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Kenneth S. Brown

Cohomology of Groups

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Preface

This book is based on a course given at Cornell University and intended primarily for second-year graduate students. The purpose of the course was to introduce students who knew a little algebra and topology to a subject in which there is a very rich interplay between the two. Thus I take neither a purely algebraic nor a purely topological approach, but rather I use both algebraic and topological techniques as they seem appropriate.

The first six chapters contain what I consider to be the basics of the subject. The remaining four chapters are somewhat more specialized and reflect my own research interests. For the most part, the only prerequisites for reading the book are the elements of algebra (groups, rings, and modules, including tensor products over non-commutative rings) and the elements of algebraic topology (fundamental group, covering spaces, simplicial and CW-complexes, and homology). There are, however, a few theorems, especially in the later chapters, whose proofs use slightly more topology (such as the Hurewicz theorem or Poincaré duality). The reader who does not have the required background in topology can simply take these theorems on faith.

There are a number of exercises, some of which contain results which are referred to in the text. A few of the exercises are marked with an asterisk to warn the reader that they are more difficult than the others or that they require more background.

I am very grateful to R. Bieri, J-P. Serre, U. Stammbach, R. Strebel, and C. T. C. Wall for helpful comments on a preliminary version of this book.

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Notational Conventions

All rings (including graded rings) are assumed to be associative and to have an identity. The latter is required to be preserved by ring homomorphisms. Modules are understood to be *left* modules, unless the contrary is explicitly stated. Similarly, group actions are generally understood to be left actions.

If a group G acts on a set X, I will usually write X/G instead of $G\backslash X$ for the orbit set, even if G is acting on the left. One exception to this concerns the notation for the set of cosets of a subgroup H in a group G. Here we are talking about the left or right translation action of H on G, and I will always be careful to put the H on the appropriate side. Thus $G/H = \{gH : g \in G\}$ and $H\backslash G = \{Hg : g \in G\}$.

A symbol such as

$$\sum_{g \in G/H} f(g)$$

indicates that f is a function on G such that $f(g)$ depends only on the coset gH of g; the sum is then taken over a set of coset representatives.

Finally, I use the "topologists' notation"

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z};$$

in particular, $\mathbb{Z}_p$ denotes the integers mod p, not the p-adic integers.

Contents

Introduction

The cohomology theory of groups arose from both topological and algebraic sources. The starting point for the topological aspect of the theory was the work of Hurewicz [1936] on "aspherical spaces." About a year earlier, Hurewicz had introduced the higher homotopy groups $\pi_n X$ of a space X ($n \geq 2$). He now singled out for study those path-connected spaces X whose higher homotopy groups are all trivial, but whose fundamental group $\pi = \pi_1 X$ need not be trivial. Such spaces are called *aspherical.*

Hurewicz proved, among other things, that the homotopy type of an aspherical space X is completely determined by its fundamental group π. In particular, the homology groups of X depend only on π; it is therefore reasonable to think of them as *homology groups of* π. [This terminology, however, was not introduced until the 1940's.] Throughout the remainder of this introduction, then, we will write $H_* \pi$ for the homology of any aspherical space with fundamental group π. (Similarly, we could define homology and cohomology groups of π with arbitrary coefficients.) As a simple example, note that $H_2(\mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z}$. [Take X to be the torus.] Although Hurewicz considered only the *uniqueness* and not the *existence* of aspherical spaces, there does in fact exist an aspherical space with any given group as fundamental group. Thus our topological definition of group homology applies to all groups

For any group π we obviously have $H_0 \pi = \mathbb{Z}$ and $H_1 \pi = \pi_{ab}$, the latter being the abelianization of π, i.e., π modulo its commutator subgroup. For $n \geq 2$, however, it is by no means clear how to describe $H_n \pi$ algebraically. The first progress in this direction was made by Hopf [1942], who expressed $H_2 \pi$ in purely algebraic terms, and who gave further evidence of its importance in topology by proving the following theorem: for any path-connected space X with fundamental group π, there is an exact sequence

(0.1) $$\pi_2 X \to H_2 X \to H_2 \pi \to 0.$$

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[To put this result in perspective, one should recall that Hurewicz had introduced homomorphisms $h_n: \pi_n X \to H_n X$ $(n \geq 2)$ and had shown that h_n is an isomorphism if $\pi_i X = 0$ for $i < n$. In particular, h_2 is an isomorphism if $\pi = \pi_1 X = 0$. When π is non-trivial, however, h_2 is in general neither injective nor surjective, and Hopf's theorem gives a precise description, in terms of π, of the extent to which surjectivity fails.]

Hopf's description of $H_2 \pi$, incidentally, went as follows: Choose a presentation of π as F/R, where F is free and $R \lhd F$; then

$$H_2 \pi = R \cap [F, F]/[R, F], \tag{0.2}$$

where $[A, B]$ for $A, B \subseteq F$ denotes the subgroup generated by the commutators $[a, b] = aba^{-1}b^{-1}$ $(a \in A, b \in B)$.

Following Hopf's paper there was a rapid development of the subject by Eckmann, Eilenberg–MacLane, Freudenthal, and Hopf. (See MacLane [1978] for some comments about this development.) In particular, one had by the mid-1940's a purely algebraic definition of group homology and cohomology, from which it became clear that the subject was of interest to algebraists as well as topologists. Indeed, the low-dimensional cohomology groups were seen to coincide with groups which had been introduced much earlier in connection with various algebraic problems. H^1, for instance, consists of equivalence classes of "crossed homomorphisms" or "derivations." And H^2 consists of equivalence classes of "factor sets," the study of which goes back to Schur [1904], Schreier [1926], and Brauer [1926]. Even H^3 had appeared in an algebraic context (Teichmüller [1940]). These are the algebraic sources of group cohomology referred to at the beginning of this introduction. (Of course, there had been nothing in this algebraic work to suggest that there was an underlying "homology theory"; this had to wait for the impetus from topology.)

It is not surprising, in view of this history, that the subject of group cohomology offers possibilities for a great deal of interaction between algebra and topology. For instance a "transfer map," motivated by a classical group-theoretic construction due to Schur [1902], was introduced into group cohomology (Eckmann [1953], Artin–Tate [unpublished]) and from there into topology, where it has become an important tool. Another example is the theory of Euler characteristics of groups. This theory was motivated by topology, but it has applications to group theory and number theory.

Our approach to the subject will be as follows: We begin in Chapters I and II by defining $H_* \pi$ from the point of view of "homological algebra." This is the point of view which had evolved by the end of the 1940's. The topological motivation, however, will always be kept in sight, and we will immediately obtain the topological interpretation of $H_* \pi$ in terms of aspherical spaces. In particular, we will prove 0.1 and 0.2.

Chapter III contains more homological algebra, involving homology and cohomology with coefficients. These arise naturally in applications, both in algebra and topology. They are also an important technical tool, since they make it possible to prove theorems by "dimension-shifting." In Chapter IV we treat the theory of group extensions, which involves the crossed homomorphisms and factor sets mentioned above.

Chapter V introduces cup and cap products (motivated by topology), and these are then used in Chapter VI in the study of the cohomology of finite groups. Much of the material in Chapter VI (such as the "Tate cohomology theory") was originally motivated by algebra (class field theory), but it turns out to be related to topological questions as well, such as the study of groups acting freely on spheres.

In Chapter VII we introduce spectral sequence techniques, which are used extensively in the remaining chapters. The reader is not expected to have previously seen spectral sequences; I give a reasonably self-contained treatment, omitting only some routine (but tedious) verifications.

Beginning with Chapter VIII the emphasis is on infinite groups, with the most interesting examples being groups of integral matrices. In Chapter VIII we discuss various finiteness conditions which can be imposed on such a group to guarantee that the homology has nice properties. Chapter IX treats Euler characteristics, which can be defined under suitable finiteness hypotheses. This theory, as we mentioned above, has interesting connections with number theory. Finally, Chapter X develops the "Farrell cohomology theory," which is a generalization to infinite groups of the Tate cohomology theory for finite groups.

CHAPTER I

Some Homological Algebra

0 Review of Chain Complexes

We collect here for ease of reference some terminology and results concerning chain complexes. Much of this will be well-known to anyone who has studied algebraic topology. The reader is advised to skip this section (or skim it lightly) and refer back to it as necessary. We will omit some of the proofs; these are either easy or else can be found in standard texts, such as Dold [1972], Spanier [1966], or MacLane [1963].

Let R be an arbitrary ring. By a *graded R-module* we mean a sequence $C = (C_n)_{n \in \mathbf{Z}}$ of R-modules. If $x \in C_n$, then we say x has *degree n* and we write $\deg x = n$. A *map of degree p* from a graded R-module C to a graded R-module C' is a family $f = (f_n : C_n \to C'_{n+p})_{n \in \mathbf{Z}}$ of R-module homomorphisms; thus $\deg(f(x)) = \deg f + \deg x$. A *chain complex* over R is a pair (C, d) where C is a graded R-module and $d: C \to C$ is a map of degree -1 such that $d^2 = 0$. The map d is called the *differential* or *boundary operator* of C. We often suppress d from the notation and simply say that C is a chain complex. We define the *cycles* $Z(C)$, *boundaries* $B(C)$, and *homology* $H(C)$ by $Z(C) = \ker d$, $B(C) = \operatorname{im} d$, and $H(C) = Z(C)/B(C)$. These are all graded modules.

One often comes across graded modules C with an endomorphism d of square zero such that d has degree $+1$ instead of -1. In this case it is customary to use superscripts instead of subscripts to denote the grading, so that $C = (C^n)_{n \in \mathbf{Z}}$ and $d = (d^n : C^n \to C^{n+1})$. Such a pair (C, d) is called a *cochain complex*. There is no essential difference between chain complexes and cochain complexes, since we can always convert one to the other by setting $C_n = C^{-n}$. We will therefore confine ourselves, for the most part, to discussing chain complexes, it being understood that everything applies to cochain complexes by reindexing as above. [Note, however, that there *is* a difference when we consider *non-negative* complexes, i.e., complexes such that C_n

[or C^n] $= 0$ for $n < 0$; if the differential is thought of as going from left to right, then a non-negative chain complex extends indefinitely to the left, whereas a non-negative cochain complex extends indefinitely to the right.] In discussing cochain complexes, one often prefixes "co" to much of the terminology; thus d may be called a coboundary operator, and we have cocycles $Z(C)$, coboundaries $B(C)$, and cohomology $H(C) = (H^n(C))_{n \in \mathbf{Z}}$.

If (C, d) and (C', d') are chain complexes, then a *chain map* from C to C' is a graded module homomorphism $f: C \to C'$ of degree 0 such that $d'f = fd$. A *homotopy* h from a chain map f to a chain map g is a graded module homomorphism $h: C \to C'$ of degree 1 such that $d'h + hd = f - g$. We write $f \simeq g$ and say that f is *homotopic* to g if there is a homotopy from f to g.

(0.1) Proposition. *A chain map $f: C \to C'$ induces a map $H(f): H(C) \to H(C')$, and $H(f) = H(g)$ if $f \simeq g$.* □

The abelian group of homotopy classes of chain maps $C \to C'$ will be denoted $[C, C']$. It is often useful to interpret $[C, C']$ as the 0-th homology group of a certain "function complex" $\mathscr{H}om_R(C, C')$, defined as follows: $\mathscr{H}om_R(C, C')_n$ is the set of graded module homomorphisms of degree n from C to C' [thus $\mathscr{H}om_R(C, C')_n = \prod_{q \in \mathbf{Z}} \operatorname{Hom}_R(C_q, C'_{q+n})$], and the boundary operator $D_n: \mathscr{H}om_R(C, C')_n \to \mathscr{H}om_R(C, C')_{n-1}$ is defined by $D_n(f) = d'f - (-1)^n fd$. [The sign here makes $D^2 = 0$. It is also consistent with other standard sign conventions, cf. exercise 3 below.] Note that the 0-cycles are precisely the chain maps $C \to C'$, and the 0-boundaries are the null-homotopic chain maps. Thus $H_0(\mathscr{H}om_R(C, C')) = [C, C']$. More generally, there is an interpretation of $H_n(\mathscr{H}om_R(C, C'))$ in terms of chain maps. Consider the complex $(\Sigma^n C, \Sigma^n d)$ defined by $(\Sigma^n C)_p = C_{p-n}$, $\Sigma^n d = (-1)^n d$; this complex is called the *n-fold suspension* of C. [If $n = 1$, we write ΣC instead of $\Sigma^1 C$.] Let $[C, C']_n = [\Sigma^n C, C']$. Then we have $H_n(\mathscr{H}om_R(C, C')) = [C, C']_n$. The elements of $[\ \ ,\ \ ]_n$ are called *homotopy classes of chain maps of degree n.*

A chain map $f: C \to C'$ is called a *homotopy equivalence* if there is a chain map $f': C' \to C$ such that $f'f \simeq \mathrm{id}_C$ and $ff' \simeq \mathrm{id}_{C'}$. And a chain map f is called a *weak equivalence* if $H(f): H(C) \to H(C')$ is an isomorphism.

(0.2) Proposition. *Any homotopy equivalence is a weak equivalence.* □

A chain complex C is called *contractible* if it is homotopy equivalent to the zero complex, or, equivalently, if $\mathrm{id}_C \simeq 0$. A homotopy from id_C to 0 is called a *contracting homotopy*. Any contractible chain complex is *acyclic*, i.e., $H(C) = 0$.

(0.3) Proposition. *C is contractible if and only if it is acyclic and each short exact sequence $0 \to Z_{n+1} \hookrightarrow C_{n+1} \xrightarrow{\bar{d}} Z_n \to 0$ splits, where $\bar{d}$ is induced by d.*

Proof. If h is a contracting homotopy, then $(h|Z): Z \to C$ splits the surjection $\bar{d}: C \to Z$. Conversely, suppose we have a splitting $s: Z \to C$, whence a graded module decomposition $C = \ker \partial \oplus \operatorname{im} s = Z \oplus \operatorname{im} s$. We then get a contracting homotopy $h: C \to C$ by setting $h|Z = s$ and $h|\operatorname{im} s = 0$. □

(0.4) Proposition. *A short exact sequence* $0 \to C' \xrightarrow{i} C \xrightarrow{\pi} C'' \to 0$ *of chain complexes gives rise to a long exact sequence in homology:*

$$\cdots \to H_n(C') \xrightarrow{H(i)} H_n(C) \xrightarrow{H(\pi)} H_n(C'') \xrightarrow{\partial} H_{n-1}(C') \to \cdots.$$

The "connecting homomorphism" ∂ is natural, in the sense that a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \end{array}$$

with exact rows yields a commutative square

$$\begin{array}{ccc} H_n(C'') & \longrightarrow & H_{n-1}(C') \\ \downarrow & & \downarrow \\ H_n(E'') & \longrightarrow & H_{n-1}(E'). \end{array}$$

□

(0.5) Corollary. *The inclusion $i: C' \to C$ is a weak equivalence if and only if C'' is acyclic.* □

This shows that the cokernel C'' of i is the appropriate object to consider if we want to measure the "difference" between $H(C)$ and $H(C')$. We now wish to define a "homotopy-theoretic" cokernel for an arbitrary chain map $f: C' \to C$, which plays the same role as the cokernel in the case of an inclusion: The *mapping cone* of $f: (C', d') \to (C, d)$ is defined to be the complex (C'', d'') with $C'' = C \oplus \Sigma C'$ (as a graded module) and $d''(c, c') = (dc + fc', -d'c')$. In matrix notation, we have

$$d'' = \begin{pmatrix} d & f \\ 0 & \Sigma d' \end{pmatrix}.$$

See exercise 2 below for the motivation for this definition.

(0.6) Proposition. *Let $f: C' \to C$ be a chain map with mapping cone C''. There is a long exact homology sequence*

$$\cdots \to H_n(C') \xrightarrow{H(f)} H_n(C) \to H_n(C'') \to H_{n-1}(C') \to \cdots.$$

In particular, f is a weak equivalence if and only if C'' is acyclic.

PROOF. There is a short exact sequence $0 \to C \to C'' \to \Sigma C' \to 0$; now apply (0.4). By checking the definition of the connecting homomorphism $H_n(\Sigma C') \to H_{n-1}(C)$, one finds that it equals $H_{n-1}(f): H_{n-1}(C') \to H_{n-1}(C)$. □

The mapping cone is also useful for studying homotopy equivalences, not just weak equivalences:

(0.7) Proposition. *A chain map $f: C' \to C$ is a homotopy equivalence if and only if its mapping cone C'' is contractible.*

PROOF. A straightforward computational proof can be found in the standard references (or can be supplied by the reader). For the sake of variety, we will sketch a conceptual proof. Suppose first that C'' is contractible. One then checks easily that the function complex $\mathscr{H}om_R(D, C'')$ is contractible for any complex D; in particular, it is acyclic. One also checks that $\mathscr{H}om_R(D, C'')$ is isomorphic to the mapping cone of $\mathscr{H}om_R(D,f): \mathscr{H}om_R(D, C') \to \mathscr{H}om_R(D, C)$. It therefore follows from (0.6) that $\mathscr{H}om_R(D,f)$ is a weak equivalence. Looking at H_0, we deduce that f induces an isomorphism $[D, C'] \to [D, C]$ for any D; hence f is a homotopy equivalence by a standard argument. Conversely, suppose f is a homotopy equivalence. Then one shows easily that $\mathscr{H}om_R(D,f)$: $\mathscr{H}om_R(D, C') \to \mathscr{H}om_R(D, C)$ is a homotopy equivalence, so its mapping cone $\mathscr{H}om_R(D, C'')$ is acyclic by 0.6. In particular, $[D, C''] = 0$ for any D, and this implies that C'' is contractible. □

Finally, we recall briefly the Künneth and universal coefficient theorems. If (C, d) (resp. (C', d')) is a chain complex of right (resp. left) R-modules, then we define their *tensor product* $C \otimes_R C'$ by $(C \otimes_R C')_n = \bigoplus_{p+q=n} C_p \otimes_R C'_q$, with differential D given by $D(c \otimes c') = dc \otimes c' + (-1)^{\deg c} c \otimes d'c'$ for $c \in C$, $c' \in C'$. The sign here can be remembered by means of the following *sign convention*: When something of degree p is moved past something of degree q, the sign $(-1)^{pq}$ is introduced. [In the present case, the differential, which is of degree -1, is moved past c, so we get the sign $(-1)^{-\deg c} = (-1)^{\deg c}$.] Note that $C \otimes_R C'$ is simply a complex of abelian groups for general R, but it is a complex of R-modules if R is commutative.

(0.8) Proposition (Künneth Formula). *Let R be a principal ideal domain and let C and C' be chain complexes such that C is dimension-wise free. There are natural exact sequences*

$$0 \to \bigoplus_{p \in \mathbf{Z}} H_p(C) \otimes_R H_{n-p}(C') \to H_n(C \otimes_R C')$$
$$\to \bigoplus_{p \in \mathbf{Z}} \mathrm{Tor}_1^R(H_p(C), H_{n-p-1}(C')) \to 0$$

and

$$0 \to \prod_{p \in \mathbf{Z}} \mathrm{Ext}_R^1(H_p(C), H_{p+n+1}(C')) \to H_n(\mathscr{H}om_R(C, C'))$$
$$\to \prod_{p \in \mathbf{Z}} \mathrm{Hom}_R(H_p(C), H_{p+n}(C')) \to 0,$$

and these sequences split. □

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