Graduate Texts in Mathematics

Edwin Hewitt
Karl Stromberg

Real and Abstract Analysis

实分析和抽象分析

Springer

世界图书出版公司
www.wpcbj.com.cn

Edwin Hewitt *Karl Stromberg*

Real and Abstract Analysis

A Modern Treatment of the Theory of
Functions of a Real Variable

Springer

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Mathematics Subject Classification (1991): 28-01, 28A20, 28A25, 46-01

Library of Congress Cataloging in Publication Data

Hewitt, Edwin, 1920–
Real and abstract analysis.
(Graduate texts in mathematics; 25)
Includes bibliographical references and indexes.
1. Function of real variables. 2. Mathematical analysis. I. Stromberg, Karl Robert, 1931– joint author. II. Title. III. Series.
QA331.5.H38 1975 515'.8 75-9898

ISBN 0-387-90138-8 Springer-Verlag New York Berlin Heidelberg
ISBN 3-540-90138-8 Springer-Verlag Berlin Heidelberg New York SPIN 10635951

图书在版编目（CIP）数据

实分析和抽象分析：英文/（美）休伊特（Hewitt，E.）著.—北京：世界图书出版公司北京公司，2009.4

书名原文：Real and Abstract Analysis

ISBN 978-7-5100-0447-6

Ⅰ.实… Ⅱ.休… Ⅲ.0211.62 Ⅳ.0211.62

中国版本图书馆 CIP 数据核字（2009）第 048130 号

书　　名：	Real and Abstract Analysis
作　　者：	Edwin Hewitt & Karl Stromberg
中 译 名：	实分析和抽象分析
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@ wpcbj. com. cn
开　　本：	24 开
印　　张：	20.5
版　　次：	2009 年 04 月
版权登记：	图字：01-2009-0681
书　　号：	978-7-5100-0447-6/O · 662　　**定　　价：** 45.00 元

This book is dedicated to

MARSHALL H. STONE

whose precept and example have
taught us both.

Preface

This book is first of all designed as a text for the course usually called "theory of functions of a real variable". This course is at present customarily offered as a first or second year graduate course in United States universities, although there are signs that this sort of analysis will soon penetrate upper division undergraduate curricula. We have included every topic that we think essential for the training of analysts, and we have also gone down a number of interesting bypaths. We hope too that the book will be useful as a reference for mature mathematicians and other scientific workers. Hence we have presented very general and complete versions of a number of important theorems and constructions. Since these sophisticated versions may be difficult for the beginner, we have given elementary avatars of all important theorems, with appropriate suggestions for skipping. We have given complete definitions, explanations, and proofs throughout, so that the book should be usable for individual study as well as for a course text.

Prerequisites for reading the book are the following. The reader is assumed to know elementary analysis as the subject is set forth, for example, in TOM M. APOSTOL's *Mathematical Analysis* [Addison-Wesley Publ. Co., Reading, Mass., 1957], or WALTER RUDIN's *Principles of Mathematical Analysis* [2nd Ed., McGraw-Hill Book Co., New York, 1964]. There are no other prerequisites for reading the book: we define practically everything else that we use. Some prior acquaintance with abstract algebra may be helpful. The text *A Survey of Modern Algebra*, by GARRETT BIRKHOFF and SAUNDERS MAC LANE [3rd Ed., MacMillan Co., New York, 1965] contains far more than the reader of this book needs from the field of algebra.

Modern analysis draws on at least five disciplines. First, to explore measure theory, and even the structure of the real number system, one must use powerful machinery from the abstract theory of sets. Second, as hinted above, algebraic ideas and techniques are illuminating and sometimes essential in studying problems in analysis. Third, set-theoretic topology is needed in constructing and studying measures. Fourth, the theory of topological linear spaces ["functional analysis"] can often be applied to obtain fundamental results in analysis, with surprisingly little effort. Finally, analysis really is *analysis*. We think that handling inequalities, computing with actual functions, and obtaining actual numbers, is indispensable to the training of every mathematician. All five of these subjects thus find a place in our book. To make the book useful to probabilists, statisticians, physicists, chemists, and engineers, we have included many "applied" topics: Hermite functions; Fourier series and integrals, including PLANCHEREL's theorem and pointwise summability; the strong law of large numbers; a thorough discussion of complex-valued measures on the line. Such applications of the abstract theory are also vital to the pure mathematician who wants to know where his subject came from and also where it may be going.

With only a few exceptions, everything in the book has been taught by at least one of us at least once in our real variables courses, at the Universities of Oregon and Washington. As it stands, however, the book is undoubtedly too long to be covered *in toto* in a one-year course. We offer the following road map for the instructor or individual reader who wants to get to the center of the subject without pursuing byways, even interesting ones.

Chapter One. Sections 1 and 2 should be read to establish our notation. Sections 3, 4, and 5 can be omitted or assigned as outside reading. What is essential is that the reader should have facility in the use of cardinal numbers, well ordering, and the real and complex number fields.

Chapter Two. Section 6 is of course important, but a lecturer should not succumb to the temptation of spending too much time over it. Many students using this text will have already learned, or will be in the process of learning, the elements of topology elsewhere. Readers who are genuinely pressed for time may omit § 6 and throughout the rest of the book replace "locally compact Hausdorff space" by "real line", and "compact Hausdorff space" by "closed bounded subset of the real line". We do not recommend this, but it should at least shorten the reading. We urge everyone to cover § 7 in detail, except possibly for the exercises.

Chapter Three. This chapter is the heart of the book and must be studied carefully. Few, if any, omissions appear possible. Chapter Three is essential for all that follows, barring § 14 and most of § 16.

After Chapter Three has been completed, several options are open. One can go directly to § 21 for a study of product measures and FUBINI's theorem. [The applications of FUBINI's theorem in (21.32) *et seq*. require parts of §§ 13–18, however.] Also §§ 17–18 can be studied immediately after Chapter Three. Finally, of course, one can read §§ 13–22 in order.

Chapter Four. Section 13 should be studied by all readers. Subheads (13.40)–(13.51) are not used in the sequel, and can be omitted if necessary. Section 14 can also be omitted. [While it is called upon later in the text, it is not essential for our main theorems.] We believe nevertheless that § 14 is valuable for its own sake as a basic part of functional analysis. Section 15, which is an exercise in classical analysis, should be read by everyone who can possibly find the time. We use Theorem (15.11) in our proof of the LEBESGUE-RADON-NIKODÝM theorem [§ 19], but as the reader will see, one can get by with much less. Readers who skip § 15 must read § 16 in order to understand § 19.

Chapter Five. Sections 17 and 18 should be studied in detail. They are parts of classical analysis that every student should learn. Of § 19, only subheads (19.1)–(19.24) and (19.35)–(19.44) are really essential. Of § 20, (20.1)–(20.8) should be studied by all readers. The remainder of § 20, while interesting, is peripheral. Note, however, that subheads (20.55)–(20.59) are needed in the refined study of infinite product measures presented in § 22.

Chapter Six. Everyone should read (21.1)–(21.27) at the very least. We hope that most readers will find time to read our presentation of PLANCHEREL's theorem (21.31)–(21.53) and of the HARDY-LITTLEWOOD maximal theorems (21.74)–(21.83). Section 22 is optional. It is essential for all students of probability and in our opinion, its results are extremely elegant. However, it can be sacrificed if necessary.

Occasionally we use phrases like "obvious on a little thought", or "a moment's reflection shows...". Such phrases mean really that the proof is not hard but is clumsy to write out, and we think that more writing would only confuse the matter. We offer a very large number of exercises, ranging in difficulty from trivial to all but impossible. The harder exercises are supplied with hints. Heroic readers may of course ignore the hints, although we think that every reader will be grateful for some of them. Diligent work on a fairly large number of exercises is vital for a genuine mastery of the book: exercises are to a mathematician what CZERNY is to a pianist.

We owe a great debt to many friends. Prof. KENNETH A. ROSS has read the entire manuscript, pruned many a prolix proof, and uncovered myriad mistakes. Mr. LEE W. ERLEBACH has read most of the text and has given us useful suggestions from the student's point of view. Prof. KEITH L. PHILLIPS compiled the class notes that are the skeleton of the book, has generously assisted in preparing the typescript for the printer, and has written the present version of (21.74)–(21.83). Valuable conversations and suggestions have been offered by Professors ROBERT M. BLUMENTHAL, IRVING GLICKSBERG, WILLIAM H. SILLS, DONALD R. TRUAX, BERTRAM YOOD, and HERBERT S. ZUCKERMAN. Miss BERTHA THOMPSON has checked the references. The Computing Center of the University of Oregon and in particular Mr. JAMES H. BJERRING have generously aided in preparing the index. We are indebted to the several hundred students who have attended our courses on this subject and who have suffered, not always in silence, through awkward presentations. We are deeply grateful to Mrs. SHANTI THAYIL, who has typed the entire manuscript with real artistry.

Our thanks are also due to the Universities of Oregon and Washington for exemption from other duties and for financial assistance in the preparation of the manuscript. It is a pleasure to acknowledge the great help given us by Springer-Verlag, in their rapid and meticulous publication of the work.

Seattle, Washington — EDWIN HEWITT

Eugene, Oregon — KARL R. STROMBERG

Table of Contents

CHAPTER ONE

Set Theory and Algebra

From the logician's point of view, mathematics is the theory of sets and its consequences. For the analyst, sets and concepts immediately definable from sets are essential tools, and manipulation of sets is an operation he must carry out continually. Accordingly we begin with two sections on sets and functions, containing few proofs, and intended largely to fix notation and terminology and to form a review for the reader in need of one. Sections 3 and 4, on the axiom of choice and infinite arithmetic, are more serious: they contain detailed proofs and are recommended for close study by readers unfamiliar with their contents.

Plainly one cannot study real- and complex-valued functions seriously without knowing what the real and complex number fields are. Therefore, in § 5, we give a short but complete construction of these objects. This section may be read, recalled from previous work, or taken on faith.

This text is *not* rigorous in the sense of proceeding from the axioms of set theory. We believe in sets, and we believe in the rational numbers. Beyond that, we have tried to prove all we say.

§ 1. The algebra of sets

(1.1) The concept of a set. As remarked above, we take the notion of set as being already known. Roughly speaking, a set [collection, assemblage, aggregate, class, family] is any identifiable collection of objects of any sort. We identify a set by stating what its members [elements, points] are. The theory of sets has been described axiomatically in terms of the notion "member of". To build the complete theory of sets from these axioms is a long, difficult process, and it is remote from classical analysis, which is the main subject of the present text. Therefore we shall make no effort to be rigorous in dealing with the concept of sets, but will appeal throughout to intuition and elementary logic. Rigorous treatments of the theory of sets can be found in *Naive Set Theory* by P. HALMOS [Princeton, N. J.: D. Van Nostrand Co. 1960] and in *Axiomatic Set Theory* by P. SUPPES [Princeton, N. J.: D. Van Nostrand Co. 1960].

(1.2) Notation. We will usually adhere to the following notational conventions. Elements of sets will be denoted by small letters: $a, b, c, \ldots, x, y, z$; $\alpha, \beta, \gamma, \ldots$ Sets will be denoted by capital Roman letters: $A, B, C, \ldots$ Families of sets will be denoted by capital script letters: $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$ Occasionally we need to consider collections of families of sets. These entities will be denoted by capital Cyrillic letters: Ж, Ч, ...

A set is often defined by some property of its elements. We will write $\{x : P(x)\}$ [where $P(x)$ is some proposition about x] to denote the set of all x such that $P(x)$ is true. We have done nothing here to sharpen the definition of a set, since "property" and "set" are from one point of view synonymous.

If the object x is an element of the set A, we will write $x \in A$; while $x \notin A$ will mean that x is not in A.

We write $\varnothing$ for the void [empty, vacuous] set; it has no members at all. Thus $\varnothing = \{x : x$ is a real number and $x^2 < 0\} = \{x : x$ is a unicorn in the Bronx Zoo$\}$, and so on.

For any object x, $\{x\}$ will denote the set whose only member is x. Similarly, $\{x_1, x_2, \ldots, x_n\}$ will denote the set whose members are precisely $x_1, x_2, \ldots, x_n$.

Throughout this text we will adhere to the following notations: N will denote the set $\{1, 2, 3, \ldots\}$ of all positive integers; Z will denote the set of all integers; Q will denote the set of all rational numbers; R will denote the set of all real numbers; and K will denote the set of all complex numbers. We assume a knowledge on the part of the reader of the sets N, Z, and Q. The sets R and K are constructed in § 5.

(1.3) Definitions. Let A and B be sets such that for all x, $x \in A$ implies $x \in B$. Then A is called a *subset of* B and we write $A \subset B$ or $B \supset A$. If $A \subset B$ and $B \subset A$, then we write $A = B$; $A \neq B$ denies $A = B$. If $A \subset B$ and $A \neq B$, we say that A is a *proper subset of* B and we write $A \subsetneqq B$. We note that under this idea of equality of sets, the void set is unique, for if $\varnothing_1$ and $\varnothing_2$ are any two void sets we have $\varnothing_1 \subset \varnothing_2$ and $\varnothing_2 \subset \varnothing_1$.

(1.4) Definitions. If A and B are sets, then we define $A \cup B$ as the set $\{x : x \in A$ or $x \in B\}$, and we call $A \cup B$ the *union of A and B*. Let $\mathscr{A}$ be a family of sets; then we define $\bigcup \mathscr{A} = \{x : x \in A$ for some $A \in \mathscr{A}\}$. Similarly if $\{A_\iota\}_{\iota \in I}$ is a family of sets indexed by iota, we write $\bigcup_{\iota \in I} A_\iota = \{x : x \in A_\iota$ for some $\iota \in I\}$. If $I = N$, the positive integers, $\bigcup_{n \in N} A_n$ will usually be written as $\bigcup_{n=1}^{\infty} A_n$. Other notations, such as $\bigcup_{n=-\infty}^{\infty} A_n$, are self-explanatory.

For given sets A and B, we define $A \cap B$ as the set $\{x : x \in A$ and $x \in B\}$, and we call $A \cap B$ the *intersection of A and B*. If $\mathscr{A}$ is any family of sets, we define $\bigcap \mathscr{A} = \{x : x \in A$ for all $A \in \mathscr{A}\}$; if $\{A_\iota\}_{\iota \in I}$ is a family of sets indexed by iota, then we write $\bigcap_{\iota \in I} A_\iota = \{x : x \in A_\iota$ for all $\iota \in I\}$. The notation $\bigcap_{n=1}^{\infty} A_n$ [and similar notations] have obvious meanings.

Example. If $A_n = \left\{x : x \text{ is a real number}, |x| < \frac{1}{n}\right\}$, $n = 1, 2, 3, \ldots,$ then $\bigcap_{n=1}^{\infty} A_n = \{0\}$.

For a set A, the family of all subsets of A is a well-defined family of sets which is known as *the power set of* A and is denoted by $\mathscr{P}(A)$. For example, if $A = \{1, 2\}$, then $\mathscr{P}(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

(1.5) Theorem. *Let A, B, C be any sets. Then we have:*

(i)	$A \cup B = B \cup A$;	(i′)	$A \cap B = B \cap A$;
(ii)	$A \cup A = A$;	(ii′)	$A \cap A = A$;
(iii)	$A \cup \varnothing = A$;	(iii′)	$A \cap \varnothing = \varnothing$;
(iv)	$A \cup (B \cup C) = (A \cup B) \cup C$;	(iv′)	$A \cap (B \cap C) = (A \cap B) \cap C$;
(v)	$A \subset A \cup B$;	(v′)	$A \cap B \subset A$;
(vi)	$A \subset B$ *if and only if* $A \cup B = B$;	(vi′)	$A \subset B$ *if and only if* $A \cap B = A$.

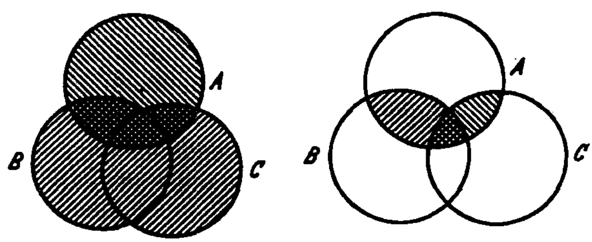
The proof of this theorem is very simple and is left to the reader.

(1.6) Theorem.

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. These and similar identities may be verified schematically; the verification of (i) follows:



$A \cap (B \cup C)$ $\qquad$ $(A \cap B) \cup (A \cap C)$

Fig. 1

A similar schematic procedure could be applied to (ii). However, we may use (i) and the previous laws as follows:

$$(A \cup B) \cap (A \cup C) = ((A \cup B) \cap A) \cup ((A \cup B) \cap C)$$
$$= (A \cap A) \cup (B \cap A) \cup (A \cap C) \cup (B \cap C) = A \cup (B \cap C)$$
$$\cup (B \cap A) \cup (A \cap C) = A \cup (B \cap C);$$

the last equality holds because $B \cap A \subset A$ and $A \cap C \subset A$. $\square$[1]

(1.7) Definition. If $A \cap B = \varnothing$, then A and B are said to be ***disjoint***. If $\mathscr{A}$ is a family of sets such that each pair of distinct members of $\mathscr{A}$ are disjoint, then $\mathscr{A}$ is said to be ***pairwise disjoint***. Thus an indexed family $\{A_\iota\}_{\iota \in I}$ is pairwise disjoint if $A_\iota \cap A_\eta = \varnothing$ whenever $\iota \neq \eta$.

(1.8) Definition. In most of our ensuing discussions the sets in question will be subsets of some fixed "universal" set X. Thus if $A \subset X$, we define

[1] The symbol $\square$ will be used throughout the text to indicate the end of a proof.

the *complement of* A [relative to X] to be the set $\{x : x \in X, x \notin A\}$. This set is denoted by the symbol A'. If there is any possible ambiguity as to which set is the universal set, we will write $X \cap A'$ for A'. Other common notations for what we call A' are $X - A$, $X \setminus A$, $X \sim A$, CA, and A^c; we will use A' exclusively.

(1.9) Theorem [DE MORGAN's laws].

(i) $(A \cup B)' = A' \cap B'$;

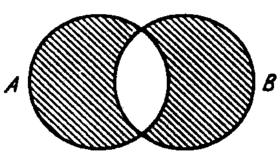
(ii) $(A \cap B)' = A' \cup B'$;

(iii) $(\bigcup_{\iota \in I} A_\iota)' = \bigcap_{\iota \in I} A'_\iota$;

(iv) $(\bigcap_{\iota \in I} A_\iota)' = \bigcup_{\iota \in I} A'_\iota$.

The proofs of these identities are easy and are left to the reader.

(1.10) Definition. For sets A and B, the *symmetric difference of A and B* is the set $\{x : x \in A$ or $x \in B$ and $x \notin A \cap B\}$, and we write $A \triangle B$ for this set. Note that $A \triangle B$ is the set consisting of those points which are in exactly one of A and B, and that it may also be defined by $A \triangle B = (A \cap B') \cup (A' \cap B)$. The symmetric difference is sketched in Fig. 2



$A \triangle B$

Fig. 2

(1.11) Definition. Let X be a set and let $\mathscr{R}$ be a nonvoid family of subsets of X such that

(i) $A, B \in \mathscr{R}$ implies $A \cup B \in \mathscr{R}$;

(ii) $A, B \in \mathscr{R}$ implies $A \cap B' \in \mathscr{R}$.

Then $\mathscr{R}$ is called a *ring of sets*. A ring of sets closed under complementation [*i.e.* $A \in \mathscr{R}$ implies $A' \in \mathscr{R}$] is called an *algebra of sets*.

(1.12) Remarks. A ring of sets is closed under the formation of finite intersections; for, if $A, B \in \mathscr{R}$, then (1.11.ii) applied twice shows that $A \cap B = A \cap (A \cap B')' \in \mathscr{R}$. By (1.11.i) and (1.11.ii), we have $A \triangle B = (A \cup B) \cap (A \cap B)' \in \mathscr{R}$. Note also that $\varnothing \in \mathscr{R}$ since $\mathscr{R}$ is nonvoid. Also $\mathscr{R}$ is an algebra if and only if $X \in \mathscr{R}$. There are rings of sets which are not algebras of sets; *e.g.*, the family of all finite subsets of N is a ring of sets but not an algebra of sets.

(1.13) Definition. A *σ-ring* [*σ-algebra*] *of sets* is a ring [algebra] of sets $\mathscr{R}$ such that if $\{A_n : n \in N\} \subset \mathscr{R}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$.

Much of measure theory deals with families of sets which form σ-rings or σ-algebras. There are σ-rings which are not σ-algebras, *e.g.*, the family of all countable subsets of an uncountable set. [For the definitions of countable and uncountable, see § 4.]

(1.14) Remarks.[1] There are many axiomatic treatments of rings and algebras of sets, and in fact some very curious entities can be interpreted

[1] This subhead is included only for its cultural interest and may be omitted by anyone who is in a hurry.

as rings or algebras of sets [see (1.25)]. Let B be any set. Suppose that to each $a \in B$ there is assigned a unique element $a^* \in B$ and that to each pair of elements $a, b \in B$ there is assigned a unique element $a \vee b \in B$ such that these operations satisfy

(i) $a \vee b = b \vee a$,

(ii) $a \vee (b \vee c) = (a \vee b) \vee c$,

(iii) $(a^* \vee b^*)^* \vee (a^* \vee b)^* = a$.

Sets B with operations $\vee$ and $*$ [or similar operations] and satisfying axioms equivalent to (i)—(iii) were studied by many writers in the period 1890—1930. They bear the generic name *Boolean algebras*, after the English mathematician GEORGE BOOLE [1815—1864]. The axioms (i)—(iii) were given by the U.S. mathematician E. V. HUNTINGTON [1874—1952] [Trans. Amer. Math. Soc. 5, 288—309 (1904)].

The reader will observe that if a, b are interpreted as sets and $\vee$ and $*$ as union and complementation, then (i)—(iii) are simple identities. Other operations can be defined in a Boolean algebra, *e.g.*, $\wedge$ [the analogue of $\cap$ for sets], which is defined by $a \wedge b = (a^* \vee b^*)^*$. A great deal of effort has been devoted to investigating abstract Boolean algebras. In the 1930's, the contemporary U.S. mathematician M. H. STONE showed that any Boolean algebra can be interpreted as an algebra of sets in the following very precise way [Trans. Amer. Math. Soc. 40, 37—111 (1936)]. Given any Boolean algebra B, there is a set X, an algebra $\mathscr{R}$ of subsets of X, and a one-to-one mapping τ of B onto $\mathscr{R}$ such that $\tau(a^*) = (\tau(a))'$ [$*$ becomes $'$] and $\tau(a \vee b) = \tau(a) \cup \tau(b)$ [$\vee$ becomes $\cup$]. Thus from the point of view of studying the operations in a Boolean algebra, one may as well study only algebras of sets.

STONE's treatment of the representation of Boolean algebras was based on a slightly different entity, namely, a *Boolean ring*. A Boolean ring is any ring S such that $x^2 = x$ for each $x \in S$. [For the definition of *ring*, see (5.3).]

STONE showed that Boolean algebras and Boolean rings having a multiplicative unit can be identified, and then based his treatment on Boolean rings. More precisely: for every Boolean ring S, there is a ring of sets $\mathscr{R}$ and a one-to-one mapping τ of S onto $\mathscr{R}$ such that

$$\tau(a + b) = \tau(a) \triangle \tau(b)$$

and

$$\tau(ab) = \tau(a) \cap \tau(b) .$$

That is, addition in a Boolean ring corresponds to the symmetric difference, and multiplication to intersection.

Proofs of the above results and a lengthy treatment of Boolean algebras and rings and of algebras and rings of sets can be found in G. BIRKHOFF, *Lattice Theory* [Amer. Math. Soc. Colloquium Publications, Vol. XXV, 2nd edition; Amer. Math. Soc., New York, N. Y., 1948].

(1.15) Exercise. Simplify as much as possible:

(a) $(A \cup (B \cap (C \cup W')))'$;

(b) $((X' \cup Y) \cap (X \cup Y'))'$;

(c) $(A \cap B \cap C) \cup (A' \cap B \cap C) \cup (A \cap B' \cap C) \cup (A \cap B \cap C')$
$\cup (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$.

(1.16) Exercise [Poretsky]. Given two sets X and Y, prove that $X = \varnothing$ if and only if $Y = X \triangle Y$.

(1.17) Exercise. Describe in words the sets $\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right)$ and $\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)$ where $\{A_1, A_2, \ldots, A_k, \ldots\}$ is any family of sets indexed by N. Also prove that the first set is a subset of the second.

(1.18) Exercise. Prove:

(a) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$;

(b) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$;

(c) $A \triangle A = \varnothing$;

(d) $\varnothing \triangle A = A$.

(1.19) Exercise. Let $\{A_\iota\}_{\iota \in I}$ and $\{B_\iota\}_{\iota \in I}$ be nonvoid families of sets. Prove that

(i) $$(\bigcup_\iota A_\iota) \triangle (\bigcup_\iota B_\iota) \subset \bigcup_\iota (A_\iota \triangle B_\iota).$$

Prove by an example that the inclusion may be proper. Can you assert anything about (i) if the $\cup$'s are changed to $\cap$'s?

(1.20) Exercise. For any sets A, B, and C, prove that

$$A \triangle B \subset (A \triangle C) \cup (B \triangle C),$$

and show by an example that the inclusion may be proper.

(1.21) Exercise. Let $\{M_n\}_{n=1}^{\infty}$ and $\{N_n\}_{n=1}^{\infty}$ be families of sets such that the sets N_n are pairwise disjoint. Define $Q_1 = M_1$ and $Q_n = M_n \cap (M_1 \cup \cdots \cup M_{n-1})'$ for $n = 2, 3, \ldots$ Prove that $N_n \triangle Q_n \subset \bigcup_{k=1}^{n} (N_k \triangle M_k)$ $(n = 1, 2, \ldots)$.

(1.22) Exercise. Consider an alphabet with a finite number of letters, say a, where $a > 1$. A *word* in this alphabet is a finite sequence of letters, not necessarily distinct. Two words are equal if and only if they have the same number of letters and if the letters are the same and in the same order. Consider all words of length l, where $l > 1$. How many words of length l have at least two repetitions of a fixed letter? How many have three such repetitions? In how many words of length l do there occur two specified distinct letters?

(1.23) Exercise.

(a) Let A be a finite set, and let $\nu(A)$ denote the number of elements of A: thus $\nu(A)$ is a nonnegative integer. Prove that

$$\nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B).$$

(b) Generalize this identity to $\nu(A \cup B \cup C)$ and to $\nu(A \cup B \cup C \cup D)$.

(c) A university registrar reported that the total enrollment in his university was 10,000 students. Of these, he stated, 2521 were married, 6471 were men, 3115 were over 21 years of age, 1915 were married men, 1873 were married persons over 21 years of age, and 1302 were married men over 21 years of age. Could this have been the case?

(d) Help the registrar. For a student body of 10,000 members, find positive integers for the categories listed in (c) that are consistent with the identity you found in (b).

(1.24) Exercise. Prove that in any Boolean ring we have the identities

(a) $x + x = 0$;

(b) $xy = yx$.

(1.25) Exercise. (a) Let B be the set of all positive integers that divide 30. For $x, y \in B$, let $x \vee y$ be the least common multiple of x and y, and let $x^* = \frac{30}{x}$. Prove that B is a Boolean algebra. Find an algebra of sets that represents B as in (1.14).

(b) Generalize (a), replacing 30 by any square-free positive integer.

(c) Generalize (b) by considering the set B of *all* square-free positive integers, defining $x \vee y$ as the least common multiple of x and y, $x \wedge y$ as the greatest common divisor of x and y, and $x \vartriangle y$ as $\frac{x \vee y}{x \wedge y}$. Show that B can be represented as a certain ring of sets but not as an algebra of sets.

§ 2. Relations and functions

In this section we take up the concepts of relation and function, familiar in several forms from elementary analysis. We adopt the currently popular point of view that relations and functions are indistinguishable from their graphs, *i.e.*, they are sets of ordered pairs. As in the case of sets, we content ourselves with a highly informal discussion of the subject.

(2.1) Definition. Let X and Y be sets. The *Cartesian product of X and Y* is the set $X \times Y$ of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

We write $(x, y) = (u, v)$ if and only if $x = u$ and $y = v$. Thus $(1, 2) \neq (2, 1)$ while $\{1, 2\} = \{2, 1\}$.

(2.2) Definition. A *relation* is any set of ordered pairs. Thus a relation is any set which is a subset of the Cartesian product of two sets. Observe that $\varnothing$ is a relation.

(2.3) Definitions. Let f be any relation. We define the *domain of f* to be the set $\operatorname{dom} f = \{x : (x, y) \in f \text{ for some } y\}$ and we define the *range of f* to be the set $\operatorname{rng} f = \{y : (x, y) \in f \text{ for some } x\}$. The symbol f^{-1} denotes the *inverse* of f: $f^{-1} = \{(y, x) : (x, y) \in f\}$.