow-up where the *blow-up set* for bell-shaped data has a finite length equal to 2π. We also study periodic blow-up patterns and their localization. This work was developed in the paper [173], written in 1991, and was a major source of inspiration in developing the idea of *reduced omega-limit sets*, an important ingredient in the sharp formulation of the S-Theorem.

In Chapter 10 we present a general theory of such degeneracy effect of convergence to Hamilton–Jacobi solutions. It applies to a class of quasilinear equations with different types of blow-up, such as *single-point*, regional or *global* blow-up. As a basic model, we classify the asymptotics of the quasilinear heat equation

$$u_t = \nabla \cdot (\ln^\sigma(1+u)\nabla u) + (1+u)(\ln(1+u))^{\beta(\sigma+1)-\sigma} \tag{13}$$

for different values of the parameters $\sigma \geq 0$ and $\beta > 1$. It is important that this equation describes all three types of blow-up: (i) regional for $\beta = 2$, (ii) *single-point* for $\beta > 2$ and (iii) *global* if $\beta \in (1, 2)$. The asymptotic blow-up patterns are proved to have different space-time structures in these three cases.

We perform in Chapter 11 the asymptotic analysis of a *fully nonlinear* parabolic equation from detonation theory. The parabolic equation

$$u_t + \tfrac{1}{2}(u_x)^2 = f(cuu_{xx}) + \ln u \quad (c > 0) \tag{14}$$

with a smooth strictly monotone increasing function, $f(s) = \ln((e^s - 1)/s)$, describes unstability of the square Zel'dovich–von Neuman–Doering (ZND) wave in detonation theory. The model is due to Buckmaster and Ludford. We study the finite time *quenching* behaviour as $t \to T$ when an initially strictly positive solution touches the singular level $u = 0$, where the diffusion-like operator degenerates and the absorption term $\ln u$ becomes singular. We establish that this behaviour is described by a singularly perturbed linear first-order equation of Hamilton–Jacobi type. It is important that the solution does not admit any proper continuation beyond quenching time, for $t > T$. This means complete collapse of the ZND-wave at the quenching point.

We add a last Chapter 12, where we briefly describe further, sometimes not very straightforward, extensions and generalizations, and give a list of related references. We show how to extend our dynamical system approach by using an extra topological structure in the metric space and hence modifying the notion of the uniform Lyapunov stability. Under a suitable assumption on the corresponding topological structure of the reduced omega-limit set of the autonomous equation, we then obtain more detailed description of the omega-limits of a class of individual orbits. Another new application is time-dependent *homogenization*-like problems for the PME or other parabolic equations with highly oscillatory coefficients.

We also demonstrate that the S-Theorem exhibits natural applications to a number of problems for *higher-order parabolic equations* with reaction/absorption-like terms, and as typical examples we treat the semilinear $2m^{\text{th}}$-order equations

$$u_t = -(-\Delta)^m u \pm |u|^{p-1} u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ \tag{15}$$

with integer $m > 1$ and exponent $p > 1$, which induce typical examples of semigroups without order-preserving properties (available for $m = 1$ only via the *Maximum Principle*).

Summing up, the nonlinear models described above play the role of key examples in explaining some crucial distinctive features of the applications of the stability theorem (Chapter 1) to a class of similar perturbed dynamical systems. Of course, such an analysis admits various extensions and generalizations to wide classes of problems, where a similar kind of perturbations occurs. We describe such generalizations in Remarks at the end of each chapter.

The equations and problems we deal with were mostly well known and were actively studied from different points of view in the last two decades in the framework of the growing theory of nonlinear partial differential equations, and the questions of (local-in-time) existence, uniqueness and regularity of solutions are documented in the literature. We present suitable references in the final section (remarks and comments on the literature) of each chapter. Though we have selected applications involving nonlinear heat equations, the abstract stability theory, on which the analysis relies, has a wider scope, and some of the examples are directed to promote such extension.

This book presents a unified approach to the study of the asymptotic behaviour of several classes of nonlinear equations. The main results were obtained by the authors during the last twelve years. These classes of asymptotic problems for nonautonomous dynamical systems were not discussed in monographs on the theory of nonlinear PDEs.

Prerequisites and use

The book assumes some knowledge of the fundamentals of partial differential equations, ordinary differential equations, and functional analysis. A certain exposure to dynamical systems will be helpful as background to understand the main result and the general philosophy. The examples of application which form the bulk of the book assume some knowledge of the main topics of nonlinear partial differential equations of evolution type and their asymptotics, e.g., global or local well-posedness and Lyapunov techniques. It is not an absolute prerequisite to read our corresponding introductory text but it explains the context and why the present method is useful. Much of the necessary material on basic theory and asymptotics of nonlinear heat equations is summarized in Chapter 2, where further references are given. More general references are [293] and [286], which deals in great detail with blow-up problems. Explanations, references and hints will be given as the text proceeds.

The book is meant for an advanced graduate level and can be taught to students in mathematics and physics interested in evolution equations and asymptotics in one semester if a proper selection of the topics is made. It can be combined with standard evolution equations and asymptotics topics into a whole year in various ways. The whole text could serve as a reference work on the S-Theorem and its applications.

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