Springer 大学数学图书——影印版

The Fundamental Theorem of Algebra

代数基本定理

Benjamin Fine
Gerhard Rosenberger 著

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内 容 提 要

本书对数学中最重要的定理——代数基本定理给出了六种证明，方法涉及到分析、代数与拓扑等数学分支。全书以一个问题为主线，纵横数学的几乎所有领域，结构严谨、文笔流畅、浅显易懂，适合高年级大学生、研究生自学和讨论，特别适合于用作短学期教材或数学选修类课程教材。

Benjamin Fine, Gerhard Rosenberger
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序　　言

在学校教书多年，当学生（特别是本科生）问有什么好的参考书时，我们所能推荐的似乎除了教材还是教材，而且不同教材之间的差别并不明显、特色也不鲜明。所以多年前我们就开始酝酿，希望为本科学生引进一些好的参考书，为此清华大学数学科学系的许多教授与清华大学出版社共同付出了很多心血。

这里首批推出的十余本图书，是从 Springer 出版社的多个系列丛书中精心挑选出来的。在丛书的筹划过程中，我们挑选图书最重要的标准并不是完美，而是有特色并包容各个学派（有些书甚至有争议，比如从数学上看也许不够严格），其出发点是希望我们的学生能够吸纳百家之长；同时，在价格方面，我们也做了很多工作，以使得本系列丛书的价格能让更多学校和学生接受，使得更多学生能够从中受益。

本系列图书按其定位，大体有如下四种类型（一本书可以属于多类，但这里限于篇幅不能一一介绍）。

一、适用面比较广、有特色并可以用作教材或参考书的图书。例如：

- Lovász et al.: Discrete Mathematics. 2003

  该书是离散数学的入门类型教材。与现有的教材（包括国外的教材）相比，它涵盖了离散数学新颖而又前沿的研究课题，同时还涉及信息科学方面既基本又有趣的应用；在着力打好数学基础的同时，也强调了数学与信息科学的关联。姚期智先生倡导和主持的清华大学计算机科学试验班，已经选择该书作为离散数学课程的教材。

二、在目前国内的数学教育中，课程主要以学科的纵向发展为主线，而对数学不同学科之间的联系讨论很少。学生缺乏把不同学科视为一个数学整体的训练，这方面的读物尤其欠缺。这是本丛书一个重要的着力点。最典型的是：

- Fine/Rosenberger: The Fundamental Theorem of Algebra. 1997

  该书对数学中最重要的定理——代数基本定理给出了六种证明，方法涉及到分析、代数与拓扑；附录中还给出了 Gauss 的证明和 Cauchy 的证明。全书以一个数学问题为主线展开，纵横数学的核心领域；结构严谨、文笔流畅、浅显易懂、引人入胜，是一本少见的能够让读者入迷的好读物，用它来引导学生欣赏和领会“数学的美”绝对不会落于空谈。该书适于自学、讨论，也是极好的短学期课程教材。

- Baker: Matrix Groups. 2001

  就内容而言，本书并不超出我国大学线性代数、抽象代数和一般拓扑学课程的内容，但是本书所讲的是李群和李代数的基础理论——这是现代数学和物理学非常重要的工具。各种矩阵群和矩阵代数是李群和李代数最典型和

最重要的例子，同时也能帮助学生建立数学不同学科之间的联系。从矩阵出发，既能把握李群和李代数的实质，又能学会计算和运用，所以这是一本不可多得的好书。

三、科学与技术的发展不断为数学提出新的研究课题，因此在数学学科的发展过程中，来自其他学科的推动力是不能忽视的。本系列中第三种类型的读物就是强调数学与其他学科的联系。例如：

- Woodhouse: Special Relativity. 2003

  该书将物理与数学有机结合，体现了物理学家伽利略的名言："大自然是一部用数学语言写成的巨著。" 不仅如此，本书作者还通过对线性代数、微积分、场论等数学的运用进一步强调并贯穿这样的观点：数学的真谛和发展存在并产生于物理或自然规律及其发现中。精读此书有助于理解物理学和数学的整体关系。

- Britton: Essential Mathematical Biology. 2003

  生命科学在本世纪一定会有很大发展，其对数学的需求和推动是可以预见的。因此生物数学在应用数学中占有日益重要的地位，数学系培养的学生至少一部分人应当对这个领域有所了解。随着生命科学的迅速发展，生物数学也发展很快。本书由浅入深，从经典的问题入手，最后走向学科前沿和近年的热点问题。该书至少可以消除学生对生物学的神秘感。

四、最后一类是适合本科学生的课外读物。这类图书对激发和引导学生学习数学的兴趣会非常有帮助，而且目前国内也急需这样的图书。例如：

- Daepp/Gorkin: Reading, Writing and Proving. 2003

  该书对初学高等数学的读者来说特别有意义。它的基本出发点是引导读者以研究的心态去学习，让读者养成独立思考的习惯，并进而成为研究型的学习者。该书将一个学习数学的过程在某种意义下程序化，努力让学习者养成一个好的学习习惯，以及学会如何应对问题。该书特色鲜明，类似的图书确实很少。

- Brzezniak/Zastawniak: Basic Stochastic Processes. 1998

  随机过程理论在数学、科学和工程中有越来越广泛的应用，本书适合国内的需要。其主要特点是：书中配有的习题是巩固和延伸学习内容的基本手段，而且有十分完整的解答，非常适合自学和作为教学参考书。这是一本难得的好书，它 1999 年出版，到 2000 年已经是第 3 次印刷，到 2003 年则第 6 次重印。

- Anglin/Lambek: The Heritage of Thales. 1995

  该书的基本内容是数学的历史和数学的哲学。数学历史是该书的线索，数学是内容的主体，引申到的是数学哲学。它不是一本史论型的著作，而是采用专题式编写方式，每个专题相对独立，所以比较易读、易懂，是本科生学习数学过程中非常好的课外读物。

本系列丛书中的大部分图书还将翻译为中文出版，以适应更多读者的需要。丛书筹划过程中，冯克勤、郑志勇、卢旭光、郑建华、王殿军、杨利军、叶俊、扈志明等很多清华大学的教授都投入了大量精力。他们之中很多人也将是后面中文版的译者。此外，我们今后还将不断努力丰富引进丛书的种类，同时也会将选书的范围在可能情况下进一步扩大到其他高水平的出版机构。

教育是科学和技术发展的基石，数学教育更是基石的基础。因为是基础所以它重要；也因为是基础所以它显示度不高，容易不被重视。只有将人才培养放到更高的地位上，中国成为创新型国家的目标才会成为可能。

本系列丛书的正式推出，圆了一个我们多年的梦，但这无疑仅仅是开始。

白峰杉

2006 年 6 月于清华园

To our families:

Linda, Carolyn, and David
Katariina, Anja, and Aila

Preface

These notes grew out of two courses, one given in the United States and one given in Germany on the Fundamental Theorem of Algebra. The purpose of these courses was to present a great deal of nonelementary mathematics, all centered on a single topic. The Fundamental Theorem of Algebra was ideal for this purpose. Analysis, algebra and topology each have developed different techniques which surround this result. These techniques lead to different proofs and different views of this important result. It is startling how much mathematics can be introduced and learned in this manner.

In the United States it was presented as a "capstone" course for upper level undergraduates. Many of the topics were familiar to the students but many were new. The goal of continually returning to a proof of the Fundamental Theorem of Algebra gave a focus to a large body of (what is at first glance) seemingly unrelated material. In addition, many nice applications, such as the insolvability of the quintic and the transcendence of e and π could be introduced. We feel that undergraduates in such a capstone course are an ideal audience for the book. Many departments in the U.S. are adopting the idea of a summary course. In addition, the book could serve as a foundation reference for beginning graduate students. We also feel that the algebra sections, Chapters 2, 3, 6, 7, could be used, with some additions from outside sources, as an alternative version of an undergraduate algebra course or as a supplement for such a course. The United States version of the course covered in one semester, with some omissions, most of the material in Chapters 1 through 7. The whole book could be covered at a relatively moderate pace in two semesters.

In Germany the material was presented to a class of potential teachers. A high school (or in Germany, *gymnasium*) teacher should be exposed to a wide range of mathematical topics. This material fulfilled this objective for this audience. It is our hope that similar teacher training courses in the U.S. might also adopt these notes. In the course in Germany, essentially the whole book was presented in two semesters.

We wish to thank Nicole Isermann for her extremely careful proofreading of the manuscript. We also wish to thank Kati Bencsath and Bruce Chandler for reading preliminary versions and making suggestions, and finally, we would like to thank Paul Halmos for his helpful suggestions.

Benjamin Fine, *Fairfield University, United States*
Gerhard Rosenberger, *Universität Dortmund, Germany*

Contents

CHAPTER 1 Introduction and Historical Remarks

The **Fundamental Theorem of Algebra** states that any complex polynomial must have a complex root. This basic result, whose first accepted proof was given by Gauss, lies really at the intersection of the theory of numbers and the theory of equations, and arises also in many other areas of mathematics. The purpose of these notes is to examine three pairs of proofs of the theorem. The first proof in each pair is fairly straightforward and depends only on what could be considered elementary mathematics. However, each of these first proofs lends itself to generalizations that in turn lead to more general results from which the Fundamental Theorem can be deduced as a direct consequence. These general results constitute the second proof in each pair.

Recall that a **complex polynomial** is a complex function of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_0, a_1, \ldots, a_n$ are complex numbers and n is a natural number. A **root**, or **zero**, of this polynomial is a complex number z_0 such that $P(z_0) = 0$.

The reasons for the different proofs of this result are due to the distinct characteristics of complex polynomials. First of all, complex polynomials are complex functions, that is, functions from C to C. As with real polynomials, complex polynomials are everywhere differentiable and so in the language of complex analysis are part of the class of **entire functions**. In this context the Fundamental Theorem of Algebra is a direct consequence of a general result called **Liouville's theorem**. This result states that an entire function that is bounded in the complex plane must be a constant. In Chapters 2 and 3 we introduce the basic results on complex numbers and complex polynomials. We then use these to present a proof of the Fundamental Theorem that utilizes only advanced calculus. This proof suggests Liouville's theorem. In Chapters 4 and 5 we then present the results from complex function theory – specifically complex differentiation, analytic functions, complex integration and Cauchy's theorem – needed to derive Liouville's theorem. From this we give our second proof of the Fundamental Theorem of algebra.

In a different direction, a complex polynomial is an algebraic object. In this context the Fundamental Theorem of Algebra can be phrased as, "the complex numbers are algebraically closed." In Chapter 6 we present the results concerning construction of field extensions and then present a proof of the Fundamental Theorem that depends only on the facts that odd-degree real polynomials have real roots and that given an irreducible polynomial $f(x)$ over a field F, a field extension $F*$ of F can be constructed such that $f(x)$ has a root in $F*$. This proof suggests the following generalization. If K is a field where odd-degree polynomials have roots and $i = \sqrt{-1}$, then $K(i)$ is algebraically closed. The proof of this generalization involves **Galois theory**. In Chapter 7 we present the basic results on group theory and Galois theory needed to understand this proof.

Finally, a complex polynomial is a topological mapping. If we adjoin the point at infinity to the complex plane we obtain a sphere, the **Riemann sphere** S^2. Since $P(\infty) = \infty$ for any complex polynomial $P(z)$, $P(z)$ can be considered as a continuous mapping $P : S^2 \to S^2$. Such topological mappings have what is termed a **winding number**, indicating how much the image of a curve C^1 on S^2 winds around when mapped to S^2. In a similar manner the function $f(z) = z^n$ winds the complex number z around the origin. In Chapter 8 we first present a proof of the Fundamental Theorem using the winding properties of $f(z) = z^n$. This is then generalized to winding numbers of functions $S^2 \to S^2$, from which the Fundamental Theorem is re-obtained. To handle this last generalization we must introduce some basic ideas and techniques in both point-set topology and algebraic topology. This is done Chapters 8 and 9. This final proof requires the most development and is therefore the least self-contained.

There are many variations of the proofs that we present. In a series of appendices we give six additional proofs, each somewhat different from those given in the main body of the notes. In Appendix A we give a modern version of Gauss's original first proof (see below). In Appendix C we present three additional proofs arising out of complex analysis. These require a more detailed analysis of Cauchy's theorem than the one given in Chapter 5. This analysis is given in Appendix B. Finally, in Appendix D we give two addditional topologically motivated proofs. These also depend on the concept of winding number but differ from the two given in Chapters 8 and 9.

We suppose that the reader has been introduced to advanced calculus as least as far as Green's theorem; has studied some abstract algebra, in particular the definitions of groups, rings and fields, and has studied some linear algebra, in particular the abstract definition of a vector space over a general field. Beyond this we have tried to make these notes as self-contained as possible. However, our goal is to arrive at an understanding of the proofs of the Fundamental Theorem. Therefore, along the way we have proved only those results that can be obtained directly and have left more difficult results (such as the proof of the fundamental theorem of Galois theory) to the references. A note here about terminology. In standard usage equations have roots, and functions and polynomials have zeros. Historically however, the word *root* has been most often connected with the Fundamental Theorem of Algebra. Therefore we use the term *root* throughout these notes, and do not really differentiate between the zero of the polynomial $P(z)$ and the root of the polynomial equation $P(z) = 0$.

The first mention of the Fundamental Theorem of Algebra, in the form that every polynomial equation of degree n has exactly n roots, was given by Peter Roth of Nurnberg in 1608. However its conjecture is generally credited to Girard who also stated the result in 1629. It was then more clearly stated by Descartes in 1637 who also distinguished between real and imaginary roots. The first published proof of the Fundamental Theorem of Algebra was then given by D'Alembert in 1746. However there were gaps in D'Alembert's proof and the first fully accepted proof was that given by Gauss in 1797 in his Ph.D. thesis. This was published in 1799. Interestingly enough, in reviewing Gauss's original proof, modern scholars tend to agree that there are as many holes in this proof as in D'Alembert's proof. Gauss, however, published three other proofs with no such holes. He published second and third proofs in 1816 while his final proof, which was essentially another version of the first, was presented in 1849.

In the main part of these notes we do not touch on Gauss's original proof, which in outline went as follows. Since $P(z)$ is a complex number for any $z \in C$ and since $z = x + iy$ with $x, y \in \mathbb{R}$ we have $P(z) = u(x, y) + iv(x, y)$. The equations $u(x, y) = 0$ and $v(x, y) = 0$ then represent curves in the plane $\mathbb{R}^2$. By a careful examination of the possible functions $u(x, y)$, $v(x, y)$ for a complex polynomial $P(z)$, Gauss showed that the curves $u(x, y) = 0$, $v(x, y) = 0$ must have a common solution (x_0, y_0). The complex number $z_0 = x_0 + iy_0$ is then a root of $P(z)$. In Appendix A we present a version of Gauss's original proof.

The Fundamental Theorem of Algebra is actually part of a general development in the theory of equations. The ability to solve quadratic equations and in essence the quadratic formula was known to the Babylonians some 3600 years ago. With the discovery of imaginary numbers, the quadratic formula then says that any degree two polynomial over C has a root in C. In the sixteenth century the Italian mathematician Niccolo Tartaglia discovered a similar formula in terms of radicals to solve cubic equations. This **cubic formula** is now known erroneously as **Cardano's**