現代控制理论及其应用论文集

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目录

分布参数系统最佳控制问题(I)

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本文是在1964—1965年间完成的。

应该特别指出的是：节4中的第一个例子，即关于梁振动问题，当时，是与七机部协作的一个任务课题的一份报告。而这个问题在1971年美国的 SIAM. Control 上载有 E. R. Barnes 的文章，内容和问题同样的，只是在具体问题方法上不同而已。

本文所用叠积分不等式方法，用来估计高阶无穷小男。这样便简化了一些繁杂的运算。

本文实际上是围绕着一类实际问题而进行的数学研究，其目的，在于提供某一类（如振动问题）控制问题的数学处理方法。当然，在数学上是属于多元函数变分学的范围。

引言

本文系研究半线性双曲型系统（二阶到高阶）的最佳控制问题，主要内容是建立极值条件——极大条件。所用方法，系采用古典变分中的乘子法并结合 РОЗОНОЭР 和 ПОНТРИЯГИН 的方法中的一些步骤。在所得结果中。除了建立了极大条件外，还顺便指出解对最佳控制的特别意义下的稳定性。

本文所讨论的内容，同时可以看成是具有约束条件的多元函数变分学。

另外，本文的第一部分是讨论右端不受限的情形，而第二部分，则是讨论右端受限的情形，这一情形，目前还很少讨论。

半线性双曲型分布参数系统最佳控制问题

一. 右端不受限的情况

§1. 二阶半线性双曲型系统柯西定解问题的最佳控制

1.1 问题的叙述. 设系统为

$$L[y]\equiv\frac{\partial^2 y}{\partial t^2}-A\frac{\partial^2 y}{\partial x^2}=f(x,t,y,u),\qquad(1\cdot1\cdot1)$$

其中

$$y=\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix},\quad A=\begin{bmatrix}a^2&0&\cdots&0\\ 0&a^2&0\cdots&0\\ \cdots&\cdots&\cdots&\cdots\\ 0&\cdots&\cdots&a^2\end{bmatrix}.\quad f=\begin{bmatrix}f^1\\ \vdots\\ f^n\end{bmatrix},\quad u=\begin{bmatrix}u^1\\ \vdots\\ u^r\end{bmatrix};$$

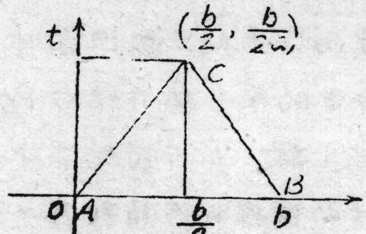
$f^i\in C(x|0\le x\le b)$, f^i关于u和y均有二阶连续偏导数, $u=u(x,t)$为取值于r维欧氏空间中某凸闭域U的控制函数, 它关于x为连续而关于t则有有限个一级不连续点。

系统(1·1·1)的容许解应满足下面的附加条件:

$$y(x,0)=\varphi(x),\qquad \partial y/\partial t\Big|_{t=0}=\psi(x)。$$
$$\varphi(x)\in C'(x),\qquad \psi(x)\in C(x).\qquad(1\cdot1\cdot2)$$

而解的定义域为

$$R:\begin{cases}0\le x\le b,\\ x=at,\\ x=-at+b.\end{cases}$$



(1·1·3)

在这些假定下, 对于任一容许控制函数

$$u=u(x,t),$$

总有一个连续解 $y=y(x,t)$ 其二级导数可积。这样的解是可

以逐步构造出来的。

给定一类似于古典变分中 Bolza 问题的泛函：

$$J=\chi\left(\frac{b}{2},\frac{b}{2a},y\left(\frac{b}{2},\frac{b}{2a}\right)\right)+\int_{\partial R}\omega(x,t,y(x,t))\,d\partial R-$$

$$-\iint_R f^0(x,t,y(x,t),u(x,t))\,dR。\qquad (1.1.4)^*$$

这样，我们便可提出如下的最佳控制问题：

在容许控制函数类中，寻求某一控制函数，使得相应的满足(1.1.1)——(1.1.3)的解，能使泛函(1.1.4)到达最小值，这样的控制称为最佳控制，这样的解称为最佳解。

1.2. 化为带偏导数的二重积分的变分问题

设 $\psi_i(x,t)$ 为关于 x 和 t 在区域 R 上（对 t 几乎处处有）有一级偏导数的可和函数，且其二阶导数可和，我们叫它为乘子（$\psi_i(x,t)$，$i=1,2,\cdots,n$）。

我们作一类似于 Hilbert 不变积分的新的泛函，而此泛函与原泛函等价，作法如下：

$$J^*=\chi\left(\frac{b}{2},\frac{b}{2a},y\left(\frac{b}{2},\frac{b}{2a}\right)\right)+\int_{\partial R}\omega(x,t,y(x,t))\,d\partial R$$

$$-\iint_R f^0(x,t,y(x,t),u(x,t))\,dR+\iint_R\left(\psi(x,t),\left[y_{tt}-Ay_{xx}-f(x,t,y,u)\right]\right)dR.\qquad(1.2.1)$$

求此泛函的极小值显然与求原来泛函极小值是一样的，这样就将原来问题转为二重积分的变分问题，在上述的那些假定和条件下。

下面，我们将进行泛函数变易的推导。

我们以 ΔJ^* 记泛函数变易：

$$\Delta J^*=J^*[y+\Delta y,\ u+\Delta u]-J^*[y,u]。$$

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* (1.1.4)的 χ，ω 均为对 y 为二次可导，f^0 则与 f^s 一样。

此为试读，需要完整PDF请访问： www.ertongbook.com

设 $u=u(x,t)$ 为最佳控制，$y(x,t)$ 为相应的最佳解，$u^*(x,t)=u(x,t)+\Delta u(x,t)$ 为变动控制，其相应的变动解为 $y^*(x,t)=y(x,t)+\Delta y(x,t)$。于是

$$\Delta J^*=\chi\left(\frac{b}{2},\frac{b}{2a},y^*\left(\frac{b}{2},\frac{b}{2a}\right)\right)-\chi\left(\frac{b}{2},\frac{b}{2a},y\left(\frac{b}{2},\frac{b}{2a}\right)\right)+$$
$$+\int_{\partial R}\left[\omega(x,t,y^*(x,t))-\omega(x,t,y(x,t))\right]d\partial R+$$
$$+\iint_R\Big(\psi(x,t),\Big(\left[y^*_{tt}-Ay^*_{xx}-f(x,t,y^*(x,t),u^*(x,t))\right]-$$
$$-\left[y_{tt}-Ay_{xx}-f(x,t,y,u)\right]\Big)\Big)dR-$$
$$-\iint_R\left[f^0(x,t,y^*,u^*)-f^0(x,t,y,u)\right]dR. \qquad (1.2.2)$$

为了以后的需要，我们可将 (1.2.2) 改写为如下的形式

$$\Delta J^*=\chi\left(\frac{b}{2},\frac{b}{2a},y^*\left(\frac{b}{2},\frac{b}{2a}\right)\right)-\chi\left(\frac{b}{2},\frac{b}{2a},y\left(\frac{b}{2},\frac{b}{2a}\right)\right)+$$
$$+\int_{\partial R}\left[\omega(x,t,y^*(x,t))-\omega(x,t,y(x,t))\right]d\partial R+$$
$$+\iint_R\Big\{\Big[\sum_{s=1}^n\psi_s(x,t)\Big(y^{s*}_{tt}-a^2y^{s*}_{xx}-f^s(x,t,y^*(x,t),u^*(x,t))\Big)\Big]-$$
$$-\Big[\sum_{s=1}^n\psi_s(x,t)\Big(y^s_{tt}-a^2y^s_{xx}-f^s(x,t,y(x,t),u(x,t))\Big)\Big]-$$
$$+\Big[f^0(x,t,y^*(x,t),u^*)-f^0(x,t,y(x,t),u(x,t))\Big]\Big\}dR.$$
$$(1.2.3)$$

记

$$\mathscr{H}=\sum_{s=1}^n\psi_s(x,t)\left(y^s_{tt}-a^2y^s_{xx}-f^s(x,t,y(x,t),u(x,t))\right)-$$
$$-f^0(x,t,y(x,t),u(x,t)). \qquad (1.2.4)$$

则有

$$\frac{\partial\mathscr{H}}{\partial y^s_{tt}}=\psi_s(x,t),\qquad \frac{\partial\mathscr{H}}{\partial y^s_{xx}}=-a^2\psi_s(x,t),$$

$$\frac{\partial\mathscr{H}}{\partial y^s_{tt}}\Delta y^s_{tt}=\frac{\partial}{\partial t}\left[\frac{\partial\mathscr{H}}{\partial y^s_{tt}}\Delta y^s_t\right]-\frac{\partial}{\partial t}\left[\frac{\partial\mathscr{H}}{\partial y^s_{tt}}\right]\Delta y^s_t=$$

$$-\frac{\partial}{\partial t}\left[\frac{\partial \mathscr{H}}{\partial y^s_{tt}}\Delta y^s_t\right]-\left[\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left[\frac{\partial \mathscr{H}}{\partial y^s_{tt}}\right]\Delta y^s\right)-\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left[\frac{\partial \mathscr{H}}{\partial y^s_{tt}}\right]\right)\Delta y^s\right];$$

$$\frac{\partial \mathscr{H}}{\partial y^s_{xx}}\Delta y^s_{xx}=\frac{\partial}{\partial x}\left[\frac{\partial \mathscr{H}}{\partial y^s_{xx}}\Delta y^s_x\right]-\frac{\partial}{\partial x}\left[\frac{\partial \mathscr{H}}{\partial y^s_{xx}}\right]\Delta y^s_x=$$

$$-\frac{\partial}{\partial x}\left[\frac{\partial \mathscr{H}}{\partial y^s_{xx}}\Delta y^s_x\right]-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{\partial \mathscr{H}}{\partial y^s_{xx}}\right]\Delta y^s\right)+\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{\partial \mathscr{H}}{\partial y^s_{xx}}\right]\right)\Delta y^s;$$

$$(1.2.5)$$

由于引进记号(1.2.4)。则(1.2.3)可写成

$$\Delta J^*=\sum_{s=1}^{n}\frac{\partial \chi}{\partial y^s}\Delta y^s\Big|_{\left(\frac{b}{2},\frac{b}{2a}\right)}+\sum_{s=1}^{n}\int_{\partial R}\frac{\partial \omega(x,t,y(x,t))}{\partial y^s}\Delta y^s d\partial R+$$

$$+\iint_R\sum_{s=1}^{n}\frac{\partial \mathscr{H}}{\partial y^s}\Delta y^s dR+\iint_R\sum_{s=1}^{n}\left[\frac{\partial \mathscr{H}}{\partial y^s_{tt}}\Delta y^s_{tt}+\frac{\partial \mathscr{H}}{\partial y^s_{xx}}\Delta y^s_{xx}\right]dR+$$

$$+\iint_R\sum_{s,k=1}^{n}\frac{\partial^2 \mathscr{H}}{\partial y^s\partial y^k}\Delta y^s\Delta y^k\Big|_{\theta_1}dR+\sum_{k,s=1}^{n}\frac{\partial^2\chi}{\partial y^s\partial y^k}\Delta y^s\Delta y^k\Big|_{\left(\frac{b}{2},\frac{b}{2a}\right)}+$$

$$+\sum_{s,k=1}^{n}\int_{\partial a}\frac{\partial^2\omega}{\partial y^s\partial y^k}\Delta y^s\Delta y^k d\partial R$$

$$+\iint_R\Big\{\sum_{s=1}^{n}\psi_s(x,t)\left[f^s(x,t,y(x,t),u^*)-f^s(x,t,y(x,t),u(x,t))\right]+$$

$$+\left[f^0(x,t,y(x,t),u^*)-f^0(x,t,y(x,t),u(x,t))\right]\Big\}dR.$$

$$(1.2.6)$$

将由(1.2.5)公式所得的结果,代入(1.2.6),则得

$$\Delta J^*=\iint_R\Big\{\sum_{s=1}^{n}\Big[(\psi_s)_{tt}-(a^2\psi_s)_{xx}-\sum_{\alpha=1}^{n}\frac{\partial f^\alpha}{\partial y_s}\psi_\alpha-\frac{\partial f^0}{\partial y_s}\Delta y^s\Big]\Big\}dR$$

$$-\iint_R\Big\{\sum_{s=1}^{n}\psi_s(x,t)\left[f^s(x,t,y(x,t),u^*)-f^s(x,t,y(x,t),u(x,t))\right]+$$

$$+\left[f^0(x,t,y(x,t),u^*)-f^0(x,t,y(x,t),u(x,t))\right]\Big\}dR+$$

$$+\iint_R\frac{\partial}{\partial t}\Big[\sum_{s=1}^{n}\psi_s\Delta y^s_t-\Big(\frac{\partial}{\partial t}\sum_{s=1}^{n}\psi_s\Big)\Delta y^s\Big]dR+$$

二阶偏导中之改变量均为$(y+\theta\Delta y)$。

$$+\iint_R \frac{\partial}{\partial x}\left[\sum_{s=1}^{n}(-a^2\psi_s)\Delta y^s_x+\left(\frac{\partial}{\partial x}\sum_{s=1}^{n}a^2\psi_s\right)\Delta y^s\right]dR+$$

$$+\int_{\partial R}\sum_{s=1}^{n}\frac{\partial\omega}{\partial y_s}\Delta y^s d\partial R+\sum_{s=1}^{n}\frac{\partial\chi}{\partial y_s}\Delta y^s\Big|_{\left(\frac{b}{2},\frac{b}{2a}\right)}+$$

$$+\sum_{K,s=1}^{n}\frac{\partial^2\chi}{\partial y^s\partial y^K}\Big|_{\theta_0}\Delta y^s\Delta y^K\Big|_{\left(\frac{b}{2},\frac{b}{2a}\right)}+\int_{\partial R}\sum_{s,K=1}^{n}\frac{\partial^2\omega}{\partial y^s\partial y^K}\Delta y^s\Delta y^K\Big|_{\theta_1}d\partial R+$$

$$+\iint_R\sum_{K,s=1}^{n}\psi_s(x,t)\sum_{j=1}^{n}\frac{\partial^2 f^s}{\partial y^K\partial y^j}\Delta y^K\Delta y^j\Big|_{\theta_2}dR+$$

$$+\iint_R\sum_{K,j=1}^{n}\frac{\partial^2 f^0}{\partial y^K\partial y^j}\Delta y^K\Delta y^j\Big|_{\theta_2}dR$$

$$-\iint_R\sum_{s,r=1}^{n}\psi_s\left[\frac{\partial f^s(x,t,y,u^*)}{\partial y^r}-\frac{\partial f^0(x,t,y,u)}{\partial y^r}\right]\Delta y^r dR$$

$$-\iint_R\sum_{r=1}^{n}\left[\frac{\partial f^0(x,t,y,u^*)}{\partial y^r}-\frac{\partial f^0(x,t,y,u)}{\partial y^r}\right]\Delta y^r dR,\quad(1.2.7)$$

（注意到(1.1.1)中$\frac{\partial f^K}{\partial y^r}$，满足Lipschitz条件）

现在再将(1.2.7)中的第三、第四两个积分详细计算之：

$$\iint_R\left\{\frac{\partial}{\partial t}\left[\sum_{s=1}^{n}\left(\psi_s\frac{\partial\Delta y^s}{\partial t}-\frac{\partial\psi_s}{\partial t}\Delta y^s\right)\right]-a^2\frac{\partial}{\partial x}\left[\sum_{s=1}^{n}\left(\psi_s\frac{\partial\Delta y^s}{\partial x}-\frac{\partial\psi_s}{\partial x}\Delta y^s\right)\right]\right\}dR=$$

$$=\int_{\partial R}\left\{\left[\sum_{s=1}^{n}\left(\psi_s\frac{\partial\Delta y^s}{\partial t}-\frac{\partial\psi_s}{\partial t}\Delta y^s\right)\right]\cos(n,t)-a^2\left[\sum_{s=1}^{n}\left(\psi_s\frac{\partial\Delta y^s}{\partial x}-\frac{\partial\psi_s}{\partial x}\Delta y^s\right)\right]\cos(n,x)\right\}d\partial R$$

$$=\int_{AB+BC+CA}\left[\sum_{s=1}^{n}\left(\psi_s\frac{\partial\Delta y^s}{\partial t}-\frac{\partial\psi_s}{\partial t}\Delta y^s\right)\right]\cos(n,t)\,d\partial R-$$

$$-a^2\int_{AB+BC+CA}\left[\sum_{s=1}^{n}\left(\psi_s\frac{\partial\Delta y^s}{\partial x}-\frac{\partial\psi_s}{\partial x}\Delta y^s\right)\right]\cos(n,x)\,d\partial R=$$

$$=2a^2\sum_{s=1}^{n}\psi_s\left(\frac{b}{2},\frac{b}{2a}\right)\Delta y^s\left(\frac{b}{2},\frac{b}{2a}\right)+2a^2\int_{CA}\sum_{s=1}^{n}\frac{\partial\psi_s}{\partial x}\Delta y^s dx-$$

$$-2a^2\int_{BC}\sum_{s=1}^{n}\frac{\partial\psi_s}{\partial x}\Delta y^s dx,\quad(1.2.8)$$

将计标所得结果（1.2.8）代入（1.2.7）并整理之，得：

$$\Delta J^* = \int_{\partial R} \sum_{s=1}^{n} \frac{\partial \omega}{\partial y^s} \Delta y^s d\partial R + 2a \int_{CA} \sum_{s=1}^{n} \frac{\partial \psi_s}{\partial x} \Delta y^s dx -$$

$$- \int_{BC} 2a \sum_{s=1}^{n} \frac{\partial \psi_s}{\partial x} \Delta y^s dx + \left[\sum_{s=1}^{n} \left(\frac{\partial \chi}{\partial y^s} + 2a \psi_s \right) \Delta y^s \right]_{\left(\frac{b}{2}, \frac{b}{2a}\right)} +$$

$$+ \iint_R \left\{ \sum_{s=1}^{n} \left[(\psi_s)_{tt} - (a^2 \psi_s)_{xx} - \sum_{\alpha=1}^{n} \frac{\partial f^\alpha}{\partial y^s} \psi_\alpha - \frac{\partial f^0}{\partial y^s} \right] \right\} \Delta y^s dR +$$

$$- \iint_R \left\{ \sum_{s=1}^{n} \psi_s \left[f^s(x,t,y,u^*) - f^s(x,t,y,u) \right] + \left[f^0(x,t,y,u^*) - f^0(x,t,y,u) \right] \right\} dR +$$

$$+ \int_{\partial R} \sum_{k,s=1}^{n} \frac{\partial^2 \omega}{\partial y^k \partial y^s} \bigg|_{\theta_1} \Delta y^k \Delta y^s d\partial R + \left(\sum_{k,s=1}^{n} \frac{\partial^2 \chi}{\partial y^s \partial y^k} \bigg|_{\theta_0} \Delta y^s \Delta y^k \right) \bigg|_{\left(\frac{b}{2}, \frac{b}{2a}\right)} +$$

$$+ \iint_R \left[\sum_{s=1}^{n} \psi_s \sum_{k,j=1}^{n} \frac{\partial^2 f^s}{\partial y^k \partial y^j} \bigg|_{\theta_2} \Delta y^k \Delta y^j + \sum_{k,j=1}^{n} \frac{\partial^2 f^0}{\partial y^k \partial y^j} \bigg|_{\theta_2} \Delta y^k \Delta y^j \right] dR$$

$$- \iint_R \sum_{r,s=1}^{n} \psi_s \left[\frac{\partial f^s(x,t,y,u^*)}{\partial y^r} - \frac{\partial f^s(x,t,y,u)}{\partial y^r} \right] \Delta y^r dR -$$

$$- \iint_R \sum_{r=1}^{n} \left[\frac{\partial f^0(x,t,y,u^*)}{\partial y^r} - \frac{\partial f^0(x,t,y,u)}{\partial y^r} \right] \Delta y^r dR, \qquad (1.2.9)$$

现在，我们引进共轭系统（即乘子 $\psi_s(x,t)$ 所满足之方程及有关的附加条件）

$$L^*[\psi] \equiv \psi_{tt} - (A\psi)_{xx} = \frac{\partial f}{\partial y} \psi \qquad (1.2.10)$$

或

$$L^*[\psi_s] = (\psi_s)_{tt} - (a^2 \psi_s)_{xx} = \sum_{\alpha=1}^{n} \frac{\partial f^\alpha}{\partial y^s} \psi_\alpha + \frac{\partial f^0}{\partial y^s},$$

$$(s = 1, 2, \cdots, n)$$

$$(\psi_0 = -1)。$$

附加条件为：

$$\frac{\partial \psi_s}{\partial x}=\frac{1}{2a}\frac{\partial \omega}{\partial y^s}, \quad (x,t)\in \overline{CA}.$$

$$\frac{\partial \psi_s}{\partial x}=-\frac{1}{2a}\frac{\partial \omega}{\partial y^s}, \quad (x,t)\in \overline{BC}, \tag{1.2.11}$$

$$\psi_s(x,t)_{(\frac{b}{2},\frac{b}{2a})}=-\frac{1}{2a}\frac{\partial \omega}{\partial y^s}{}_{(\frac{b}{2},\frac{b}{2a})},$$

$$(s=1,2,\cdots,n)$$

而(1.2.10)－(1.2.11) 的解是适定的且为绝对连续的。

因此，倘若我们的乘子就是由（1.2.10)—(1.2.11)确定的话，那么(1.2.9) 便变为

$$\begin{aligned}\Delta J^*=&-\iint_R\Big\{\sum_{s=1}^n\psi_s(x,t)\big[f^s(x,t,y(x,t),u^*)-f^s(x,t,y(x,t),u(x,t))\big]+\\&+f^0(x,t,y(x,t),u^*)-f^0(x,t,y(x,t),u(x,t))\Big\}dR+\\&+\sum_{s,k=1}^n\frac{\partial^2\chi}{\partial y^s\partial y^k}\Big|_{\theta_0}\Delta y^s\Delta y^k\Big|_{(\frac{b}{2},\frac{b}{2a})}+\sum_{s,k=1}^n\int_{\partial R}\frac{\partial^2\omega}{\partial y^s\partial y^k}\Big|_{\theta_1}\Delta y^s\Delta y^k d\partial R\\&+\iint_R\Big[\sum_{s=1}^n\psi_s\sum_{k,j=1}^n\frac{\partial^2 f^s}{\partial y^k\partial y^j}\Big|_{\theta_2}\Delta y^k\Delta y^j+\sum_{k,j=1}^n\frac{\partial^2 f^0}{\partial y^k\partial y^j}\Big|_{\theta_2}\Delta y^k\Delta y^j\Big]dR\\&-\iint_R\sum_{r,s=1}^n\psi_s\Big[\frac{\partial f^s(x,t,y,u^*)}{\partial y^r}-\frac{\partial f^s(x,t,y,u)}{\partial y^r}\Big]\Delta y^r dR-\\&-\iint_R\sum_{r=1}^n\Big[\frac{\partial f^0(x,t,y,u^*)}{\partial y^r}-\frac{\partial f^0(x,t,y,u)}{\partial y^r}\Big]\Delta y^r dR。\end{aligned}\tag{1.2.12}$$

由于，若 $u=u(x,t)$ 为最佳控制函数，则必有 $\Delta J^*\geq 0$。因此，估计（1.2.12）中的 Δy^s 的界，就成为我们最注意的事了。因为倘若能估出（1.2.12）中自第二项以后的值，均为高阶无穷小量的话，则显然可由（1.2.12）中第一个积分中的被积函数，得到最大原则。这样，下面我们便来进行 Δy^s 的估计。

1.3. 解的增长估计

系统（1.1.1）的增量方程及相应的初始条件为：

$$\frac{\partial^2 \Delta y_s}{\partial t^2} - A\frac{\partial^2 \Delta y_s}{\partial x^2} = f^s(x,t,y^*,u^*) - f^s(x,t,y,u), \quad (1.3.1)$$

$$\Delta y_s(x,0) = 0, \qquad \left.\frac{\partial \Delta y_s}{\partial t}\right|_{t=0} = 0, \quad (1.3.2)$$

根据达朗培尔公式，可得如下的积分表达式：

$$\Delta y_s = \frac{1}{2a}\int_0^t \int_{x-a(t-t')}^{x+a(t-t')} \left[f^s(x',t',y^*,u^*) - f^s(x',t',y,u)\right]dx'dt', \quad (1.3.3)$$

$$\|\Delta y(x,t)\| \leq \frac{M}{2a}\int_0^x \int_0^t \left[\|\Delta y(\xi,\eta)\| + \|\Delta u(\xi,\eta)\|\right] d\xi\, d\eta.^* \quad (1.3.4)$$

由(1.3.4)，我们可以推出

$$\|\Delta y(x,t)\| \leq C\int_0^x \int_0^t \|\Delta u(\xi,\eta)\| d\xi\, d\eta. \quad (1.3.5)$$

这个事实证明如下：

令

$$Y_0(x,t) = \frac{M}{2a}\int_0^x \int_0^t \|\Delta u(\xi,\eta)\| d\xi\, d\eta. \quad (1.3.6)$$

则(1.3.4)可写为：

$$\|\Delta y(x,t)\| \leq Y_0(x,t) + \int_0^x \int_0^t \frac{M}{2a}\|\Delta y(\xi,\eta)\| d\xi\, d\eta. \quad (1.3.7)$$

利用逐步逼近方法，作

$$\|\Delta y(x,t)\|_{(0)} = Y_0(x,t), \quad (1.3.8_0)$$

$$\|\Delta y(x,t)\|_{(1)} = Y_0(x,t) + \int_0^x \int_0^t \frac{M}{2a} Y_0(\xi,\eta) d\xi\, d\eta. \quad (1.3.8_1)$$

$$\|\Delta y(x,t)\|_{(2)} = Y_0(x,t) + \int_0^x \int_0^t \frac{M}{2a} Y_0(\xi,\eta) d\xi\, d\eta +$$

---

* 这里，为方便计，可将三角形域扩大为矩形域，只要将原来定义在Δ域上的函数，当其在Δ形外和矩形内使其为零即可。

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$$+\int_0^x\int_0^t \frac{M}{2a}\int_0^\xi\int_0^\eta \frac{M}{2a} Y_0(\xi_1,\eta_1)\,d\xi_1 d\eta_1\, d\xi\, d\eta =$$

$$= Y_0(x,t)+\int_0^x\int_0^t \frac{M}{2a} Y_0(\xi,\eta)\,d\xi\,d\eta +$$

$$+\int_0^x\int_0^t \frac{M}{2a}\,d\xi_1 d\eta_1 \int_{\xi_1}^x\int_{\eta_1}^t \frac{M}{2a} Y(\xi_1,\eta_1)\,d\xi\,d\eta =$$

$$= Y_0(x,t)+\int_0^x\int_0^t \frac{M}{2a} Y_0(\xi,\eta)\,d\xi\,d\eta+\int_0^x\int_0^t \frac{M}{2a} Y_0(\xi_1,\eta_1)\,d\xi_1 d\eta_1 \int_{\xi_1}^x\int_{\eta_1}^t \frac{M}{2a}\,d\xi\,d\eta =$$

$$= Y_0(x,t)+\int_0^x\int_0^t \frac{M}{2a} Y_0(\xi,\eta)\,d\xi\,d\eta+\int_0^x\int_0^t \frac{M}{2a} Y_0(\xi,\eta)\int_\xi^x\int_\eta^t \frac{M}{2a}\,d\xi_1 d\eta_1\, d\xi\,d\eta =$$

$$= Y_0(x,t)+\frac{M}{2a}\int_0^x\int_0^t Y_0(\xi,\eta)\left[1+\int_\xi^x\int_\eta^t \frac{M}{2a}\,d\xi_1 d\eta_1\right] d\xi\,d\eta. \qquad (1.3.8_2)$$

. . . .

$$\|\Delta y(x,t)\|_{(n+1)} = Y_0(x,t)+\frac{M}{2a}\int_0^x\int_0^t Y_0(\xi,\eta)\left[\sum_{K=0}^{n}\frac{1}{(K!)^2}\left(\int_\xi^x\int_\eta^t \frac{M}{2a}\,d\xi_1 d\eta_1\right)^K\right] d\xi\,d\eta. \qquad (1.3.8_{n+1})$$

于是显然有

$$\|\Delta y(x,t)\|_{(n)} \leq Y_0(x,t)+\frac{M}{2a}\int_0^x\int_0^t Y_0(\xi,\eta)\, e^{\int_\xi^x\int_\eta^t \frac{M}{2a}d\xi_1 d\eta_1} d\xi\,d\eta, \qquad (1.3.9)$$

也很容易看出当 $(n)\to\infty$ 时，$\|\Delta y(x,t)\|_{(n)}$ 是一致收敛的。于是有

$$\lim_{(n)\to\infty}\|\Delta y(x,t)\|_{(n)} = \|\Delta y(x,t)\|$$

从而得到

$$\|\Delta y(x,t)\| \leq Y_0(x,t)+\frac{M}{2a}\int_0^x\int_0^t Y_0(\xi,\eta)\, e^{\int_\xi^x\int_\eta^t \frac{M}{2a}d\xi_1 d\eta_1} d\xi\,d\eta. \qquad (1.3.10)$$

再利用狄利赫里公式，能得

$$\int_0^x\int_0^t Y_0(\xi,\eta)\, e^{\int_\xi^x\int_\eta^t \frac{M}{2a}d\xi_1 d\eta_1} d\xi\,d\eta \leq C\int_0^x\int_0^t \|\Delta U(\xi,\eta)\|\,d\xi\,d\eta. \qquad (1.3.11)$$

将(1.3.11)代入(1.3.10)，即得

$\|\Delta u(\xi,\eta)\| d\xi d\eta$ *

1.4. 最佳控制必要条件

根据前几段的分析，我们能得到如下的结论：

定理1 设容许控制 $u=u(x,t)$ 为(1.1.1)—(1.1.4)的最佳控制，则对于满足(1.1.10)—(1.2.11)的绝对连续向量函数 $\psi(x,t)=(\psi_0(x,t),\cdots,\psi_n(x,t))$，$(\psi_0=-1)$。使得最大条件

$$(\psi(x,t), f(x,t,y(x,t),u(x,t)))(=)\max_{u\in U}(\psi(x,t), f(x,t,y(x,t),u)) \quad \text{成立。} \tag{1.4.1}$$

证：若不然，即当 $u=u(x,t)$ 时，而(1.4.1)不成立，则必在 R 上有某一点 (x',t') 使(1.4.1)不成立，因而必在 (x',t') 某一测度为甚小的 ε 邻域上不成立；因此，由(1.2.12)和(1.3.5)(用一次许瓦兹不等式)得知存在 $\alpha>0$，使

$$\Delta J^* = -\alpha\, mes(\varepsilon) + o(mes(\varepsilon)) < 0, \tag{1.4.2}$$

而这是与最佳控制的假设是矛盾的，于是定理I得证。

从 Δy 的估计式中，显然可以得解对最佳控制关于积分区域小测度的稳定性。

另外，对于Laplace双曲型二阶半线性系统，像

$$\frac{\partial^2 y}{\partial x\partial t} + B(x,t)\frac{\partial y}{\partial x} + C(x,t)\frac{\partial y}{\partial t} = f(x,t,y,u),$$

$$\frac{\partial^2 y}{\partial x\partial t} + f(x,t,y,y_t,y_x,u)=0,$$

均可用同样方法来处理如上所述的最佳控制问题，这里不一一予以讨论。在下一节中，我们只将高阶系统加以讨论。

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* 这个方法同样推广了Gronwall不等式，而与[8]的证法不同

§2. 一般形式高阶半线性双曲系统的最佳控制问题

2·1. 问题的叙述　设系统为

$$\frac{\partial^{\alpha} y}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}+f(t,x_1,\cdots,x_n,y,y_t,y_t^2,\cdots,y_t^{\nu_0},y_{x_1}^{\nu_1},\cdots y_{x_n}^{\nu_n}\cdots u(t,x_1,\cdots,x_n))=0$$

(2·1·1)*

$$(\alpha_0+\alpha_1+\cdots+\alpha_n=\alpha.\quad 0\leq t\leq T;\quad 0\leq x_i\leq b_i),$$

附加条件为：

$$y(0,x_1,\cdots,x_n)=\varphi_0(x_1,x_2,\cdots,x_n),\qquad (2\cdot1\cdot2)$$

$$y(t,x_1,\cdots,x_{j-1},0,x_{j+1},\cdots,x_n)=\varphi_j(t,x_1,\cdots,x_{j-1},\cdots,x_n)\qquad(2\cdot1\cdot3)$$

其中φ_0. φ_j对每一个变元有相应的导数．u对t有有限个一级间断点，f关于x连续关于y. y_t, $\cdots$, $y_{x_n}^{\alpha_n}$, u有连续二阶偏导数。

给定泛函

$$J=\int_0^t\int_0^{x_1}\cdots\int_0^{x_n}f^0(\tau,x,y,y_\tau,\cdots,y_t^{\alpha_0},\cdots,y_{x_n}^{\alpha_n},u(\tau,\bar{x}))\,d\tau d\bar{x}_1\cdots dx_n.$$

(2·1·4)

最佳问题为

$$(B)\begin{cases}(2\cdot1\cdot1),\\(2\cdot1\cdot2),\\(2\cdot1\cdot3),\\(2\cdot1\cdot4)=min.\end{cases}$$

2·2. 几个辅助公式的推导．

设$\mathscr{H}(t,x_1,\cdots,x_n,y,y_t^{\nu_0}\cdots y_{x_1}^{\nu_1}\cdots y_{x_n}^{\nu_n},\cdots u(x,t),\psi(x,t))$为关于$x$连续关于$y$, y_t, $\cdots y_x$, $\cdots y_{x_1}^{\nu_n}$有连续二阶偏导数，其中

$$y_t'=\frac{\partial y}{\partial t},\quad y_t^2=\frac{\partial^2 y}{\partial t^2},\ \cdots,\ y_{x_n}^{\nu_n}=\frac{\partial^{\nu_n}y}{\partial x_n^{\nu_n}}\ \cdots.$$

---

* f中的偏导数的阶数小于α。

下面是几个辅助公式的推导：

$$\frac{\partial \mathcal{H}}{\partial y_{x_1}}\Delta y_{x_1}=\frac{\partial}{\partial x_1}\left[\frac{\partial \mathcal{H}}{\partial y_{x_1}}\Delta y\right]-\frac{\partial}{\partial x_1}\left[\frac{\partial \mathcal{H}}{\partial y_{x_1}}\right]\Delta y. \qquad (2\cdot2\cdot1)$$

. . . .

$$\frac{\partial \mathcal{H}}{\partial y_{x_n}}\Delta y_{x_n}=\frac{\partial}{\partial x_n}\left[\frac{\partial \mathcal{H}}{\partial y_{x_n}}\Delta y\right]-\frac{\partial}{\partial x_n}\left[\frac{\partial \mathcal{H}}{\partial y_{x_n}}\right]\Delta y. \qquad (2\cdot2\cdot2)$$

$$\frac{\partial \mathcal{H}}{\partial y_{x_i x_K}}\Delta y_{x_i x_K}=\frac{\partial}{\partial x_i}\left[\frac{\partial \mathcal{H}}{\partial y_{x_i x_K}}\Delta y_{x_K}\right]-\frac{\partial}{\partial x_K}\left[\frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{H}}{\partial y_{x_i x_K}}\right)\Delta y\right]+$$

$$+\frac{\partial^2}{\partial x_i \partial x_K}\left[\frac{\partial \mathcal{H}}{\partial y_{x_i x_K}}\right]\Delta y; \quad (i, K=1.2.\cdots, n), \quad (2\cdot2\cdot3)$$

. . . .

$$\frac{\partial \mathcal{H}}{\partial\left(\frac{\partial^K y}{\partial x_s^{\alpha_s}\cdots\partial x_p^{\alpha_p}}\right)}\ \frac{\partial^K \Delta y}{\partial x_s^{\alpha_s}\cdots\partial x_p^{\alpha_p}}=$$

$$=\frac{\partial}{\partial x_s}\left[\frac{\partial \mathcal{H}}{\partial\left(\frac{\partial^K y}{\partial x_s^{\alpha_s}\cdots\partial x_p^{\alpha_p}}\right)}\ \frac{\partial^{K-1}\Delta y}{\partial x_s^{\alpha_s-1}\cdots\partial x_p^{\alpha_p}}\right]-\frac{\partial}{\partial x_s}\left(\frac{\partial \mathcal{H}}{\partial\left(\frac{\partial^K y}{\partial x_s^{\alpha_s}\cdots\partial x_p^{\alpha_p}}\right)}\right)\frac{\partial^{K-1}\Delta y}{\partial x_s^{\alpha_s-1}\cdots\partial x_p^{\alpha_p}}$$

$$=\frac{\partial}{\partial x_s}\left[\mathcal{H}_{y^K_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}}\Delta y^{K-1}_{x_s^{\alpha_s-1}\cdots x_p^{\alpha_p}}\right]-\frac{\partial}{\partial x_s}\left[\frac{\partial}{\partial x_s}\left(\mathcal{H}_{y^K_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}}\right)\Delta y^{K-2}_{x_s^{\alpha_s-2}\cdots x_p^{\alpha_p}}\right]+$$

$$+\cdots+(-1)^{(\alpha_s-1)}\frac{\partial}{\partial x_s}\left\{\left[\frac{\partial}{\partial x_s}\frac{\partial}{\partial x_s}\cdots\frac{\partial}{\partial x_s}\left(\mathcal{H}_{y^K_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}}\right)\right]\Delta y^{K-\alpha_s}_{x_{s+1}^{\alpha_{s+1}}\cdots x_p^{\alpha_p}}+\right.$$

$$\left.+(-1)^{\alpha_s}\frac{\partial^{\alpha_s}}{\partial x_s^{\alpha_s}}\left(\mathcal{H}_{y^K_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}}\right)\Delta y^{K-\alpha_s}_{x_{s+1}^{\alpha_{s+1}}\cdots x_p^{\alpha_p}}\right\}=$$

$$=\frac{\partial}{\partial x_s}\left[\mathcal{H}_{y^K_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}}\Delta y^{K-1}_{x_s^{\alpha_s-1}\cdots x_p^{\alpha_p}}\right]\cdots(-1)^{\alpha_s-1}\frac{\partial}{\partial x_s}\left[\frac{\partial^{\alpha_s-1}}{\partial x_s^{\alpha_s-1}}\left(\mathcal{H}_{y^K_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}}\right)\Delta y^{K-\alpha_s}_{x_{s+1}^{\alpha_{s+1}}\cdots x_p^{\alpha_p}}\right]$$

$$+(-1)^{\alpha_s}\frac{\partial}{\partial x_{s+1}}\left[\frac{\partial^{\alpha_s}}{\partial x_s^{\alpha_s}}\left(\mathcal{H}_{y^K_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}}\right)\Delta y^{K-\alpha_s-1}_{x_{s+1}^{\alpha_{s+1}-1}\cdots x_p^{\alpha_p}}\right]+\cdots+$$

$$+(-1)^{\alpha_s+\alpha_s+1}\frac{\partial}{\partial x_{s+1}}\left\{\frac{\partial^{\alpha_{s+1}+\alpha_s-1}}{\partial x_s^{\alpha_s}\cdots\partial x_{s+1}^{\alpha_{s+1}-1}}\left(\mathscr{H}y^{K}_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}\right)\Delta y^{K-\alpha_s-\alpha_{s+1}}_{x_{s+2}^{\alpha_{s+2}}\cdots x_p^{\alpha_p}}\right\}+$$

$$+\cdots+(-1)^{\alpha_s+\cdots+\alpha_{p-1}}\frac{\partial}{\partial x_p}\left\{\frac{\partial^{\alpha_s+\cdots+\alpha_{p-1}}}{\partial x_s^{\alpha_s}\cdots\partial x_p^{\alpha_{p-1}}}\left(\mathscr{H}y^{K}_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}\right)\Delta y\right\}+$$

$$+(-1)^{\alpha_s+\cdots+\alpha_p}\frac{\partial^{\alpha_s+\cdots+\alpha_p}}{\partial x_s^{\alpha_s}\cdots\partial x_p^{\alpha_p}}\left(\mathscr{H}y^{K}_{x_s^{\alpha_s}\cdots x_p^{\alpha_p}}\right)\Delta y, \qquad (2\cdot2\cdot4)$$

（一般的，可将t^{α_0}再补上，此处未标在内）

$$\begin{pmatrix}K=1.2,\cdots,n; & \alpha_0+\alpha_s+\cdots+\alpha_{p-1}+\alpha_p=K, \\ S=1.2,\cdots,n; & p=1.2,\cdots,n.\end{pmatrix}$$

下面，我们将作新泛函，并计算其改变量。

2·3. 泛函改变量

作出与原泛函等价的新泛函

$$J^*=\int_0^t\int_0^{x_1}\cdots\int_0^{x_n}\left[f^\circ+(\psi, y_{t^{\alpha_0}x_1^{\alpha_1}\cdots x_n^{\alpha_n}}+f)\right]d\xi_n\cdots d\xi_1 d\tau, \quad (2.3.1)$$

于是

$$\Delta J^*=\int_0^t\int_0^{x_1}\cdots\int_0^{x_n}\Big\{\big[f^\circ(t,x,y+\Delta y, y_t+\Delta y_t,\cdots,u+\Delta u)-f^\circ(t,x,y,y_t\cdots u)\big]+$$

$$+\Big(\psi(t,x),\Big(\Big[\frac{\partial^\alpha(y+\Delta y)}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}+f(t,x,y+\Delta y,y_t+\Delta y_t\cdots,u+\Delta u)\Big]-$$

$$-\Big[\frac{\partial^\alpha y}{\partial t^{\alpha_0}\cdots\partial x_n^{\alpha_n}}+f(t,x,y,y_t,\cdots,u)\Big]\Big)\Big)\Big\}d\xi_n\cdots d\xi_1 d\tau, \quad (2\cdot3\cdot2)$$

我们将(2·3·1)中的被积函数看作为$\mathscr{H}$，那么(2·3·2)中的被积函数即为$\mathscr{H}$的改变量$\Delta\mathscr{H}$。于是按公式(2·2·1)—(2·2·4)计算$\Delta\mathscr{H}$，并整理之，得

$$\Delta J^*=\int_0^t\int_0^{x_1}\cdots\int_0^{x_n}\Big\{\frac{\partial^\alpha\psi}{\partial t^{\alpha_0}\cdots\partial x_n^{\alpha_n}}-\Big(\psi,\frac{\partial f(t,x,y,y_t\cdots u)}{\partial y}\Big)+\frac{\partial f^\circ(t,x,y\cdots u)}{\partial y}$$

$$-\frac{\partial}{\partial t}\Big(\frac{\partial f^\circ(t,x,y,y_t,\cdots,u)}{\partial y_t}\Big)-\frac{\partial}{\partial t}\Big(\psi,\frac{\partial f(t,x,y,y_t,\cdots,u)}{\partial y_t}\Big)-$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial f^0(t,x,y,y_t\cdots u)}{\partial y_x}\right)-\frac{\partial}{\partial x}\left(\psi,\frac{\partial f(t,x,y,y_\tau,\cdots u}{\partial y_x}\right)+\cdots+$$

$$+(-1)^{\alpha_0+\cdots+\alpha_p}\frac{\partial^{\alpha_0+\cdots+\alpha_p}}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_p^{\alpha_p}}\left(\frac{\partial f^0(t,x,y,y_t,\cdots u)}{\partial\left(\frac{\partial^k y(t,x)}{\partial t^{\alpha_0}\cdots\partial x_n^{\alpha_n}}\right)}\right)-$$

$$-(-1)^{\alpha_0+\cdots+\alpha_p}\frac{\partial^{\alpha_0+\cdots+\alpha_p}}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\left(\psi,\frac{\partial f(t,x,y,y_t,\cdots u)}{\partial\left(\frac{\partial^k y(t,x)}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_p}}\right)}\right)\Bigg\}\Delta y\,d\xi_1\cdots d\xi_n\,d\tau+$$

$$+\int_0^t\int_0^{x_1}\cdots\int_0^{x_n}\Bigg\{\frac{\partial}{\partial t}\left(\frac{\partial f^0}{\partial y_t},\Delta y\right)+\frac{\partial}{\partial t}\left(-\psi\frac{\partial f}{\partial y_t},\Delta y\right)+\left(\frac{\partial}{\partial x},\left(\frac{\partial f^0}{\partial y_x},\Delta y\right)\right)-$$

$$-\left(\frac{\partial}{\partial x},\left(\psi\frac{\partial f}{\partial y_x},\Delta y\right)\right)+\cdots+\left((-1)^{\alpha-1}\frac{\partial}{\partial x_p}\left[\frac{\partial^{\alpha-2}}{\partial t^{\alpha_0}\cdots\partial x_{p-1}^{\alpha_{p-1}}}\left(\psi,\frac{\partial f}{\partial\left(\frac{\partial^k y}{\partial t^{\alpha_0}\cdots\partial x_n^{\alpha_p}}\right)}\right)\right]\Delta y\right)+$$

$$+\int_0^t\int_0^{x_1}\cdots\int_0^{x_n}\Big[f^0(t,x,y,y_t,\cdots,u+\Delta u)-f^0(t,x,y,y_t,\cdots,u)+$$

$$+\big(\psi(x,t),f(t,x,y,y_t,\cdots,u+\Delta u)-f(t,x,y,y_t,\cdots,u)\big)\Big]d\xi_1\cdots d\xi_n\,d\tau+$$

$$+(***),\qquad(2\cdot3\cdot3)$$

其中（＊＊＊）表含有Δy，Δy_t，Δy_x，…，$\Delta y^{\alpha-1}_{t^{\alpha_0}\cdots x_p^{\alpha_p}}$的二次式积分之全体。跟以前一样，倘若$\psi(t,x)$满足

$$\begin{cases}\dfrac{\partial^\alpha\psi}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_p^{\alpha_p}}-\left(\psi,\dfrac{\partial f(t,x,y,y_t,\cdots,u}{\partial y}\right)+\dfrac{\partial f^0(t,x,y,\cdots u)}{\partial y}-\cdots+\cdots\\ \qquad+\cdots-(-1)^{\alpha_0+\cdots+\alpha_p}\dfrac{\partial^{\alpha_0+\cdots+\alpha_p}}{\partial t^{\alpha_0}\cdots\partial x_p^{\alpha_p}}\left(\psi,\dfrac{\partial f(t,x,y,\cdots u)}{\partial\left(\frac{\partial^k y(t,x)}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_p^{\alpha_p}}\right)}\right)=0,\qquad(2\cdot3\cdot4)\\ R(\psi)_\Gamma=0,\ \text{（由(2·3·3)第二个积分为零所导出的边界条件）}\qquad(2\cdot3\cdot5)\end{cases}$$

于是（2·3·3）便成为

$$\Delta J^*=\int_0^t\int_0^{x_1}\cdots\int_0^{x_n}\Big\{f^0(t,x,y,y_\tau,\cdots u+\Delta u)-f^0(t,x,y,y_\tau,\cdots u)+$$

$$+\big(\psi,f(t,x,y,\cdots,u+\Delta u)-f(t,x,y,y_t,\cdots,u)\big)\Big\}d\xi_1\cdots d\xi_n\,d\tau+(***)\qquad(2\cdot3\cdot6)$$

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