理論力學問題詳解

上册

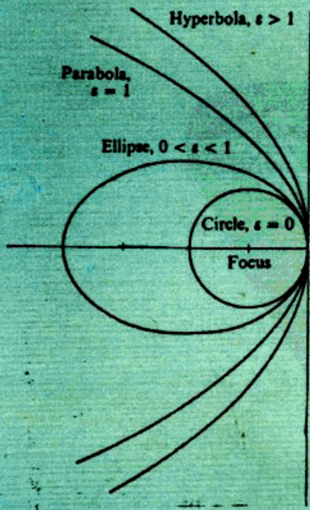
CLASSICAL DYNAMICS

OF PARTICLES AND SYSTEMS

SECOND EDITION

JERRY B. MARION

UNIVERSITY OF MARYLAND
COLLEGE PARK



曉園出版社

前　　言

研習理工的同學，都有一種認識，那就是：一本書的習題往往是該書的精華所在，藉着習題的印證，才能對書中的原理原則澈底的吸收與瞭解。

有鑒於此，曉園出版社特地聘請了許多在本科上具有相當研究與成就的人士，精心出版了一系列的題解叢書，為各該科目的研習，作一番介紹與舖路的工作。

一個問題的解答方法，常因思惟的角度而異。曉園題解叢書，毫無疑問的都是經過一番精微的思考與分析而得。其目的在提供對各該科目研讀時的參考與比較；而對於一般的自修者，則有啓發與提示的作用。希望讀者能藉着這一系列題解叢書的幫助，而在本身的學問進程上有更上層樓的成就。

理論力學問題詳解

Problems

1-1. Find the transformation matrix which rotates a rectangular coordinate system through an angle of 120° about an axis making equal angles with the original three coordinate axes.

1-2. Prove Eqs. (1.10) and (1.11) from trigonometric considerations.

1-3. Show
(a) $(\mathsf{AB})' = \mathsf{B}'\mathsf{A}'$.
(b) $(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1}$.

1-4. Show by direct expansion that $|\lambda|^2 = 1$. For simplicity, take λ to be a two-dimensional transformation matrix.

1-5. Show that Eq. (1.15) can be obtained by using the requirement that the transformation leave unchanged the length of a line segment.

1-6. Consider a unit cube with one corner at the origin and three adjacent sides lying along the three axes of a rectangular coordinate system. Find the vectors which describe the diagonals of the cube. What is the angle between any pair of diagonals?

1-7. Let **A** be a vector from the origin to a point P fixed in space. Let **r** be a vector from the origin to a variable point $Q(x_1, x_2, x_3)$. Show that

$$\mathbf{A} \cdot \mathbf{r} = A^2$$

is the equation of a plane perpendicular to **A** and passing through the point P.

1-8. Show that the *triple scalar product* $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ can be written as

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Show also that the product is unaffected by an interchange of the scalar and vector product operations or by a change in the order of **A**, **B**, **C**, as long as they are in cyclic order; that is,

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}, \qquad \text{etc.}$$

We may therefore use the notation **ABC** to denote the triple scalar product. Finally, give a geometrical interhretation of **ABC** by computing the volume of the parallelepiped defined by the three vectors **A**, **B**, **C**.

1-9. Let **a**, **b**, **c** be three constant vectors drawn from the origin to the points A, B, C. What is the distance from the origin to the plane defined by the points A, B, C? What is the area of the triangle ABC?

1-10. If **X** is an unknown vector which satisfies the following relations involving the known vectors **A** and **B** and the scalar φ,

$$\mathbf{A} \times \mathbf{X} = \mathbf{B}; \qquad \mathbf{A} \cdot \mathbf{X} = \varphi$$

express **X** in terms of **A**, **B**, φ, and the magnitude of **A**.

1-11. Obtain the cosine law of plane trigonometry by interpreting the product $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B})$ and the expansion of the product.

1-12. Obtain the sine law of plane trigonometry by interpreting the product $\mathbf{A} \times \mathbf{B}$ and the alternate representation $(\mathbf{A} - \mathbf{B}) \times \mathbf{B}$.

1-13. Derive the following expressions by using vector algebra:

(a) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(b) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

1-14. Show that

(a) $\sum_{i,j} \varepsilon_{ijk} \delta_{ij} = 0$ (b) $\sum_{j,k} \varepsilon_{ijk} \varepsilon_{ljk} = 2\delta_{il}$ (c) $\sum_{i,j,k} \varepsilon_{ijk} \varepsilon_{ijk} = 6$

1-15. Show that (see also Problem 1-8)

$$\mathbf{ABC} = \sum_{i,j,k} \varepsilon_{ijk} A_i B_j C_k$$

1-16. Evaluate the sum $\sum_k \varepsilon_{ijk} \varepsilon_{lmk}$ (which contains 81 terms) by considering the result for all possible combinations of i, j, l, m, viz.,

(a) $i = j$ (b) $i = l$

(c) $i = m$ (d) $j = l$

(e) $j = m$ (f) $l = m$

(g) $i \neq l$ or m (h) $j \neq l$ or m

Show that

$$\sum_k \varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

and then use this result to prove

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

1-17. Use the ε_{ijk} notation and derive the identity

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}(\mathbf{ABD}) - \mathbf{D}(\mathbf{ABC})$$

1-18. Let **A** be an arbitrary vector and let **e** be a unit vector in some fixed direction. Show that

$$\mathbf{A} = \mathbf{e}(\mathbf{A} \cdot \mathbf{e}) + \mathbf{e} \times (\mathbf{A} \times \mathbf{e})$$

What is the geometrical significance of each of the two terms of the expansion?

1-19. Find the components of the acceleration vector **a** in spherical and in cylindrical coordinates.

1-20. A particle moves with $v = \text{const.}$ along the curve $r = k(1 + \cos\theta)$ (a *cardioid*). Find $\ddot{\mathbf{r}} \cdot \mathbf{e}_r = \mathbf{a} \cdot \mathbf{e}_r$, $|\mathbf{a}|$, and $\dot{\theta}$.

1-21. If **r** and $\dot{\mathbf{r}} = \mathbf{v}$ are both explicit functions of time, show that

$$\frac{d}{dt}[\mathbf{r} \times (\mathbf{v} \times \mathbf{r})] = r^2\mathbf{a} + (\mathbf{r} \cdot \mathbf{v})\mathbf{v} - (v^2 + \mathbf{r} \cdot \mathbf{a})\mathbf{r}$$

1-22. Show that

$$\mathbf{grad}(\ln |\mathbf{r}|) = \frac{\mathbf{r}}{r^2}$$

1-23. Find the angle between the surfaces defined by $r^2 = 9$ and $x + y + z^2 = 1$ at the point $(2, -2, 1)$.

1-24. Show that $\mathbf{grad}(\varphi\psi) = \varphi\, \mathbf{grad}\, \psi + \psi\, \mathbf{grad}\, \varphi$.

1-25. Show that

(a) $\mathbf{grad}\, r^n = nr^{(n-2)}\,\mathbf{r}$

(b) $\mathbf{grad}\, f(r) = \dfrac{\mathbf{r}}{r}\dfrac{df}{dr}$

(c) $\nabla^2(\ln r) = \dfrac{1}{r^2}$

1-26. Show that

$$\int (2\mathbf{r} \cdot \dot{\mathbf{r}} + 2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\, dt = r^2 + \dot{r}^2 + \text{const.}$$

where **r** is the vector from the origin to the point (x_1, x_2, x_3). The quantities r and $\dot{r}$ are the magnitudes of the vectors **r** and $\dot{\mathbf{r}}$, respectively.

1-27. Show that

$$\int \left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2} \right) dt = \frac{\mathbf{r}}{r} + \mathbf{C}$$

where $\mathbf{C}$ is a constant vector.

1-28. Evaluate the integral

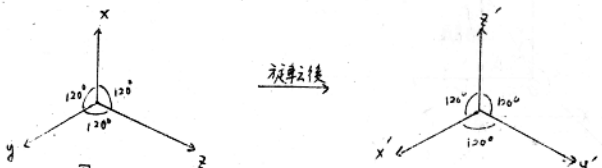
$$\int \mathbf{A} \times \ddot{\mathbf{A}}\, dt$$

1-29. Show that the volume common to the intersecting cylinders defined by $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ is $V = 16a^3/3$.

習題 一

1-1 試寫出對於一個和三直角座標軸成等角的軸做 120° 旋轉的變換矩陣

解:



圖一　　圖二.

註: 圖一、二 都是座標軸在垂直於轉軸的平面上的投影

經過旋轉後　x' 軸在原來 y 軸的位置

y' 軸在原來 z 軸的位置

z' 軸在原來 x 軸的位置

$\cos(x', x) = 0$, $\quad \cos(x', y) = 1$ $\quad \cos(x', z) = 0$

$\cos(y', x) = 0$ $\quad \cos(y', y) = 0$ $\quad \cos(y', z) = 1$

$\cos(z', x) = 1$ $\quad \cos(z', y) = 0$ $\quad \cos(z', z) = 0$

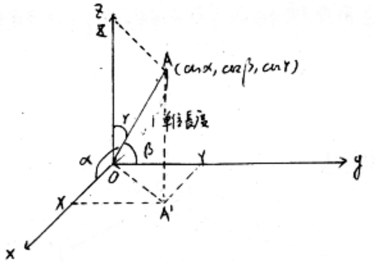
故 變換矩陣 $X' = \Lambda X$

為:
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

1-2. 利用三角法證明 (1.10)式及(1.11)式

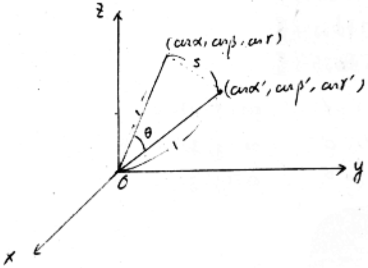
(1.10)式: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(1.11)式: $\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$



如圖所示，由畢氏定理知 $OA^2 = OZ^2 + OA'^2 = OZ^2 + OX^2 + OY^2$

$$\therefore 1 = \left(\frac{OZ}{OA}\right)^2 + \left(\frac{OX}{OA}\right)^2 + \left(\frac{OY}{OA}\right)^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$



由餘弦定理知

$$S^2 = 1^2 + 1^2 - 2\times1\times1\cos\theta \cdots (1)$$

又 $S^2 = (\cos\alpha - \cos\alpha')^2 + (\cos\beta - \cos\beta')^2 + (\cos\gamma - \cos\gamma')^2$

$$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\alpha' + \cos^2\beta' + \cos^2\gamma' - 2(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')$$

$$= 2 - 2(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma') \cdots (2)$$

比較(1) and (2)

得 $\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$

1-3 (a) 證明 $(AB)^t = B^t A^t$ 　A,B為矩陣

令 $C = B^t A^t$

則 $C_{ij} = \sum_k (B^t)_{ik} (A^t)_{kj}$

$= \sum_k B_{ki} A_{jk}$

$= \sum_k A_{jk} B_{ki}$

$= (AB)_{ji}$

$= (AB)^t{}_{ij}$

故 $C = (AB)^t$ 即 $(AB)^t = B^t A^t$

(b) 證明 $(AB)^{-1} = B^{-1}A^{-1}$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$

$(AB)(B^{-1}A^{-1}) = A.(BB^{-1})A^{-1} = AA^{-1} = I$

故 $B^{-1}A^{-1}$ 為 AB之反元素

即 $B^{-1}A^{-1} = (AB)^{-1}$

1-4 用直接展開證明 $|\lambda|^2 = 1$ 為簡單起見，只考慮二維空間的變換矩陣 λ

設 $\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$

$\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} = 0$ 　or　 $\lambda_{11}\lambda_{12} + \lambda_{21}\lambda_{22} = 0$

$\lambda_{11}^2 + \lambda_{12}^2 = \lambda_{21}^2\ \lambda_{22}^2 = 1$ 　　 $\lambda_{11}^2 + \lambda_{21}^2 = \lambda_{12}^2 + \lambda_{22}^2 = 1$

$|\lambda|^2 = |\lambda|\,|\lambda|$

$= |\lambda|\,|\lambda^t|$

$= |\lambda\lambda^t|$

$= \det \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{bmatrix} = \det \begin{bmatrix} \lambda_{11}^2 + \lambda_{12}^2 & \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} \\ \lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12} & \lambda_{21}^2 + \lambda_{22}^2 \end{bmatrix}$

$= \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

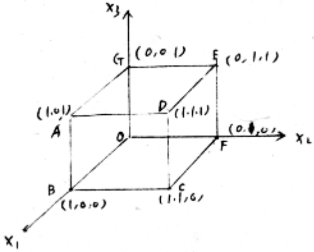
1-5 轉換時:線段長度不變. 以此法証明 (1-15)

[證] $x_i' = \sum_j \lambda_{ij} x_j$

$$\because \sum_k x_k^2 = \sum_i x_i'^2 = \sum_{i,j,k} (\lambda_{ij}x_j)(\lambda_{ik}x_k) = \sum_{j,k}\sum_i (\lambda_{ij}\lambda_{ik}) x_j x_k$$

$$\therefore \sum_{j,k} \delta_{jk} x_j x_k = \sum_{jk} \left(\sum_i \lambda_{ij}\lambda_{ik}\right) x_j x_k \qquad \therefore \delta_{jk} = \sum_i \lambda_{ij}\lambda_{ik}$$

1-6 有一立方体其一角頂与原点重合, 三鄰邊与三個垂直座標軸重合 求其各对角線, 以向量表示. 並求其夾角.



對角線 $\overrightarrow{AF} = -\hat{i} + \hat{j} - \hat{k}$

$\overrightarrow{BE} = -\hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{AF}$ 與 $\overrightarrow{BE}$ 之夾角設為 θ

則由: $\dfrac{\overrightarrow{AF} \cdot \overrightarrow{BE}}{|\overrightarrow{AF}|\,|\overrightarrow{BE}|} = \cos\theta$

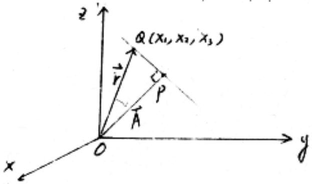
$$\cos\theta = \frac{1+1-1}{\sqrt{3}\,\sqrt{3}} = \frac{1}{3}$$

故 $\theta = \cos^{-1}\frac{1}{3}$

其他各对角線之間的夾角其求法与上面所示者相同

(一) 令 $\vec{A}$ 為自原點至空間某定點 P 之向量，使 $\vec{r}$ 為自原點至空間動點 Q 之向量

証明 $\vec{A}\cdot\vec{r}=A^2$ 表示一平面與 A 向量垂直，且通過 P



設過 P 垂直於 $\vec{A}$ 的平面為 E，Q 為 E 上任何一點

(i) Q = P 時 $\vec{r}=\vec{A}$

而 $\vec{A}\cdot\vec{r}=\vec{A}\cdot\vec{A}=A^2$

(ii) Q ≠ P 時

$$\vec{r}=\vec{A}+\overrightarrow{PQ}$$

$$\vec{A}\cdot\vec{r}=\vec{A}\cdot(\vec{A}+\overrightarrow{PQ})$$

$$=\vec{A}\cdot\vec{A}+\vec{A}\cdot\overrightarrow{PQ}$$

$$=A^2 \quad (\because \overrightarrow{PQ}\subset E \text{ 而 } \vec{A} \text{ 為 E 之法線})$$

由(i)(ii)，得 對於 E 上任何一點均滿足 $\vec{A}\cdot\vec{r}=A^2$ 之方程式

反之若空間任何一點 Q，若

$$\vec{A}\cdot\vec{r}=A^2$$

則 $\vec{A}\cdot(\vec{A}+\overrightarrow{PQ})=A^2$

$$A^2+\vec{A}\cdot\overrightarrow{PQ}=A^2$$

$$\Rightarrow \vec{A}\cdot\overrightarrow{PQ}=0$$

$\vec{A}\perp\overrightarrow{PQ}$ 故 $Q\in E$

1-8. 証明三矢純積量(Triple scalar product) $\vec{A}\times\vec{B}\cdot\vec{C}$ 寫為

$$\vec{A}\times\vec{B}\cdot\vec{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

並証明上式不因点乘及叉乘之換或、$\vec{A},\vec{B},\vec{C}$ 之輪換而變，即 $(\vec{A}\times\vec{B})\cdot\vec{C} = \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = (\vec{C}\times\vec{A})\cdot\vec{B}$ 等. 故吾人可用 $(\vec{A}\vec{B}\vec{C})$ 表示 triple scalar product，此外並解釋其幾何意义 計算由 $\vec{A}\vec{B}\vec{C}$ 所定義之平行六面体体積.

[証]: $$(\vec{A}\times\vec{B})\cdot\vec{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$(\vec{A}\times\vec{B})\cdot\vec{C} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \cdot \vec{C}$$

$$= C_1 \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} - C_2 \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} + C_3 \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}$$

$$= \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

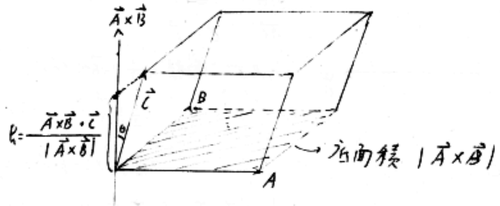
$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{B}\times\vec{C})\cdot\vec{A} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = (\vec{A}\times\vec{B})\cdot\vec{C}$$

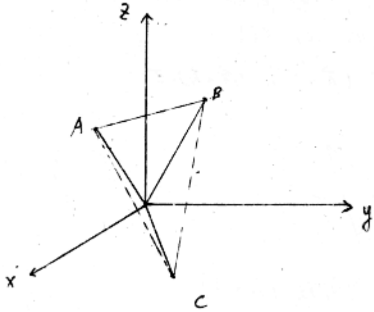
$$\vec{B}\cdot(\vec{C}\times\vec{A}) = (\vec{C}\times\vec{A})\cdot\vec{B} = \vec{C}\cdot(\vec{A}\times\vec{B}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

$$(\vec{C}\times\vec{A})\cdot\vec{B} = (\vec{A}\times\vec{B})\cdot\vec{C}$$



幾何意義

1-9 若 $\vec{a}, \vec{b}, \vec{c}$ 為三定向量，各自原點至A, B, C 者，求自原點至ABC所定之平面之距離及 $\triangle ABC$ 之面積



$\overrightarrow{AB} = \vec{b} - \vec{a}$

$\overrightarrow{AC} = \vec{c} - \vec{a}$

$\triangle ABC$ 之面積 $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \frac{1}{2}|(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})|$

$= \frac{1}{2}|\vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{a}\times\vec{c}|$

$= \frac{1}{2}|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}|$

由 $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ 所構成的 parallepipe

其体積為 $|\vec{a}\times\vec{b}\cdot\vec{c}|$

三角錐的体積為 $\frac{1}{3}|\vec{a}\times\vec{b}\cdot\vec{c}|$

O到平面ABC之距離

$$= \frac{1}{3}\frac{|\vec{a}\times\vec{b}\cdot\vec{c}|}{\triangle ABC\text{之面積}}$$

$$= \frac{2|\vec{a}\times\vec{b}\cdot\vec{c}|}{3|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$$

1-10 $\vec{X}$ 為未知向量，$\vec{A}, \vec{B}$ 為已知向量，純量 φ 滿足下式

$\vec{A}\times\vec{X} = \vec{B} \qquad \vec{A}\cdot\vec{X} = \varphi$

試以 $\vec{A}, \vec{B}, \varphi$ 及 $\vec{A}$ 之大小表示 $\vec{X}$

解：$|\vec{A}\times\vec{X}| = |\vec{A}||\vec{X}|\sin\theta = |\vec{B}|$

$\vec{A}\cdot\vec{X} = |\vec{A}||\vec{X}|\cos\theta = \varphi$

$\vec{X} = |\vec{X}|\cos\theta\frac{\vec{A}}{|\vec{A}|} + |\vec{X}|\sin\theta\frac{\vec{B}\times\vec{A}}{|\vec{B}||\vec{A}|}$

$$\vec{X} = |\vec{X}|\cos\theta\,\frac{\vec{A}}{|\vec{A}|} + |\vec{X}|\sin\theta\,\frac{\vec{B}\times\vec{A}}{|\vec{B}||\vec{A}|}$$

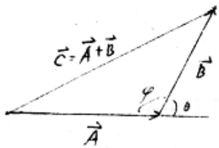
$$= \frac{\varphi}{|\vec{A}|}\,\frac{\vec{A}}{|\vec{A}|} + \frac{|\vec{B}|}{|\vec{A}|}\cdot\frac{\vec{B}\times\vec{A}}{|\vec{B}||\vec{A}|}$$

$$= \frac{\varphi\vec{A}}{A^2} + \frac{\vec{B}\times\vec{A}}{A^2}$$

$$= \frac{\vec{B}\times\vec{A}+\varphi\vec{A}}{A^2}$$

1-11 由 $(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$ 以求餘弦定律

解:



求証: $|\vec{C}|^2 = |\vec{A}|^2+|\vec{B}|^2-2|\vec{A}||\vec{B}|\cos\varphi$

$$|\vec{C}|^2 = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$$

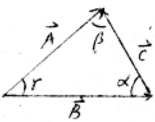
$$= \vec{A}\cdot\vec{A}+\vec{B}\cdot\vec{B}+2\vec{A}\cdot\vec{B}$$

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = -|\vec{A}||\vec{B}|\cos\varphi$$

代入得 $|\vec{C}|^2=|\vec{A}|^2+|\vec{B}|^2-2|\vec{A}||\vec{B}|\cos\varphi$

此即餘弦定律

1-12 由 $(\vec{A}\times\vec{B})$ 及 $(\vec{B}+\vec{C})\times\vec{B}$ 之恒等以求正弦定律



$$\vec{A}\times\vec{B} = (\vec{B}+\vec{C})\times\vec{B}$$

$$= \vec{B}\times\vec{B}+\vec{C}\times\vec{B} = \vec{C}\times\vec{B} = \vec{C}\times\vec{A}$$

$$|\vec{A}||\vec{B}|\sin\gamma = |\vec{C}||\vec{B}|\sin\alpha = |\vec{C}||\vec{A}|\sin\beta$$

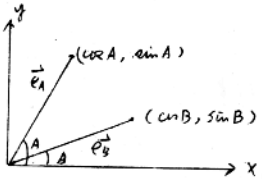
$$\frac{\sin\gamma}{|\vec{C}|} = \frac{\sin\beta}{|\vec{B}|} = \frac{\sin\alpha}{|\vec{A}|}$$

1-13 以向量代數導出下列各式

(a) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(b) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

解



$$\vec{e}_A \cdot \vec{e}_B = 1\cdot 1\ \cos(A-B)$$
$$= \cos A\cos B + \sin A\sin B$$

$\therefore \cos(A-B) = \cos A\cos B + \sin A\sin B$

$$\vec{e}_A \times \vec{e}_B = -\sin(A-B)\hat{k}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \end{vmatrix}$$
$$= -(\sin A\cos B - \cos A\sin B)\hat{k}$$

故 $\sin(A-B) = \sin A\cos B - \cos A\sin B$

1-14 証明

(a) $\sum_{ij} \epsilon_{ijk}\delta_{ij} = 0$

(b) $\sum_{jk} \epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il}$

(c) $\sum_{ijk} \epsilon_{ijk}\epsilon_{ijk} = 6$