應用彈性力學

APPLIED ELASTICITY

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HIGHER EDUCATION PRESS
WILEY EASTERN LIMITED

（京） 112號

高等學校教材

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高等教育出版社出版

WILEY EASTERN LIMITED

高等教育出版社發行

國防工業出版社印刷廠印裝

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開本 787×1092 1/20 印張 19.2 字數500 000

1992 年8月第1版 1992 年 8月第1次印刷

印數0 001—2 087

ISBN 7-04-004076-X/TB · 211

定價 9.50元

Preface

This book is intended as a textbook for the course of "Theory of Elasticity" offered in universities and colleges of engineering. It can also be used as a reference book for engineers.

In this book plane problems are discussed first. After the reader has obtained a preliminary knowledge of the basic theories and problem-solving methods of the course, spatial problems are dealt with, followed by discussions on problems in plates and shells. On the whole, this arrangement meets the principle of proceeding gradually from the basic and the easy, to the complex and difficult materials, and is helpful for teaching and learning.

In elucidating fundamental principles and deducing basic equations, the author has endeavoured to render mathematical derivation explicit and concise, and to avoid using abstruse and overelaborate mathematical reasoning which is likely to overshadow physical concepts. Therefore, the reader will find it easy to grasp the quintessence of the contents.

In expounding problem-solving approaches, the author not only dwells on the nature of the approaches and their applications in engineering problems, but also emphatically points out the mode of thinking in analysing problems and the proper way of solving problems. Hopefully, the reader could be inspired to draw inferences about other problems from a relevant example in the book, thus developing the ability to analyse and solve problems independently.

In preparing this book, the author has quoted substantially from the book "Theory of Elasticity", which he wrote in Chinese in 1979. This early book has been widely used in China's universities and colleges of engineering and is currently published in the third edition (a revised edition).

ZHILUN XU

CONTENTS

1
Introduction

1.1. CONTENTS OF THEORY OF ELASTICITY

The theory of elasticity, often called elasticity for short, is the branch of solid mechanics which deals with the stresses and deformations in elastic solids produced by external forces or changes in temperature.

For students of various engineering disciplines, the purpose of studying elasticity is to analyse the stresses and displacements of structural or machine elements within the elastic range and thereby to check the sufficiency of their strength, stiffness and stability. Although this purpose is the same as that of studying mechanics of materials and structural mechanics, these three branches of solid mechanics do differ from one another both in the objects studied and in the methods of analysis used.

Mechanics of materials deals essentially with the stresses and displacements of a structural or machine element in the shape of a bar, straight or curved, which is subjected to tension, compression, shear, bending, or torsion. Structural mechanics, on the basis of mechanics of materials, deals with the stresses and displacements of a structure in the form of a bar system, such as a truss or a rigid frame. As to the structural elements which are not in the form of a bar, such as blocks, plates, shells, dams and foundations, they are analysed only in the theory of elasticity. Moreover, in order to analyse a bar element thoroughly and precisely, it is necessary to apply the theory of elasticity.

Although bar-shaped elements are studied both in mechanics of materials and in theory of elasticity, the methods of analysis used in the two subjects are not entirely the same. When such an element subjected to external loads is studied in mechanics of materials, some assumptions are usually made on the strain condition or the stress distribution. These assumptions simplify the mathematical derivation to a certain extent, but often inevitably reduce the degree of accuracy of the results obtained. In elasticity, however, the study of a bar-shaped element usually does not need those assumptions. Thus the results obtained are more accurate and may be used to check the approximate results obtained in mechanics of materials.

For example, when the problem of bending of a straight beam under transverse loads is analysed in mechanics of materials, it is assumed that a plane section of the beam remains plane after bending. This assumption leads to the linear distribution of bending stresses. In the theory of elasticity, however, one can solve the problem without this assumption and prove that if the depth of the beam is not much smaller than the span length, the stress distribution will be far from linear variation, as shown in Fig. 7.3.1, and the maximum tensile stress is seriously undervalued in mechanics of materials.

Another example is the calculation of stresses in a prismatical tension member with a hole. It is assumed in mechanics of materials that the tensile stresses are uniformly distributed across the net section of the member, whereas the precise analysis in the theory of elasticity shows that the stresses are by no means uniform, but are concentrated near the hole; the maximum stress at the edge of the hole is far greater than the average stress across the net section, as shown in Fig. 4.9.2.

Before the twentieth century, bar systems were formally analysed only in structural mechanics and not in elasticity. In spite of this convention, in this century many engineers used a joint application of the two branches of solid mechanics, with the mutual infiltration of the two as a result. The utilization of various methods of analysis in structural mechanics greatly strengthened the theory of elasticity and thus enabled engineers to obtain the solutions of many complicated problems in elasticity. Although these solutions are approximate theoretically, they prove to be scientifically accurate for engineering designs. For example, using the finite element method developed in the last thirty years, we can solve a problem in elasticity by the discretization of the body concerned and then the application of the displacement method, the force method, or the mixed method in structural mechanics. This is a brilliant example of the joint application of the two branches of solid mechanics.

Moreover, in the design of a structure, we can utilize the different branches of solid mechanics for different members of the structure, and even for different parts of a single member, to get the most satisfactory results with the least amount of work.

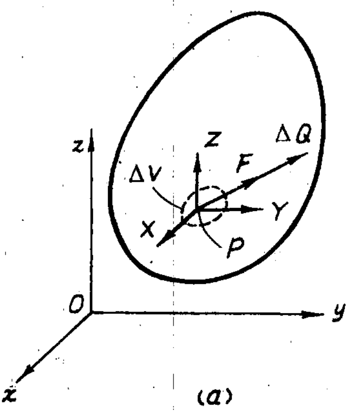
The students should not pay too much attention to the fuzzy and temporary dividing lines between the three courses in solid mechanics. On the contrary, they are advised to note all the possibilities of joint application of the three courses.

1.2 SOME IMPORTANT CONCEPTS IN THEORY OF ELASTICITY

The concepts most frequently encountered in elasticity are those about external forces, stresses, deformations and displacements. Even though the students are already familiar with these concepts, it is still necessary to review them concisely and introduce their notations and sign conventions used in elasticity.

There are two kinds of external forces which may act on bodies, namely the body forces and the surface forces.

External forces, or the loads, distributed over the volume of the body, such as gravitational forces, or inertia forces in the case of a body in motion are called body forces. In order to indicate clearly the magnitude and direction of the body force acting at a certain point P in the body, we take an elementary volume ΔV of the body around point P, as shown in Fig. 1.2.1. Let the body force acting on ΔV be ΔQ, so the average intensity of the body force will be $\Delta Q/\Delta V$. As ΔV is continuously contracted, the quantity ΔQ, and hence the quantity $\Delta Q/\Delta V$, will continuously change in magnitude, direction and point of application. Now, if we assume that the body forces are continuously distributed over the volume of the body and contract ΔV toward point P, the quantity $\Delta Q/\Delta V$ will approach a certain limit F:



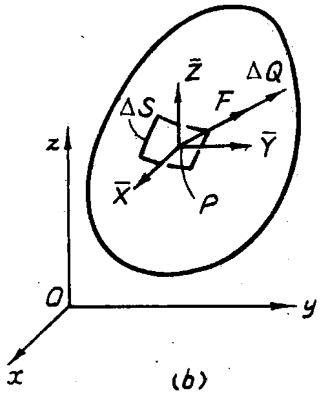
(a)

Fig. 1.2.1

$$\lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V} = F.$$

This vector quantity F is the intensity of body force at P. Since ΔV is a scalar quantity, the limiting direction of ΔQ will give the direction of F. The projections of F on the x, y and z axes will be denoted by X, Y and Z respectively and called the body force components at P. Such a component is considered positive or negative according as it acts in the positive or negative direction of the corresponding coordinate axis. Its dimension is [force] [length]$^{-3}$.

External forces, or the loads, distributed over the surface of a body, such as the pressure of one body on another or hydrostatic pressure, are called surface forces. To indicate clearly the magnitude and direction of the surface force at a certain point P on the surface, we take an elementary area ΔS of the surface around point P, as shown in Fig. 1.2.2. Let the surface force acting on ΔS be ΔQ, so the average intensity will be $\Delta Q/\Delta S$. If we assume that the surface forces are continuously distributed over the surface of the body and contract ΔS toward point P, the quantity $\Delta Q/\Delta S$ will approach a certain limit F:



(b)

Fig. 1.2.2

$$\lim_{\Delta S \to 0} \frac{\Delta Q}{\Delta S} = F.$$

This vector quantity F is the intensity of surface force at P. The projections of F on the x, y and z axes will be denoted by $\bar{X}, \bar{Y}$ and $\bar{Z}$ respectively and called the surface force components at P. Such a component is considered positive or negative according as it acts in the positive or negative direction of the coordinate axis. Its dimension is [force] [length]$^{-2}$.

Under the action of external forces, internal forces will be produced between the parts of a body. To study the internal forces at a certain point P of the body, let us imagine the body to be divided into two parts A and B by a section mn passing through this point, as shown in Fig 1.2.3, and take an elementary area ΔA on the section around P. Let ΔQ be the internal force acted by part B on part A across ΔA, so the average intensity of the internal force, or the average stress, will be $\Delta Q/\Delta A$. If we assume the internal force is continuously distributed over the section and contract ΔA toward point P, the quantity $\Delta Q/\Delta A$ will approach a certain limit S:

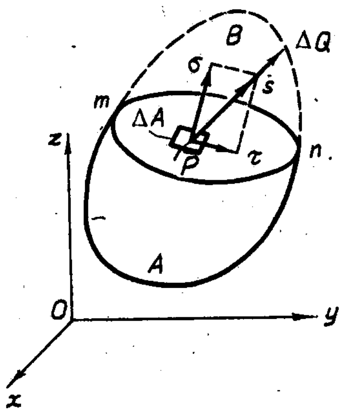


Fig. 1.2.3

$$\lim_{\Delta A \to 0} \frac{\Delta Q}{\Delta A} = S.$$

The vector quantity S is said to be the stress on the section *mn* at point P and its direction is the limiting direction of ΔQ. Usually, the stress is resolved into a normal component σ called the normal stress and a tangential component τ called the shearing stress.

Generally speaking, the stresses on different sections passing through the same point in a body are different. In order to describe the stress condition at the point P, that is, the magnitudes and directions of the stresses on all those sections, we isolate an elementary parallelepiped PABC (Fig. 1.2.4) from the body, with its edges parallel to the coordinate axes. Let the lengths of PA, PB and PC be $\Delta x, \Delta y$ and Δz respectively. The stress on each of the six sides is resolved into three components along the coordinate axes, one normal stress and two shearing stresses. To indicate the acting plane and the direction of a normal stress, we associate the stress with a coordinate subscript. For instance, σ_x indicates the normal stress acting on a plane perpendicular to the x axis and also in the x direction. As to a shearing stress, we should associate it with two coordinate subscripts. For instance, τ_{xy} indicates the shearing stress acting on a plane perpendicular to the x axis, but in the y direction.

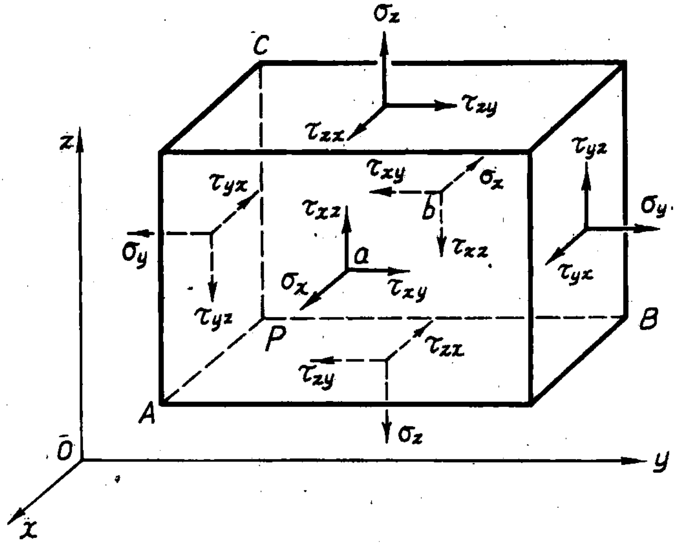


Fig. 1.2.4

If the outward normal to a side of the element is in the positive direction of a coordinate axis, a stress component on this side will be considered positive as it acts in the positive direction of the corresponding axis; if the outward normal to a side of the element is in the negative direction of a coordinate axis, a stress component on this side will be considered positive as it acts in the negative direction of the corresponding axis. Stresses in the directions opposite to those stated above are considered negative. Thus, all the stresses shown in Fig. 1.2.4 are positive. It should be noted that the above sign convention and that used in mechanics of materials do not always give the same result for a shearing stress, though the two conventions do give the same result for a normal stress, that is, positive for tension and negative for compression. The dimension of stresses is [force] [length]$^{-2}$.

The six shearing stresses do not all differ from each other but are mutually equal in pairs. For example, taking the moments of all the stresses about the line ab, which is parallel to the x axis and passes through the centroid of the element, we can set up an equilibrium equation as

$$2\tau_{yz}\Delta z\Delta x\frac{\Delta y}{2}-2\tau_{zy}\Delta y\Delta x\frac{\Delta z}{2}=0,$$

which yields $\tau_{yz}=\tau_{zy}$ after simplification. Together with the other two similar equations, we have

$$\tau_{yz}=\tau_{zy},\ \tau_{zx}=\tau_{xz},\ \tau_{xy}=\tau_{yx}. \tag{1.2.1}$$

Thus, it is proved that the shearing stresses which act on two perpendicular planes and are perpendicular to the intersecting line of the two planes are equal in magnitude and have the same sign. Hence, the subscript letters of the notation of a shearing stress may be interchanged at will.

Here we have ignored the variation of stresses in the element (they are taken as being uniformly distributed), and we have also ignored the action of body forces. However, it will be shown later that Eqs. (1.2.1) still hold even if the stresses are not uniform and the body forces are taken into consideration.

Incidentally, if the sign convention in mechanics of materials is used, Eqs. (1.2.1) will be replaced by

$$\tau_{yz}=-\tau_{zy},\ \tau_{zx}=-\tau_{xz},\ \tau_{xy}=-\tau_{yx}, \tag{1.2.2}$$

which are not so simple as Eqs. (1.2.1). However, it may also be noted that the sign convention in mechanics of materials must be adopted in the use of Mohr's circle of stress.

It will be shown later that the normal and shearing stresses on any section through point P can be evaluated, if the stress components σ_x, σ_y, σ_z, $\tau_{yz}=\tau_{zy}$, $\tau_{zx}=\tau_{xz}$ and $\tau_{xy}=\tau_{yx}$ at that point are known. Consequently, the six stress components precisely define the stress condition at that point.

By deformation we mean the change of the shape of a body. Since the shape of a body may be expressed by the lengths and angles of its parts, its deformation may be expressed by the changes in lengths and angles of the parts. To study the deformation condition at a certain point P of the body, we consider again the elementary parallelepiped shown in Fig. 1.2.4. During its deformation, generally speaking, the lengths of its three edges PA, PB and PC, and also the three right angles between them, will change by certain amounts, however small. A change in length per unit length is called a normal strain and the change of a right angle, expressed in radian, is called a shearing strain. A normal strain will be denoted by ε, with ε_x denoting that of the edge PA along the x axis, etc. It is considered positive for elongation and negative for contraction, in consistency with the sign convention for normal stresses. A shearing strain will be denoted by γ, with γ_{xy} denoting the change of the right angle between PA and PB along the x and y axes, etc. It is considered positive for a decrease of the right angle and negative for an increase of the right angle, in consistency with the sign convention for shearing stresses. It is noted that all the strains are dimensionless.

It will be shown later that, if the six strains ε_x, ε_y, ε_z, γ_{yz}, γ_{zx} and γ_{xy} are known at any point of a body, we can evaluate the normal strain of any line segment at the point and also the change of the angle made by any two line segments at the point. Hence, the six strains, called the strain components at the point, precisely define the strain condition at that point.

By displacement, we mean the change of position. The displacement at any point of a body is expressed by its projections on the x, y and z axes, denoted by u, v and w, respectively. These three projections are called the displacement components at the point. Such a component is considered positive or negative according as it is in the positive or negative direction of the corresponding coordinate axis. The dimension of a displacement or its components is [length].

Generally speaking, all the components of body forces, surface forces, stresses, strains and displacements at a point vary with the position of the point considered. Consequently they are functions of coordinates in space.

1.3 BASIC ASSUMPTIONS

To evaluate the stresses, strains and displacements in an elasticity problem, we must derive a series of basic equations and boundary conditions. During the process of derivation, however, if we consider all the influential factors in an all-round way, the results obtained will be so complicated that practically no solutions can be found. Therefore, we have to make some basic assumptions about the properties of the body considered and on the range of out study. Under such assumptions, we can neglect some of the influential factors of minor importance temporarily, thus simplifying the basic equations and the boundary conditions, In this text, we will comply with the following assumptions in classical elasticity :

(1) The body is continuous, i.e., the whole volume of the body is filled with continuous matter, without any void. Only under this assumption, can the physical quantities in the body, such as stresses, strains and displacements, be continuously distributed and thereby expressed by continuous functions of coordinates in space. In reality, all engineering materials are composed of elementary particles and do not accord with the assumption of continuity. However, it may be conceived that this assumption will lead to no significant errors so long as the dimensions of the body are very large in comparison with those of the particles and with the distances between neighboring particles.

(2) The body is perfectly elastic, i.e., it wholly obeys Hooke's law of elasticity, which shows the linear relations between the stress components and the strain components. Under this assumption, the elastic constants will be independent of the magnitudes of these components. The justification for this assumption lies in the physical behavior of nearly all materials in engineering construction.

(3) The body is homogeneous so that the elastic properties are the same throughout the body. Thus, the elastic constants will be independent of the location in the body. Under this assumption, we may analyse an elementary volume isolated from the body and then apply the results of analysis to the entire body.

(4) The body is isotropic so that the elastic properties are the same in all directions. Thus, the elastic constants will be independent of the orientation of coordinate axes.

Most engineering materials do not satisfy the above two assumptions completely. Structural steel, for instance, when studied with a microscope, is seen to consist of crystals of various kinds and various orientations. It seems that the material is far from being homogeneous and isotropic. However, since the dimensions of any single crystal are very small in comparison with those of the entire body, and since the crystals are orientated at random, the behavior of a piece of steel, on average, appears to justify the assumptions of homogeneity and isotropy. This is the reason why the solutions in elasticity based on these assumptions can be applied to steel structures with very great accuracy so long as none of the members has been subjected to the process of rolling which may produce a definite orientation of the crystals. In contrast with steel, wood is definitely not isotropic, since the elastic properties of wood in the direction of the grain differ greatly from those in the perpendicular directions. In assuming isotropic material, we shall of course exclude the treatment of wooden structures.

(5) The displacements and strains are small, i.e., the displacement components of all points of the body during deformation are very small in comparison with its original dimensions, and the strain components and the rotations of all line elements are much smaller than unity. Thus, when we formulate the equilibrium equations relevant to the deformed state, we may use the lengths and angles of the body before deformation. In addition, when we formulate the geometrical equations involving strains and displacements, we may neglect the squares and products of the small quantities. These two measures are necessary to linearize the algebraic and differential equations in elasticity for their easier solution.

1.4 PROBLEMS

1.4.1 Discuss the applicability of solutions in elasticity to concrete and reinforced concrete structures.

1.4.2 Discuss the applicability of solutions in elasticity to soil and rock foundations.