University Mathematics Textbooks

# CALCULUS

VOLUME I

Haibo Chen Biyu Liu Xuanyun Qin Qi Zhang

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大学数学教材

# 微积分

## （上册）

## 一元微积分与无穷级数

陈海波　刘碧玉　秦宣云　张　齐

中南大学出版社

**University Mathematics Textbooks**

# CALCULUS

## (Volume Ⅰ)

## One-variable Calculus, with Infinite Series

Haibo Chen Biyu Liu Xuanyun Qin Qi Zhang

Central South University Press

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# Preface

Advanced Mathematics-Calculus is the first university mathematics course, and also one of the most important courses for science and engineering students in universities. In order to help students improve their English level and the ability to read English science references, and to make use of successful teaching experiences in Western countries, more and more universities in China have begun to use bilingual (generally Chinese and English) teaching method in classrooms. With the differences of course contents and teaching methods between China and Western countries, none seems to meet the teaching requirements set forth by the Ministry of Education of China, although many good English language calculus textbooks are available. This book is intended to serve as a text for the bilingual teaching course in calculus that is usually taken by first-year students who study science and engineering.

This is a student textbook covering essential topics in calculus usually taught in the early stages of science and engineering students in China. The requirements of such students have influenced its content and presentation helped in many ways by the authors' long and continuous experience of teaching mathematical methods to various degree students in Central South University. It is divided into two volumes. The first volume contains Calculus of single variable and infinite series. The second volume consists of Calculus of multivariable with analysis geometrics and ordinary differential equations.

This volume has been divided into six chapters, each covering a coherent theme. Chapter 1 deals mainly with limits and continuity for functions of single variable. Chapter 2 through 5 is the heart of this volume. Chapter 2 and 3 treat single variable differential calculus, while Chapters 4 and 5 treat single variable integral calculus with applications in geometry and physics. The final chapter, Chapter 6, consists of infinite series.

This book (Volume Ⅰ and Volume Ⅱ) is edited chiefly by Haibo Chen. Editors include Haibo Chen (Chapter 5, 11), Biyu Liu (Chapter 1, 4), Xuanyun

Qin (Chapter 2, 6), Qi Zhang (Chapter 3), Qingping Liu (Chapter 10), Xinge Liu (Chapter 7), Songhai Deng (Chapter 9) and Yeqing Ren (Chapter 8). Limited by the authors' knowledge, it is impossible to avoid some errors and unclear explanations. We would appreciate any constructive criticisms and corrections from readers.

The Authors

# Contents

# Chapter 1 Functions and Limits

In this chapter, we will introduce three fundamental concepts of calculus. There are function, limit and continuity. Functions are the objects of research in calculus and are also a bridge connecting mathematics and application. The limit is a very important idea in calculus. Continuity is an important property of functions and continuous funcations are the main objects of study in calculus.

## 1.1 Functions

This section develops the notion of a function, and shows how functions can be built up from simpler functions.

### 1.1.1 The concept of a function

**Definition 1.1.1** Let $X$ and $Y$ be sets of numbers. A function $f$ from $X$ to $Y$ is a rule (or method) for assigning one (and only one) element $y \in Y$ to each element $x \in X$.

$x$ is called the argument or independent variable, $y$ is called the value of the function at $x$, or dependent variable.

The set $X$ is called the domain of definition of the function $f$ or the domain of the function $f$. The set of all the value of the function is called the range of the function. It is a sunset of $Y$.

The domain of definition and rule are the two important factors to determine the function. The former describes the region of existence of the function, and the latter gives the method for determining the corresponding elements of the set $Y$ from the elements of the set $X$. A function is completely determined by these two factors and is independent of the forms of the expression and the kind of elements contained in the set.

When the function is given by a formula, the domain is usually understand to consist of all the numbers for which the formula is defined.

**Example 1.1.1** Find the domain of the function $y = \sqrt{4 - x^2} + \dfrac{1}{\sqrt{x - 1}}$.

**Solution** For $\sqrt{4-x^2}+\frac{1}{\sqrt{x-1}}$ to be meaningful, the square roots of $4-x^2$ and $x-1$ must make sense, and $\sqrt{x-1}\neq 0$ at the same time. Thus, the domain consists of all numbers $x$ such that

$$\begin{cases} 4-x^2 \geqslant 0 \\ x-1>0 \end{cases}$$

Or, equivalently, $1<x\leqslant 2$. That is, the domain is the interval $(1, 2]$.

**Example 1.1.2** Which is the domain of the function $f(x)=\frac{1}{1+\frac{1}{1+\frac{1}{x}}}$?

(A) $x\in\mathbf{R}$ but $x\neq 0$ (B) $x\in\mathbf{R}$ but $1+\frac{1}{x}\neq 0$

(C) $x\in\mathbf{R}$ but $x\neq 0, -1, -\frac{1}{2}$ (D) $x\in\mathbf{R}$ but $x\neq 0, -1$

**Solution** For $\frac{1}{1+\frac{1}{1+\frac{1}{x}}}$ to be meaningful, $x\neq 0$, $1+\frac{1}{x}\neq 0$ and $1+\frac{1}{1+\frac{1}{x}}\neq 0$. That is, the domain is $x\neq 0$, $x\neq -1$, $-\frac{1}{2}$. Thus we choose (C).

**Example 1.1.3** Judge whether the following pair of functions are equal?

(1) $f(x)=\lg x^2$ and $g(x)=2\lg x$;

(2) $f(x)=x$ and $g(x)=\sqrt{x^2}$;

(3) $f(x)=\sqrt[3]{x^4-x^3}$ and $g(x)=x\sqrt[3]{x-1}$.

**Solution** (1) The function $f(x)=\lg x^2$ and $g(x)=2\lg x$ are not the same function because the domain of $f(x)=\lg x^2$ is different from $g(x)=2\lg x$;

(2) The function $f(x)=x$ and $g(x)=\sqrt{x^2}$ are not the same function because the formula of $f(x)=x$ is different from $g(x)=\sqrt{x^2}$;

(3) The function $f(x)=\sqrt[3]{x^4-x^3}$ and $g(x)=x\sqrt[3]{x-1}$ are the same function because the formula and domain of $f(x)=\sqrt[3]{x^4-x^3}$ and $g(x)=x\sqrt[3]{x-1}$ are the same.

In case both the independent variable and the dependent variable are numbers, we can draw a picture of the function, called its graph.

**Graph of a function** Let $f(x)$ be a function whose the independent variable and the dependent variable are numbers. The graph of $f(x)$ consists of those points $(x, y)$ such that $y = f(x)$, denoted by $G(f)$ (See Figure 1.1.1).

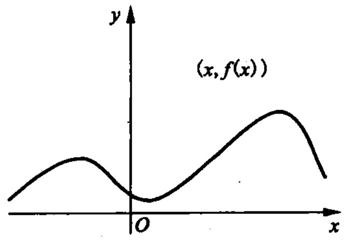


**Figure 1.1.1**

### 1.1.2 Ways of representing a function

To express a function is mainly to express its corresponding rule. There are many methods to express the corresponding rule, the following three are often used.

(1) Analytic representation

Many functions are given by an analytic representation. For example, the functions are given in Example 1.1.1 and Example 1.1.2.

(2) Method of tabulation

Sometimes, a function is given by a table that lists the independent variable and its corresponding dependent variable.

(3) Method shown by graph

The relation between $y$ and $x$ is shown by a graph. For example, the temperature curve recorded by some instruments expresses the relation between the temperature and time.

### 1.1.3 Properties of a function

**Bounded Functions**

let $D$ be the domain of the function $y = f(x)$ and the set $X \subset D$. If there exists a positive number $M$, such that

$$|f(x)| \leq M, \text{ for } \forall x \in X.$$

then the function $y = f(x)$ is bounded on $X$.

**Monotone Functions**

Let $D$ be the domain of the function $y = f(x)$ and the set $I \subset D$.

(1) If $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$, and $x_1 < x_2$, then $f(x)$ is called an increasing function on $I$;

(2) If $f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$, and $x_1 < x_2$, then $f(x)$ is called a decreasing function on $I$.

**Odd Functions and Even Functions**

Let $(-a, a)$ $(a > 0)$ be the domain of definition of the function $y = f(x)$,

(1) If $f(-x) = -f(x)$ for all $x \in (-a, a)$, then $f(x)$ is called an odd function;

(2) If $f(-x) = -f(x)$ for all $x \in (-a, a)$, then $f(x)$ is called an even function.

The graph of an odd function is symmetric with respect to the origin (See Figure 1.1.3), and graph of an even function is symmetric with respect to the y-axis (See Figure 1.1.2).

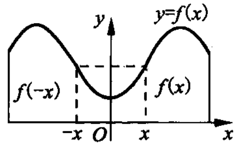


**Figure 1.1.2**

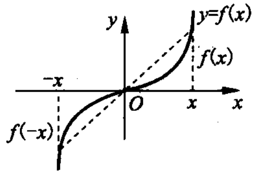


**Figure 1.1.3**

**Periodic Functions**

Let $D$ be the domain of definition of the function $y=f(x)$, if there exists the number $T\neq 0$ such that $f(x+T)=f(x)$, for all $x\in\mathbf{R}$, $x\pm T\in D$, then $f(x)$ is called a periodic function and $T$ is called a period of $f(x)$ (See Figure 1.1.4).

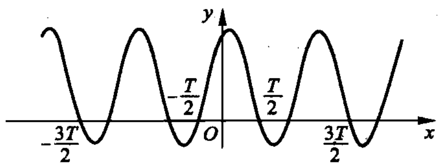


**Figure 1.1.4**

**Example 1.1.4** Determine whether the function $f(x)=\ln(x+\sqrt{1+x^2})$ is odd or even?

**Solution** This function is defined on $(-\infty, +\infty)$, also

$$f(-x)=\ln(-x+\sqrt{1+x^2})=\ln\left(\frac{1}{x+\sqrt{1+x^2}}\right)$$

$$=-\ln(x+\sqrt{1+x^2})=-f(x)$$

Therefore, $f(x)$ is an odd function.

**Example 1.1.5** If there exists a constant $c\neq 0$, such that $f(x+c)=-f(x)$, for all $x\in(-\infty, +\infty)$, prove that $f(x)$ is a periodic function.

**Proof** Since $f(x+c)=-f(x)$ for all $x\in(-\infty, +\infty)$, then

$$f(x+2c)=f[(x+c)+c]=-f(x+c)=f(x)$$

Thus $f(x)$ is a periodic function and $2c$ is a period of $f(x)$.

### 1.1.4 Operation rule for functions

(1) Rational operation rule for functions

**Definition 1.1.2** The sum, difference, product and quotient of two functions $f(x)$ and $g(x)$ are defined by the following rule on the domain of definition $D_f \cap D_g$, where $D_f$ and $D_g$ are the domain of definition of the function $f(x)$ and $g(x)$, respectively.

$$(f+g)(x)=f(x)+g(x)$$
$$(f-g)(x)=f(x)-g(x)$$
$$(f\cdot g)(x)=f(x)\cdot g(x)$$
$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)} \quad (g(x)\neq 0)$$

(2) Composition of operations rule for functions

**Definition 1.1.3 (Composition of function)** Let $X$, $Y$ and $Z$ be sets of numbers. Let $g$ be a function from $X$ to $Y$, and let $f$ be a function from $Y$ to $Z$, then the function that assigns each element $x$ in $X$ to the element $f(g(x))$ in $Z$ is called the composition of $f$ and $g$. It is denoted $fog$ ($fog$ is read as "$f$ circle $g$" or as "$f$ composed with $g$"), or $y=(fog)(x)=f[g(x)]$.

The range $R(g)$ of $g$ must be the subset of the domain $Y$ of $f$, that is $R(g)\subset Y$, and $u=g(x)$ is called the middle variable.

The above figure (Figure 1.1.5) depicts the notion of a composite function.

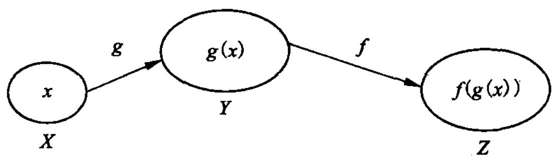


**Figure 1.1.5**

**Example 1.1.6** Suppose that $f(\sin\frac{x}{2})=1+\cos x$, find $f(\cos\frac{x}{2})$.

**Solution** Since $f(\sin\frac{x}{2})=1+\cos x=2\cos^2\frac{x}{2}=2(1-\sin^2\frac{x}{2})$,

so, we have $f(u)=2(1-u^2)$. Therefore

$$f(\cos\frac{x}{2})=2(1-\cos^2\frac{x}{2})=2\sin^2\frac{x}{2}=1-\cos x.$$

**Example 1.1.7** Suppose that $f(x+\frac{1}{x})=x^2+\frac{1}{x^2}$, find $f(x)$.

**Solution** Since $f(x+\frac{1}{x}) = x^2+\frac{1}{x^2} = (x+\frac{1}{x})^2 - 2$,

Therefore, $f(x) = x^2 - 2$.

(3) Inverse functions

**Definition 1.1.4 (Inverse functions)** Suppose that $D(f)$ is the domain of definition of the function $f$, $R(f)$ is the range of $f$, the inverse function $f^{-1}$ from $R(f)$ to $D(f)$ is a rule (that is $x = f(y)$) for assigning one element $y \in D(f)$ to each element $x \in R(f)$.

The domain of definition of $f$ is the range of its inverse function $f^{-1}$, and the domain of definition of $f^{-1}$ is the range of $f$.

For a function $f$, if its inverse function exists, then

$$y = f(x),\ x \in D(f),\ y \in R(f)$$

$$x = f^{-1}(y),\ D(f^{-1}) = R(f),\ x \in R(f^{-1}) = D(f)$$

Hence the function $y = f(x)$ and its inverse function $x = f^{-1}(y)$ are represented by the same relation, so their graph is the same curve in the $xOy$ plane.

But we are accustomed to express the independent variable by $x$, and the dependent variable by $y$, so that the inverse function $x = f^{-1}(y)$ is usually expressed by $y = f^{-1}(x)$. In this case, the graph of the function $y = f(x)$ and the inverse function $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$ (Figure 1.1.6).

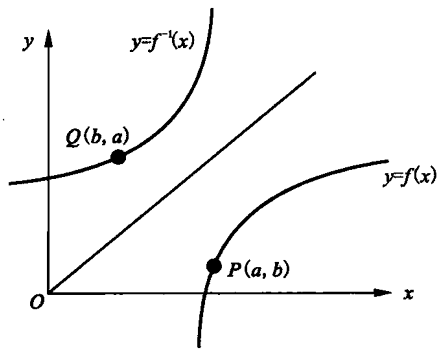


**Figure 1.1.6**

**Example 1.1.8** Find the inverse function of the following functions.

(1) $y = \frac{2x-3}{3x+2}$; (2) $y = \ln(x+2) + 1$.

**Solution**

(1) First change the position of $x$ and $y$ in the analytic representation, we obtain, $x=\frac{2y-3}{3y+2}$, that is $3xy+2x=2y-3$.

Thus $y=\frac{3+2x}{2-3x}$ is the inverse function of $y=\frac{2x-3}{3x+2}$.

(2) Since $x=\ln(y+2)+1$ that is $y+2=e^{x-1}$

Thus $y=e^{x-1}-2$ is the inverse function of $y=\ln(x+2)+1$.

### 1.1.5 Piecewise defined function

It should be noted that the analytic representation of a function sometimes consists of several components on different subsets of the domain of definition of the function. A function expressed by this kind of representation is called a piecewise defined function.

**Example 1.1.9** (sign function)

$$y=\operatorname{sgn}x=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

It's domain of definition is $D(f)=(-\infty, +\infty)$, it's range is $R(f)=\{-1, 0, 1\}$.

**Example 1.1.10** (The greatest integer function)

The function whose value at any number $x$ is the largest integer smaller than or equal to $x$ is called the greatest integer function, denoted by $y=[x]$ $(x\in \mathbf{R})$.

For instance $[-2.5]=-3$, $[0]=0$, $[\sqrt{2}]=1$, $[\pi]=3$.

**Example 1.1.11** (The Dirichlet's function)

The function whose value at any rational number is 1 and at any irrational number is 0 is called the Dirchlet's function, denoted by

$$y=D(x)=\begin{cases}1, & x \text{ is a rational number}\\ 0, & x \text{ is an irrational number}\end{cases}$$

It's domain of definition is $D(f)=(-\infty, +\infty)$ and is range is $R(f)=\{0, 1\}$.

**Example 1.1.12** (Integer variable function)

The function whose domain of definition is the set of positive integer is called an integer variable function, and denoted by $y=f(n)$, $n\in \mathbf{N}^+$.

If we rewrite $f(n)=a_n$, $n\in \mathbf{N}^+$, then the integer variable function can also be denoted as follows:

$$a_1, a_2, \cdots, a_n, \cdots$$

For this reason, an integer variable function is also called a sequence.

**Example 1.1.13**

Suppose that $f(x)=\begin{cases}1, & |x|<1;\\ 0, & |x|=1;.\\ -1, & |x|>1.\end{cases}$ and $g(x)=e^x$, fin $f(g(x))$, $g(f(x))$.

**Solution**

$$f(g(x))=\begin{cases}1, & |e^x|<1\\ 0, & |e^x|=1\\ -1, & |e^x|>1\end{cases}=\begin{cases}1, & x<0\\ 0, & x=0;\\ -1, & x>0\end{cases}$$

$$g(f(x))=\begin{cases}e, & |x|<1\\ 1, & |x|=1.\\ e^{-1}, & |x|>1\end{cases}$$

### 1.1.6 Elementary functions

Those functions which we have learned in high school, such as power function: $y=x^{\mu}$ ($\mu$ is a number), exponential function: $y=a^x$ ($a>0$, $a\neq 1$), logarithm function: $y=\log_a x$ ($a>0$, and $a\neq 1$); trigonometric function; $y=\sin x$, $y=\cos x$, $y=\tan x$, $y=\cot x$, inverse trigonometric function: $y=\arcsin x$, $y=\arccos x$, $y=\arctan x$, $y=\text{arccot}\, x$, are all described by analytic representation. The above five types of functions and constants are called by the joint name basic elementary functions.

**Definition 1.1.5 (Elementary function)**

A function formed from the six kinds of basic elementary functions by a finite number of rational operations and compositions of functions which can be expressed by a single analytic expression is called an elementary functions.

For example, $\dfrac{1+x^2\sin x}{\arccos x}$, $\ln(x+\sqrt{1+x^2})$ are both elementary functions. But $f(x)=\begin{cases}e^x, & x\leqslant 0\\ \sin x, & x>0\end{cases}$ is not an elementary function because it can not be described by one analytic expression.

How to decompose a composite function into the combination of some basic elementary functions is very important. For example, $y=\sin\sqrt{2^{\ln\cos x^2}}$ is composed by the following basic elementary functions:

$$y=\sin u,\ u=\sqrt{v},\ v=2^w,\ w=\ln s,\ s=\cos t,\ t=x^2$$

**Definition 1.1. 6 (Hyperbolic function)**

Hyperbolic sine: $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $x \in \mathbf{R}$,

Hyperbolic cosine: $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $x \in \mathbf{R}$,

Hyperbolic tangent: $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, $x \in \mathbf{R}$,

Hyperbolic cotangent: $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, $x \in \mathbf{R}$.

They are called by a joint name hyperbolic functions.

There are some identities for hyperbolic functions which are similar to those for trigonometric functions.

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$
$$\cosh^2 x - \sinh^2 x,\ \sin 2hx = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

The inverse function of a hyperbolic function is an inverse hyperbolic function. They are

Inverse hyperbolic sine: $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, $x \in \mathbf{R}$

Inverse hyperbolic cosine: $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \in [1, +\infty)$

Inverse hyperbolic tangent: $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $x \in (-1, 1)$.

### 1.1.7 Implicit functions

In applied problems we often need to investigate a class of functions in which the correspondence rule between the dependent variable $y$ and the independent variable $x$ is defined by an equation of the form $F(x, y) = 0$, where $F(x, y)$ denotes an expression in $x$ and $y$.

**Definition 1.1.7** If there is a function $y = f(x)$, which is defined on some interval $I$, such that $F(x, f(x)) \equiv 0$, $x \in I$, then $y = f(x)$, $x \in I$ is called an implicit function defined by the equation $F(x, y) = 0$.

## Exercise 1.1

1. Find the domain of definition and range of the following function:

$$y = f(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$