理論力學問題詳解

下 册

CLASSICAL DYNAMICS

OF PARTICLES AND SYSTEMS

SECOND EDITION

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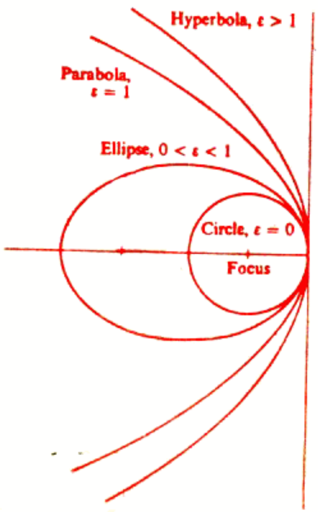

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理論力學題解

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Problems

9-1. In an elastic collision of two particles, with masses m_1 and m_2, the initial velocities are $\mathbf{u}_1$ and $\mathbf{u}_2 = \alpha\mathbf{u}_1$ ($\alpha \neq 0$). If the initial kinetic energies of the two particles are equal, find the conditions on u_1/u_2 and m_1/m_2 so that m_1 will be at rest after the collision.

9-2. Show that

$$\frac{T_1}{T_0} = \frac{m_1^2}{(m_1 + m_2)^2} \cdot S^2$$

where

$$S \equiv \cos\psi + \frac{\cos(\theta - \psi)}{(m_1/m_2)}$$

9-3. Show that T_1/T_0 can be expressed in terms of $m_2/m_1 \equiv \alpha$ and $\cos\psi \equiv y$ as

$$\frac{T_1}{T_0} = (1 + \alpha)^{-2}\left[2y^2 + \alpha^2 - 1 + 2y\sqrt{\alpha^2 + y^2 - 1}\right]$$

Plot T_1/T_0 as a function of ψ for $\alpha = 1, 2, 4$, and 12. These plots correspond to the energies of protons or neutrons after scattering from hydrogen ($\alpha = 1$), deuterium ($\alpha = 2$), helium ($\alpha = 4$), and carbon ($\alpha = 12$), or of alpha particles scattered from helium ($\alpha = 1$), oxygen ($\alpha = 4$), etc.

9-4. A particle of mass m_1 with initial laboratory velocity u_1 collides with a particle of mass m_2 which is at rest in the laboratory system. The particle m_1 is scattered through a laboratory angle ψ and has a final velocity v_1, where $v_1 = v_1(\psi)$. Find the surface such that the time of travel of the scattered particle from the point of collision to the surface is independent of the scattering angle. Consider the cases (a) $m_2 = m_1$, (b) $m_2 = 2m_1$, (c) $m_2 = \infty$. Suggest an application of this result in terms of a detector for nuclear particles.

9-5. Show that the equivalent of Eq. (9.44) expressed in terms of θ, rather than ψ, is

$$\sigma(\theta) = \sigma(\psi) \cdot \frac{1 + x\cos\theta}{(1 + 2x\cos\theta + x^2)^{3/2}}$$

9-6. Calculate the differential cross section $\sigma(\theta)$ and the total cross section σ_t for the elastic scattering of a particle from an impenetrable sphere; i.e., the potential is given by

$$U(r) = \begin{cases} 0, & r > a \\ \infty, & r < a \end{cases}$$

9-7. If, in the previous problem, the energy lost by the scattered particle to the sphere is ε, show that

$$d\sigma_{\text{C.M.}}(\varepsilon) = \frac{\pi a^2}{\varepsilon_{\max}}\, d\varepsilon$$

Thus, show that in the center-of-mass system the energies of the scattered particles are distributed uniformly.

9-8. Show that the Rutherford scattering cross section (for the case $m_1 = m_2$) can be expressed in terms of the recoil angle as

$$\sigma_{\text{lab}}(\zeta) = \frac{k^2}{T_0^2} \cdot \frac{1}{\cos^3 \zeta}$$

9-9. Consider the case of Rutherford scattering in the event that $m_1 \gg m_2$ (i.e., the mass of the incident particle is much greater than that of the target). Obtain an approximate expression for the differential cross section in the *laboratory* coordinate system.

9-10. Consider the case of Rutherford scattering in the event that $m_2 \gg m_1$. Obtain an expression for the differential cross section in the center-of-mass system that is correct to first order in the quantity m_1/m_2. Compare this result with Eq. (9.53).

9-11. A fixed force center scatters a particle of mass m according to the force law $F(r) = k/r^3$. If the initial velocity of the particle is u_0, show that the differential scattering cross section is

$$\sigma(\theta) = \frac{k\pi}{2mu_0^2\theta^2 \sin\theta}$$

The integral of this expression gives an infinite result for the total cross section. However, if the force vanishes for $r > r_0$, show that there is some minimum scattering angle θ_0; express the result in terms of m, u_0, r_0, and k. Then, show that the total cross section is

$$\sigma_t = 2\pi \int_{\theta_0}^{\pi} \sigma(\theta) \sin\theta \, d\theta = \pi r_0^2$$

9-12. It is found experimentally that in the elastic scattering of neutrons by protons ($m_n \cong m_p$) at relatively low energies, the energy distribution of the recoiling protons in the lab system is constant up to a maximum energy which is the energy of the incident neutrons. What is the angular distribution of the scattering in the C.M. system?

9-13. Show that the energy distribution of particles recoiling from an elastic collision is always directly proportional to the differential scattering cross section in the C.M. system.*

* This result is due to H. H. Barschall and M. H. Kanner, *Phys. Rev.* **58**, 590 (1940).

習題九

9-1 m_1, m_2 作彈性碰撞. 初速各為 u_1, $u_2=\alpha u_1$ $(\alpha\neq 0)$. 若起初動能相等且 m_1 在碰撞後靜止 u_1/u_2, m_1/m_2 應如何?

Sol. 由動量守恆得 $m_1u_1+m_2\alpha u_1=m_2u_2' \Rightarrow u_2'=\frac{m_1+m_2\alpha}{m_2}u_1$ --- ①

由動能守恆得 $\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2\alpha^2u_1^2=\frac{1}{2}m_2u_2'^2$ --- ②

①代入②, 並消去 u_1^2 $\quad m_1+\alpha^2m_2=\frac{(m_1+\alpha m_2)^2}{m_2}$

展開 $m_1m_2+\alpha^2m_2^2=m_1^2+2\alpha m_1m_2+\alpha^2m_2^2$

$m_1m_2(1-2\alpha)=m_1^2 \quad \therefore m_1/m_2=1-2\alpha$ --- ③

又原先動能相等. $\frac{1}{2}m_1u_1^2=\frac{1}{2}m_2\alpha^2u_1^2$ --- ④

③代入④ $\quad 1-2\alpha=\alpha^2 \quad \therefore \alpha=-1\pm\sqrt{2}$

故 $u_1/u_2=\frac{1}{\alpha}=\frac{1}{-1\pm\sqrt{2}}=1\mp\sqrt{2}$

$m_1/m_2=1-2\alpha=3\mp2\sqrt{2}$

9-2 若 $S\equiv\cos\psi+\frac{\cos(\theta-\psi)}{(m_1/m_2)}$ 證明 $\frac{T_1}{T_0}=\frac{m_1^2}{(m_1+m_2)^2}S^2$

Sol Text 上證 $T_1/T_0=\frac{m_1^2}{(m_1+m_2)^2}\left[\cos\psi\pm\sqrt{\left(\frac{m_2}{m_1}\right)^2-\sin^2\psi}\right]^2$ 之步驟付之闕如. 今証之

由已証之 $T_1/T_0=1-\frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta)$ 出發

先將 $\cos\theta$ 以 ψ 之函數表之

$\tan\psi = \dfrac{\sin\theta}{\cos\theta + (m_1/m_2)}$ 平方之，将 $\sin^2\theta$ 化成 $1-\cos^2\theta$ 得

$$\cos^2\theta\,(\sec^2\psi) + 2\left(\frac{m_1}{m_2}\right)\tan^2\psi\cos\theta + \left(\frac{m_1}{m_2}\right)^2\tan^2\psi - 1 = 0$$

此为 $\cos\theta$ 之二次方程式，解之得

$$\cos\theta = \left[-\frac{m_1}{m_2}\tan^2\psi \pm \sqrt{\left(\frac{m_1}{m_2}\right)^2\tan^4\psi - \left(\frac{m_1}{m_2}\right)^2\tan^2\psi\sec^2\psi + \sec^2\psi}\;\right]\Big/\sec^2\psi$$

$$= \left[-\frac{m_1}{m_2}\tan^2\psi \pm \sqrt{-\left(\frac{m_1}{m_2}\right)^2\tan^2\psi + \sec^2\psi}\;\right]\Big/\sec^2\psi$$

$$= \frac{m_1}{m_2}\left[-\sin^2\psi \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\;\cos\psi\right]$$

故 $$T_1/T_0 = 1 - \frac{2m_1m_2}{(m_1+m_2)^2}\left\{1 - \frac{m_1}{m_2}\left[-\sin^2\psi \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\;\cos\psi\right]\right\}$$

$$= \frac{1}{(m_1+m_2)^2}\left\{(m_1+m_2)^2 - 2m_1m_2 + 2m_1^2\left[-\sin^2\psi \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\;\cos\psi\right]\right\}$$

$$= \frac{1}{(m_1+m_2)^2}\left[m_1^2 + m_2^2 + 2m_1^2(-\sin^2\psi) \pm 2m_1^2\sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\;\cos\psi\right]$$

$$= \frac{m_1^2}{(m_1+m_2)^2}\left[1 - 2\sin^2\psi + \left(\frac{m_2}{m_1}\right)^2 + 2\sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\;\cos\psi\right]$$

$$= \frac{m_1^2}{(m_1+m_2)^2}\left[\cos2\psi + \left(\frac{m_2}{m_1}\right)^2 + 2\sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\;\cos\psi\right]$$

但 $$\left[\cos\psi \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\right]^2 = \cos2\psi \pm 2\sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\;\cos\psi + \left(\frac{m_2}{m_1}\right)^2$$

$\therefore$ $$T_1/T_0 = \frac{m_1^2}{(m_1+m_2)^2}\left[\cos\psi \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\right]^2$$ 当 $m_1>m_2$ 取正號

由此式出发証 Prob 9.2

把 $\tan\psi = \dfrac{\sin\theta}{\cos\theta + m_1/m_2}$ 变为 $\dfrac{\sin\psi}{\cos\psi} = \dfrac{\sin\theta}{\cos\theta + m_1/m_2}$

展開 $\sin\psi\cos\theta + \left(\frac{m_1}{m_2}\right)\sin\psi = \sin\theta\cos\psi$ $\therefore \dfrac{m_2}{m_1} = \dfrac{\sin\psi}{\sin(\theta-\psi)}$

$$\left(\frac{m_2}{m_1}\right)^2-\sin^2\psi=\frac{\sin^2\psi}{\sin^2(\theta-\psi)}-\sin^2\psi=\frac{\sin^2\psi\cos^2(\theta-\psi)}{\sin^2(\theta-\psi)}=\left(\frac{m_2}{m_1}\right)^2\cos^2(\theta-\psi)$$

$$\therefore\ T_1/T_0=\frac{m_1^2}{(m_1+m_2)^2}\left[\cos\psi+\frac{m_2}{m_1}\cos(\theta-\psi)\right]^2=\frac{m_1^2}{(m_1+m_2)^2}\,S^2$$

9-3 証明 T_1/T_0 可以 $m_2/m_1\equiv\alpha$ 及 $\cos\psi\equiv y$ 表為

$$T_1/T_0=(1+\alpha)^{-2}\left[2y^2+\alpha^2-1+2y\sqrt{\alpha^2+y^2-1}\right]$$

作 T_1/T_0 之圖 当 $\alpha=1,2,4,12$

Sol. 由 Text $\quad T_1/T_0=\frac{m_1^2}{(m_1+m_2)^2}\left[\cos\psi+\sqrt{\left(\frac{m_2}{m_1}\right)^2-\sin^2\psi}\right]^2$

$$=\frac{1}{\left(1+\frac{m_2}{m_1}\right)^2}\left[\cos^2\psi-\sin^2\psi+\left(\frac{m_2}{m_1}\right)^2+2\sqrt{\left(\frac{m_2}{m_1}\right)^2-\sin^2\psi}\,\cos\psi\right]$$

$$=(1+\alpha)^{-2}\left[2y^2-1+\alpha^2+2\sqrt{\alpha^2+y^2-1}\ y\right]$$

$\alpha=1 \quad T_1/T_0=y^2=\cos^2\psi$

$\alpha=2 \quad T_1/T_0=3^{-2}\left[2y^2+3+2y\sqrt{3+y^2}\right]$

$\qquad\qquad =3^{-2}\left[2\cos^2\psi+3+2\cos\psi\sqrt{3+\cos^2\psi}\right]$

$\alpha=4 \quad T_1/T_0=5^{-2}\left[2y^2+15+2y\sqrt{15+y^2}\right]$

$\alpha=12 \quad T_1/T_0=13^{-2}\left[2y^2+143+2y\sqrt{143+y^2}\right]$

当 $\alpha\gg1 \quad T_1/T_0\doteq\alpha^2/(1+\alpha)^2\doteq1$

9-4 Lab 系统中 質點 m_1 以初速 u_1 与靜止之質點 m_2 碰撞 m_1 被散射. Lab 中散射角為 ψ. 末速 $v_1=v_1(\psi)$. 求散射後, 同一时间所能到達之曲面. 考慮 (a) $m_2=m_1$ (b) $m_2=2m_1$ (c) $m_2=\infty$ 之情形.

Sol 由能量守恆可得 $\frac{1}{2}m_1V_1^2+\frac{1}{2}m_2V_2^2=\frac{1}{2}m_1u_1^2 \cdots\cdots(1)$

由線動量守恆可得 $m_1V_1\cos\psi+m_2V_2\cos\xi=m_1u_1 \cdots\cdots(2)$

$m_1V_1\sin\psi=m_2V_2\sin\xi \cdots\cdots(3)$

由(2)式可得：$m_2V_2\cos\xi=m_1u_1-m_1V_1\cos\psi \cdots\cdots(4)$

消去ξ，$(3)^2+(4)^2$ 可得 $m_2^2V_2^2=m_1^2u_1^2+m_1^2V_1^2-2m_1^2u_1V_1\cos\psi$

$\therefore V_2^2=\left(\frac{m_1}{m_2}\right)^2(u_1^2-2u_1V_1\cos\psi+V_1^2) \cdots\cdots(5)$

(5)代入(1)式可得：

$$\frac{1}{2}m_1V_1^2+\frac{1}{2}m_2\left(\frac{m_1}{m_2}\right)^2(u_1^2-2u_1V_1\cos\psi+V_1^2)=\frac{1}{2}m_1u_1^2$$

整理後可得 $(m_1+m_2)V_1^2-(2m_1\cos\psi)u_1V_1+(m_1-m_2)u_1^2=0$

解上式可得 $V_1=\frac{m_1\cos\psi\pm\sqrt{m_2^2-m_1^2\sin^2\psi}}{m_1+m_2}u_1$

(a) 若$m_2=m_1$，則 $V_1=\frac{m_1\cos\psi\pm\sqrt{m_1^2(1-\sin^2\psi)}}{2m_1}u_1$

$$=\frac{2m_1\cos\psi}{2m_1}u_1$$

$$=u_1\cos\psi$$

(b) 若$m_2=2m_1$，則 $V_1=\frac{\cos\psi\pm\sqrt{4-\sin^2\psi}}{3}u_1$

(c) 若$m_2=\infty$，則 $V_1=-u_1\sin\psi$

9-5 証明 $\sigma(\theta)=\sigma(\psi)\frac{1+x\cos\theta}{(1+2x\cos\theta+x^2)^{3/2}}$

Sol 由(9-43)式可得

$$\sigma(\psi)=\sigma(\theta)\frac{\left[x\cos\psi+\sqrt{1-x^2\sin^2\psi}\right]^2}{\sqrt{1-x^2\sin^2\psi}}$$

由(9-24a)式可得 $\frac{T_1}{T_0}=\left(\frac{V_1}{u_1}\right)^2=1-\frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta)$

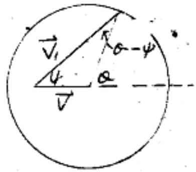
令 $\frac{m_1}{m_2}=x$，則得 $\frac{u_1}{V_1}=\left[1-\frac{2x}{(1+x)^2}(1-\cos\theta)\right]^{-\frac{1}{2}}$

$$=\left\{\frac{1+x^2+2x\cos\theta}{(1+x)^2}\right\}^{-\frac{1}{2}}$$

$$\frac{\sin\psi}{\sin\theta}=\frac{V_1'}{V_1}=\frac{m_2}{m_1+m_2}\frac{u_1}{V_1}=\frac{1}{1+x}\left[1-\frac{2x}{(1+x)^2}(1-\cos\theta)\right]^{-\frac{1}{2}}$$

$$=\frac{1}{1+x}\left\{\frac{(1+x)^2}{1+2x\cos\theta+x^2}\right\}^{\frac{1}{2}}$$

$$=\frac{1}{\sqrt{1+2x\cos\theta+x^2}}\cdots\cdots(1)$$



但 $\frac{\sin(\theta-\psi)}{\sin\theta}=\frac{V}{V_1}=\frac{m_1}{m_1+m_2}\frac{u_1}{V_1}$

$$=\frac{x}{1+x}\left[1-\frac{2x}{(1+x)^2}(1-\cos\theta)\right]^{-\frac{1}{2}}$$

$$=\frac{x}{\sqrt{1+2x\cos\theta+x^2}}\cdots\cdots(2)$$

比較(1)及(2)式可得 $\sin(\theta-\psi)=\frac{x}{\sqrt{1+2x\cos\theta+x^2}}\sin\theta$

$$\therefore \cos(\theta-\psi)=\left[1-\sin^2(\theta-\psi)\right]^{\frac{1}{2}}=\sqrt{1-\frac{x^2\sin^2\theta}{1+2x\cos\theta+x^2}}=\frac{1+x\cos\theta}{\sqrt{1+2x\cos\theta+x^2}}$$

$$\therefore \sigma(\theta)=\sigma(\psi)=\frac{\cos(\theta-\psi)\sin^2\psi}{\sin^2\theta}=\sigma(\psi)\frac{1+x\cos\theta}{(1+2x\cos\theta+x^2)^{3/2}}$$

9-6 若 $V(r)=\begin{cases}0 & r>a\\ \infty & r<a\end{cases}$ 求 $\sigma(\theta)$, σ_t

Sol $\Theta=\int_{r_{min}}^{\infty}\frac{(b/r^2)dr}{\sqrt{1-(b^2/r^2)-(V/T_0')}}=\int_a^{\infty}\frac{(b/r^2)dr}{\sqrt{1-(b^2/r^2)}}\Bigg|_{r=b\sec\theta}$

$$=\int_{\sec^{-1}\frac{a}{b}}^{\pi/2}\frac{[b/(b^2\sec^2\theta)]b\sec\theta\tan\theta}{\sin\theta}d\theta=\int d\theta=\frac{\pi}{2}-\sec^{-1}\frac{a}{b}$$

$$\theta=\pi-2\Theta=2\sec^{-1}\frac{a}{b}\ \therefore \sec\frac{\theta}{2}=\frac{a}{b}\ \therefore b=a\cos\frac{\theta}{2}$$

$$\sigma(\theta)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{a\cos\frac{\theta}{2}}{\sin\theta}\left(\frac{a}{2}\sin\frac{\theta}{2}\right)=\frac{a^2}{4}$$

$$\sigma_t=\int\sigma(\theta)\,d\Omega=\int_0^\pi\frac{a^2}{4}2\pi\sin\theta\,d\theta=\pi a^2$$

9-7 前題中，若能量損失為ϵ，求証 $d\sigma_{C.M}(\epsilon)=\frac{\pi a^2}{\epsilon_{max}}d\epsilon$，因此在C.M中，能量之分佈是均匀的。

Sol 入射質點所損失的能量 = m_2 所獲得之能量

$$\therefore \epsilon=T_2=\frac{1}{2}m_2v_2'^2=\frac{1}{2}m_2(v_2'^2+V^2-2v_2V\cos\theta)$$

$$=\frac{1}{2}m_2\left[\left(\frac{m_1u_1}{m_1+m_2}\right)^2+\left(\frac{m_1u_1}{m_1+m_2}\right)^2-2\left(\frac{m_1u_1}{m_1+m_2}\right)^2\cos\theta\right]$$

$$=m_2\left(\frac{m_1u_1}{m_1+m_2}\right)^2(1-\cos\theta)$$

$$=\frac{2m_2m_1^2u_1^2}{(m_1+m_2)^2}\sin^2\frac{\theta}{2}=\epsilon_{max}\sin^2\frac{\theta}{2}$$

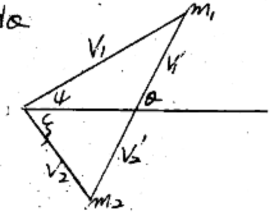
微分上式：

$$d\epsilon=\epsilon_{max}\left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(\frac{1}{2}\right)\right]d\theta=\frac{1}{2}\epsilon_{max}\sin\theta\,d\theta$$

$$\because d\sigma_{C.M}(\epsilon)=\frac{1}{2}\pi a^2\sin\theta\,d\theta$$

$$\therefore d\sigma_{C.M}=\frac{\pi a^2}{\epsilon_{max}}d\epsilon$$

$$\Rightarrow\frac{d\sigma_{C.M}(\epsilon)}{d\epsilon}=\frac{\pi a^2}{\epsilon_{max}}=\text{const}$$



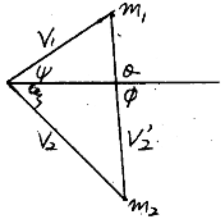
故証得能量之均匀分佈性

9-8 Rutherford 的散射公式，當 $m_1=m_2$，$\sigma_{lab}(\xi)=\frac{K^2}{T_0^2}\frac{1}{\cos^3\xi}$ 証之！

Sol 微分截面為

$$\sigma_{lab}(\xi)2\pi\sin\xi\,d\xi=-\sigma(\psi)2\pi\sin\psi\,d\psi$$

∴ $\sigma_{lab}(\xi)=-\sigma(\psi)\frac{|\sin\psi\, d\psi|}{\sin\xi\, d\xi}$



但 $\psi+\xi=\frac{\pi}{2} \Rightarrow \frac{d\psi}{d\xi}=-1$

∵ $\sin\psi=\sin(\frac{\pi}{2}-\xi)=\cos\xi$,

$\cos\psi=\cos(\frac{\pi}{2}-\xi)=\sin\xi$

∴ $\sigma_{lab}(\xi)=\sigma(\psi)\frac{\sin\psi}{\cos\psi}$

$m_1=m_2$時, $\sigma(\psi)=\frac{K^2}{T_0^2}\frac{\cos\psi}{\sin^4\psi}$

∴ $\sigma_{lab}(\xi)=\frac{K^2}{T_0^2}\frac{\cos\psi}{\sin^4\psi}\cdot\frac{\sin\psi}{\sin\xi}=\frac{K^2}{T_0^2}\frac{1}{\sin^3\psi}=\frac{K^2}{T_0^2}\frac{1}{\cos^3\xi}$

9-9 $m_1\gg m_2$ 時·以近似法表 $\sigma_{lab}(\psi)$

Sol. ∵ $\sigma(\psi)d\Omega=\sigma(\theta)d\Omega'$

∴ $\sigma(\psi)2\pi\sin\psi\, d\psi=\sigma(\theta)2\pi\sin\theta\, d\theta$, $\sigma(\psi)=\sigma(\theta)\frac{\sin\theta}{\sin\psi}\frac{d\theta}{d\psi}$

今求 θ·ψ 之關係·

∵ $\tan\psi=\frac{\sin\theta}{\cos\theta+(m_1/m_2)}$ 当 $m_1\gg m_2$ $\tan\psi\doteq\frac{m_2}{m_1}\sin\theta=\sin\theta/\chi$

又 $T_0'=\frac{m_2}{m_1+m_2}T_0\doteq\frac{m_2}{m_1}T_0=T_0/\chi$ $\sin\psi\doteq\sin\theta/\chi$

∴ $\sigma(\psi)=\sigma(\theta)\frac{\sin\theta\, d\theta}{\sin\psi\, d\psi}=\left[\frac{K^2}{(4T_0')^2}\frac{1}{\sin^4\theta/2}\right]\frac{\chi\cos\psi}{\cos\theta}=\frac{K^2\chi^2\cos\psi}{(4T_0/\chi)^2\sin^4\frac{\theta}{2}\cos\theta}$

$=\frac{K^2\chi^4\cos\psi}{16T_0^2\sin^4\frac{\theta}{2}\cos\theta}$ $\theta=\sin^{-1}(\chi\sin\psi)$

9-10 当 $m_2\gg m_1$ 以近似法表 $\sigma_{cm}(\theta)$. 準確至 m_1/m_2 一次項

Sol. $\sigma(\theta)=\frac{k^2}{(4T_0')^2}\frac{1}{\sin^4\frac{\theta}{2}}$ $\because T_0'=\frac{m_2}{m_1+m_2}T_0 \doteq \frac{T_0}{1+(m_1/m_2)}$

$\therefore \sigma(\theta)=\frac{k^2}{(4T_0)^2\sin^4\frac{\theta}{2}}(1+\frac{m_1}{m_2})^2 \doteq \frac{k^2}{4T_0^2\sin^4\frac{\theta}{2}}(1+2\frac{m_1}{m_2})$

上式中已用到 $(1+x)^2 \simeq 1+2x$

9-11 一力中心以 $F(r)=k/r^3$ 散射质点 m. 若质点之初速 u_0, 证 $\sigma(\theta)=\frac{k\pi}{2mu_0^2\theta^2\sin\theta}$, 且 $\sigma_t=\infty$. 但若力在 $r>r_0$ 处消失, 则有一最小散射角 θ_0, 以 m. u_0. r_0 及 k 表示之. 證 $\sigma_t=2\pi\int_{\theta_0}^{2\pi}\sigma(\theta)\sin\theta d\theta=\pi r_0^2$

Sol. $F=k/r^3$ $\qquad U=-\int_{\infty}^{r}Fdr=\frac{k}{2r^2}$

$$\Theta=\int_{r_{min}}^{\infty}\frac{(b/r^2)\,dr}{\sqrt{1-(b^2/r^2)-(U/T_0')}}=\int\frac{(b/r^2)dr}{\sqrt{1-(b^2/r^2)-\frac{k}{2r^2T_0'}}}=\int\frac{(b/r)dr}{\sqrt{r^2-(b^2+\frac{k}{2T_0'})}}\Bigg|_{r=a\sec\theta}$$

$$=\int\frac{b(a\tan\theta\sec\theta\,d\theta)}{a\sec\theta\,(a\tan\theta)}=\int\frac{b}{a}d\theta=\frac{b}{a}\theta\Big|_{\sec^{-1}\frac{a}{a}}^{\pi/2}=\frac{b\pi}{2a}$$

$$\theta=\pi-2\Theta=\pi(1-\frac{b}{a})=\pi\left(1-\frac{b}{\sqrt{b^2+\frac{k}{2T_0'}}}\right)$$

解 b. $\qquad b^2=\frac{k(\pi-\theta)^2}{2T_0'\theta^2(2\pi-\theta)^2}$

微分 $\qquad 2b\,db=\frac{k}{2T_0'}\frac{-2\pi^2(\pi-\theta)}{(2\pi-\theta)^2\theta^2}d\theta$

$$\sigma(\theta)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{k\pi^2(\pi-\theta)}{mu_0^2\sin\theta(2\pi-\theta)^2\theta^2}$$

$\theta\to 0 \quad \sigma(\theta)\to\infty \quad \sigma_t=\infty$

若 $F=0$ 当 $r>r_0$ $\qquad U=-\int_a^r Fdr=-\int_{r_0}^{r}\frac{k}{r^3}dr=\frac{k}{2r^2}-\frac{k}{2r_0^2}$

$$\Theta = \int_{r_{min}}^{r_0} \frac{(b/r^2)\,dr}{\sqrt{1-\frac{b^2}{r^2}-\left(\frac{K}{2r^2}-\frac{K}{2r_0^2}\right)/T_0}} + \int_{r_0}^{\infty} \frac{(b/r^2)\,dr}{\sqrt{1-b^2/r^2}}$$

$$= \frac{b}{\sqrt{b^2+\frac{K}{mu_0^2}}} \sec^{-1}\frac{\sqrt{1+\frac{K}{mu_0^2 r_0^2}}\,r_0}{\sqrt{b^2+\frac{K}{mu_0^2}}} + \frac{\pi}{2} - \sec^{-1}\frac{r_0}{b}$$

$$\theta = \pi - 2\Theta = 2\sec^{-1}\frac{r_0}{b} - \frac{2b}{\sqrt{b^2+\frac{K}{mu_0^2}}} \sec^{-1}\frac{\sqrt{1+\frac{K}{mu_0^2 r_0^2}}\,r_0}{\sqrt{b^2+\frac{K}{mu_0^2}}}$$ 當 $b=r_0,\ \theta=\theta_0=0$

$\therefore \sigma_t = 2\pi\int_0^{\pi} \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| d\theta = \pi r_0^2$

9-12 中子在低能量碰撞质子，质子能量分佈為常數，則在 C.M 中，散射之角分佈如何？

Sol. $dN = I\sigma(\psi)\,d\Omega = I\sigma(\psi)\,2\pi\sin\psi\,d\psi$ $\quad n \longrightarrow p$

$dN/dT_2 =$ 常數，令 $dN/d(T_2/T_0) = c$

$\because m_1 \doteq m_2 \quad T_2/T_0 \doteq \sin^2\psi \quad \therefore \frac{dN}{d\psi} = c\,\frac{d(T_2/T_0)}{d\psi} \doteq 2c\sin\psi\cos\psi$

$\therefore I\sigma(\psi)\,2\pi\sin\psi = 2c\sin\psi\cos\psi$

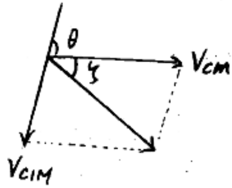
$$\sigma(\psi) = \frac{c\cos\psi}{\pi I}$$

又 $\sigma(\theta) = \sigma(\psi)\frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta} = \frac{c\cos\psi\sin\psi\,d\psi}{\pi I\,\sin\theta\,d\theta}$

$\because m_1 \doteq m_2 \quad \psi = \theta/2$

$$\therefore \sigma(\theta) = \frac{c}{4\pi I}$$

9-13 証明反跳之能量分佈等於 $\sigma_{cm}(\theta)$ 乘以常數



Sol. $\because \dfrac{T_2}{T_0} = \dfrac{4m_1m_2}{(m_1+m_2)^2}\cos^2\zeta$ 且

$2\zeta = \pi - \theta$

$\therefore \dfrac{T_2}{T_0} = \dfrac{4m_1m_2}{(m_1+m_2)^2}\sin^2\dfrac{\theta}{2} = \dfrac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta)$

$d(T_2/T_0) = \dfrac{2m_1m_2}{(m_1+m_2)^2}\sin\theta\, d\theta$

$\because dN = I\,\sigma(\theta)\,d\Omega = I\,\sigma(\theta)\,2\pi\sin\theta\, d\theta$

$\therefore dN/d(T_2/T_0) = K\cdot\sigma(\theta)$ $\quad K=$常數

Problems

10-1. Show that the transformation equations connecting the K' and K systems [Eqs. (10.22)] can be expressed as

$$x_1' = x_1 \cosh \alpha - ct \sinh \alpha$$
$$x_2' = x_2 ; \qquad x_3' = x_3$$
$$t' = t \cosh \alpha - \frac{x_1}{c} \sinh \alpha$$

where $\tanh \alpha = v/c$. Show that the Lorentz transformation corresponds to a rotation through an angle $i\alpha$ in four-dimensional space.

10-2. Show that the equation

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = 0$$

is invariant under a Lorentz transformation but not under a Galilean transformation. (This is the wave equation which describes the propagation of light waves in free space.)

10-3. Show that the expression for the FitzGerald–Lorentz contraction [Eq. (10.48)] can also be obtained if the observer in the K' system measures the time necessary for the rod to pass a fixed point in his system and then multiplies the result by v.

10-4. What are the apparent dimensions of a cube of side l (in its own proper frame) which moves with a uniform velocity v directly *toward* or *away from* an observer?

10-5. Consider two events that take place at different points in the K system at the same instant t. If these two points are separated by a distance Δx, show that in the K' system the events are not simultaneous but are separated by a time interval $\Delta t' = -v\gamma\, \Delta x/c^2$.

10-6. Two clocks, located at the origins of the K and K' systems (which have a relative velocity v), are synchronized when the origins coincide. After a time t, an observer at the origin of the K system observes the K' clock by means of a telescope. What does the K' clock read?

10-7. In his 1905 paper (see the translation in Lo23), Einstein states: "We conclude that a balance-clock at the equator must go more slowly, by a very small amount, than a precisely similar clock situated at one of the poles under otherwise identical conditions." Neglect the fact that the equator clock does not undergo uniform motion and show that after a century the clocks will differ by approximately 0.0025 sec.

10-8. Consider a relativistic rocket whose velocity with respect to a certain inertial frame is v and whose exhaust gases are emitted with a constant velocity V with respect to the rocket. Show that the equation of motion is

$$m_0 \frac{dv}{dt} + V \frac{dm_0}{dt}(1 - \beta^2) = 0$$

where $m_0 = m_0(t)$ is the mass of the rocket in its rest frame and $\beta = v/c$.

10-9. Consider an inertial frame K which contains a number of particles with rest masses $m_{0,\alpha}$, ordinary momentum components $p_{\alpha,j}$, and total energies E_α. The center-of-mass system of such a group of particles is defined to be that system in which the net ordinary momentum is zero. Show that the velocity components of the center-of-mass system with respect to K are given by

$$\frac{v_j}{c} = \frac{\sum\limits_\alpha p_{\alpha,j} c}{\sum\limits_\alpha E_\alpha}$$

10-10. A common unit of energy used in atomic and nuclear physics is the *electron volt* (eV), the energy acquired by an electron in falling through a potential difference of one volt: 1 MeV $= 10^6$ eV $= 1.602 \times 10^{-6}$ erg. In these units the rest mass of an electron is $m_e c^2 = 0.511$ MeV and that of a proton is $m_p c^2 = 931$ MeV. Calculate the kinetic energy and the quantities β and γ for an electron and for a proton each of which has a momentum of 100 MeV/c. Show that the electron is "relativistic" whereas the proton is "nonrelativistic."

10-11. The energy of a light quantum (or *photon*) is expressed by $E = h\nu$, where h is Planck's constant and ν is the frequency of the photon. The momentum of the photon is $h\nu/c$. Show that if the photon scatters from a free electron (of mass m_e), the scattered photon will have an energy

$$E' = E\left[1 + \frac{E}{m_e c^2}(1 - \cos\theta)\right]^{-1}$$