计算工程与科学理论及应用

陈大鹏论文选

Computational Engineering and Sciences
Theories and Applications

陈大鹏 等 著

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——陈大鹏论文选

陈大鹏　等　著

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Alternative Ways for Formulation of Hybrid Stress Elements

0 Introduction

An element stiffness matrix by assumed compatible displacements can be derived not only by the conventional potential energy principle but also, indirectly, by generalized variational principles such as the Hu-Washizu principle and the Heilinger-Reissner principle. Tang and his colleagues[1, 2] have initiated a general and convenient method for constructing what they called quasi-conforming elements. They indicated that their method can be derived using the Hu-Washizu principle and pointed out that the compatible element is only a special case. Fraeijs de Veubeke[3] had cited his limitation principle and indicated that if no restrictions are applied to the assumed stress distribution the Hellinger-Reissner principle will yield the same element stiffness matrix as that by the assumed displacement method. Compatible elements are often found to be too rigid for finite element analyses and incompatible elements have been suggested by many authors[4-6]. There is, however, the lack of a rational procedure for constructing shape functions that will guarantee the resulting element to pass the patch test. The present note is to show a way to formulate incompatible elements for solid continuum and for plate bending problems by the Hellinger-Reissner principle. It turns out that the resulting scheme is equivalent to that recently suggested by Tong[7,8] for constructing hybrid stress elements. In Tong's scheme the inversion of a large flexibility matrix can be avoided. A review is also given of the quasi-conforming elements by Tang et al., indicating another possible derivation of hybrid stress elements that does not require the inversion of a large matrix.

1 Mixed Formulation by Hellinger-Reissner Principle

To derive the element stiffness matrix it is only necessary to express the boundary displacements n in terms of the nodal displacements q. When the strain energy U of the element is expressed in terms of q in the form of

$$U=\frac{1}{2}q^{\mathrm{T}}kq \tag{1}$$

then k is the element stiffness matrix.

When the displacements u in an element is not compatible with the boundary displacements $\bar{u}$ the Hellinger-Reissner principle takes the form

$$\pi_{\mathrm{R}}=\int_{V}\left[-\frac{1}{2}\sigma^{\mathrm{T}}S\sigma-\sigma^{\mathrm{T}}(\mathrm{D}u)\right]\mathrm{d}V-\int_{\partial V}T^{\mathrm{T}}(u-\bar{u})\,\mathrm{d}S=\text{Stationary} \tag{2}$$

Where σ = stresses, S= elastic compliance, T= boundary traction that is related to σ, V= volume of the element, and, ∂V = entire boundary of the element, and the strain-displacement relation is expressed as

$$\varepsilon=\mathrm{D}u \tag{3}$$

In the finite element formulation we separate element displacements u into two parts—the compatible part u_{c} which is expressed in terms of q and the additional part u_{a} which is expressed in terms of internal displacement parameters λ that can be statically condensed in the element level. Here, u_{a} may be incompatible along the boundary or it may be bubble functions which are zero along the boundary. If u_{a} is incompatible, equation (2) should be used in the formulation. By realizing that

$$\int_{V}\sigma(\mathrm{D}u_{\mathrm{a}})\mathrm{d}V=-\int_{V}(\mathrm{D}\sigma)^{\mathrm{T}}u_{\mathrm{a}}\mathrm{d}V+\int_{\partial V}T^{\mathrm{T}}u_{\mathrm{a}}\mathrm{d}S$$

and $u_{\mathrm{a}}=u=\bar{u}$ on ∂V, we have

$$\pi_{\mathrm{R}}=\int_{V}\left[-\frac{1}{2}\sigma^{\mathrm{T}}S\sigma+\sigma^{\mathrm{T}}(\mathrm{D}u_{\mathrm{c}})-(\mathrm{D}^{\mathrm{T}}\sigma)^{\mathrm{T}}u_{\mathrm{a}}\right]\mathrm{d}V \tag{4}$$

We also note that when u_{c} are bubble functions for which $u_{\mathrm{a}}=0$ on ∂V, the boundary integral term in equation (2) no longer appears, so equation (4) still holds. Since the equation

$$\mathrm{D}^{\mathrm{T}}\sigma=0 \tag{5}$$

represents the stress equilibrium conditions, the last term in the integral in equation (4) actually plays the role of the conditions of constraint with the corresponding Lagrange multipliers.

In the finite element implementation, we assume

$$\sigma = P\beta \tag{6}$$

where
$$P = \begin{pmatrix} c_1 & & \\ & c_2 & \\ & & \ddots \end{pmatrix} \tag{7}$$

and c_i are row vectors. Further, we assume

$$u_c = Nq \tag{8}$$

and
$$u_a = L\lambda \tag{9}$$

from which
$$\mathrm{D}u_c = Bq \tag{10}$$

and
$$\mathrm{D}^{\mathrm{T}}\sigma = E\beta \tag{11}$$

The functional π_R thus takes the form

$$\pi_R = -\frac{1}{2}\beta^{\mathrm{T}}H\beta + \beta^{\mathrm{T}}Gq - \beta^{\mathrm{T}}R_1\lambda \tag{12}$$

where
$$\left.\begin{aligned} H &= \int_V P^{\mathrm{T}}SP\mathrm{d}V \\ G &= \int_V P^{\mathrm{T}}B\mathrm{d}V \\ R_1 &= \int_V E^{\mathrm{T}}L\mathrm{d}V \end{aligned}\right\} \tag{13}$$

and

From the first variation of π_R with respect to β and λ we obtain

$$\beta = H^{-1}(Gq - R_1\lambda) \tag{14}$$

and
$$R_1^{\mathrm{T}}\beta = 0 \tag{15}$$

By eliminating λ and substituting β into

$$U = \frac{1}{2}\beta^{\mathrm{T}}H\beta \tag{16}$$

we obtain
$$k = G^{\mathrm{T}}MG - G^{\mathrm{T}}MR_1(R_1^{\mathrm{T}}MR_1)^{-1}R_1^{\mathrm{T}}MG \tag{17}$$

where
$$M = H^{-1} \tag{18}$$

We note that equation (17) can also be obtained by first obtaining a stiffness matrix with q and λ as nodal displacements and then eliminating

λ by static condensation.

The key point to be mentioned here is that if, for all the stress components which are coupled in the stress-strain relation, the same row vector c_i is used in equation (17), the inversion of H can be reduced to that of a matrix of the order of the c_i matrix[7-10] and the matrix $R^{\mathrm{T}}MR_1$ is, in general, also of much smaller order than that of H.

It is seen that equation (4) can also be changed to

$$\pi_{\mathrm{R}^*} = \int_V \left[-\frac{1}{2}\sigma^{\mathrm{T}} S\sigma - (\mathrm{D}^{\mathrm{T}}\sigma)^{\mathrm{T}} u_{\mathrm{c}} - \mathrm{D}^{\mathrm{T}}\sigma^{\mathrm{T}} u_{\mathrm{a}} \right] \mathrm{d}V + \int_{\partial V} T^{\mathrm{T}} u_{\mathrm{c}} \mathrm{d}S \tag{19}$$

If u_{a} is properly chosen so that, under the present variational formulation, equation (5) is identically satisfied, the second term in the volume integral in equation (19) can be omitted; the resulting variational functional then becomes

$$\pi_{\mathrm{R}^{**}} = \int_V \left[-\frac{1}{2}\sigma^{\mathrm{T}} S\sigma - (\mathrm{D}^{\mathrm{T}}\sigma)^{\mathrm{T}} u_{\mathrm{a}} \right] \mathrm{d}V + \int_{\partial V} T^{\mathrm{T}} u_{\mathrm{c}} \mathrm{d}S \tag{20}$$

$$\pi_{\mathrm{R}^{**}} = -\frac{1}{2}\beta^{\mathrm{T}} H\beta - \beta^{\mathrm{T}} R_1\lambda + \beta^{\mathrm{T}} G_1 q \tag{21}$$

Although in the original stress assumption, the equilibrium condition is not incorporated, the variational process now enforces the condition. The resulting element by either equation (4) or equation (20) can thus be named a hybrid stress model.

Equation (21) represents exactly the same procedure suggested by Tong[7,8] for the formulation of hybrid stress elements. For a 3-D solid problem with stress distribution ($\sigma = P\beta$) expressed in terms of Cartesian co-ordinates as in Reference[8], the equations of equilibrium [equation(5)] which are of polynomial type can be satisfied pointwise by setting the coefficient of each polynomial term to zero. This results in constraint equations of the form

$$R^{\mathrm{T}}\beta = 0 \tag{22}$$

Tong, then, suggested that for the hybrid formulation one need to take the first and last term in equation (21) and to introduce the Lagrange multiplier term $\beta^{\mathrm{T}}R\lambda$. The number of Lagrange multipliers λ, then, must be equal to the number of independent equations in equation (22). Here in the present formulation by π_{R}, the number of terms in u_{a} should also be the same number. We note that we cannot reduce $\pi_{\mathrm{R}^{**}}$ to π_{R^*} in a reverse process. Another remark is that if the stresses σ are expressed in terms of variables in a general co-ordinate system such as the natural co-ordinates for isoparametric elements, then, in general, the equations of equilibrium are only approximately satisfied. In such a case, however, equations (4) and (19) still hold.

We should remark that the incompatible element Q_6 by Wilson et al.[4] was not derived by using equation (4). Thus, although in the case of a rectangular shape, element Q_6 was shown to be identical to the hybrid stress element by Pian[11, 12], it does not pass the patch test for an element of a general geometric shape.

The next example for the hybrid stress element is the plate bending problem for which it is not an easy task to construct, within the element, a compatible displacement field in terms of nodal displacements, q, while it is simple to express boundary displacements $\overline{u}$ in terms of $\hat{q}$. In this case again, equation (2) should be employed. It is obvious that the element displacements u should be at least of the same order as that of $\overline{u}$. If u is entirely independent of $\overline{u}$, say u is equal to $L\lambda$, then the number of λ should be about the same as that of q and the total number of displacement parameters will be at least double that of nodal displacement q. Then, in order to avoid rank deficiency of the resulting element stiffness matrix, higher order stress approximation must be made. In practice, we let

$$u = \hat{u} + u_{\mathrm{a}} \tag{23}$$

where $\hat{u}$ is an incompatible displacement in terms of q, and u_{a} consists of only a small number of terms with parameters λ.

Integrating equation (2), we obtain

$$\pi_{\mathrm{R}^{\cdot}} = \int_V \left[-\frac{1}{2}\sigma^{\mathrm{T}} S \sigma - (\mathrm{D}^{\mathrm{T}}\sigma)^{\mathrm{T}} \hat{u} - (\mathrm{D}^{\mathrm{T}}\sigma)^{\mathrm{T}} u_{\mathrm{a}} \right] \mathrm{d}V + \int_{\partial V} T^{\mathrm{T}} \overline{u}\, \mathrm{d}S \tag{24}$$

Here, by proper choice of u_{a}, the equilibrium equations can be identically satisfied by the variational process, and we may again write simply

$$\pi_{\mathrm{R}^{\cdot\cdot}} = \int_V \left[-\frac{1}{2}\sigma^{\mathrm{T}} S \sigma - (\mathrm{D}^{\mathrm{T}}\sigma)^{\mathrm{T}} u_{\mathrm{a}} \right] \mathrm{d}V + \int_{\partial V} T^{\mathrm{T}} \overline{u}\, \mathrm{d}S \tag{25}$$

This is precisely what Tong[7] has used to arrive at his new scheme for the construction of plate bending elements under the assumed stress hybrid model. In one of Tong's examples, the assumed moment distribution is quadratic and there exists only one equilibrium constraint for β, hence there should be only one λ. In face, in this case, u_a may be any function in terms of one parameter and may even be a constant.

2 Formulation by Hu-Washizu Principle

The quasi-conforming elements initiated by Tang[1] can be derived by the Hu-Washizu principle. For an element with boundary displacements $\bar{u}$ the variational functional is

$$\pi_{HW} = \int_V \left(\frac{1}{2}\varepsilon^T C\varepsilon - \sigma^T\varepsilon + \sigma^T \mathrm{D}u \right) \mathrm{d}V + \int_{\partial V} T^T (u - \bar{u})\,\mathrm{d}S \tag{26}$$

where

$$C = S^{-1} \tag{27}$$

and strains ε, stresses σ, element displacements u and boundary displacements $\bar{u}$ are independent.

The key step in the finite element formulation is that both σ and ε are approximated by the same function, i.e.

$$\varepsilon = P\alpha \tag{28}$$

and

$$\sigma = P\beta \tag{29}$$

where P is given by equation (7).

The integral of $\sigma^T\varepsilon$ over the element volume can be written as $\beta^T \bar{H}\alpha$ where $\bar{H}$ is symmetric and positive definite and can be expressed, in general, in the form of

$$H = \begin{pmatrix} B_1 & & & & \\ & B_2 & & & \\ & & \ddots & & \\ & & & B_i & \\ & & & & \ddots \end{pmatrix} \tag{30}$$

where

$$B_i = \int_V c_i^T c_i \mathrm{d}V \tag{31}$$

Thus, the inversion of $\bar{H}$ is reduced to that of the individual submatrices B_i. In fact, in many cases it is possible to choose reference co-ordinates for the c_i functions such that B_i are all diagonal matrices.

The treatment of displacement u and $\bar{u}$ are the same as those discussed earlier and the stress equilibrium condition can be introduced accordingly. For example, similar to the implementation of equation (4)

$$\pi_{\mathrm{HW}} = \frac{1}{2}\alpha^{\mathrm{T}} J\alpha - \beta^{\mathrm{T}} H\alpha + \beta^{\mathrm{T}} Gq - \beta^{\mathrm{T}} R_1\lambda \tag{32}$$

where

$$J = \int_V P^{\mathrm{T}} CP\mathrm{d}V \tag{33}$$

Variation of π_{HW} with respect to β thus leads to

$$\alpha = \bar{H}^{-1}(Gq - R\lambda) \tag{34}$$

Substituting α into the strain energy expression the element stiffness matrix with q and λ as nodal displacements can be obtained and λ can then be eliminated by static condensation. The resulting element stiffness matrix can also be expressed by equation (17) with

$$M = \bar{H}^{-1} J\bar{H}^{-1} \tag{35}$$

If the row vectors c_i are the same for all stress components which are coupled in the stress-strain relation, it can be shown that the present result is identical to that obtained by π_{R}. Tang and his colleagues have applied the quasi-conforming elements to 2-D and 3-D isoparametric elements and to plates and shells[1,2,13,14]. Many of their resulting elements are identical to hybrid stress elements.

3 Conclusion

The introduction of additional internal displacement modes in mixed finite element formulations by the Hellinger-Reissner principle and the Hu-Washizu principle can lead to element stiffness matrices that are equivalent to the assumed stress hybrid method. The new approach will yield more flexible and more efficient methods. For example, because exact equilibrium is not required, the assumed stresses may be expanded in terms of natural co-ordinates, such as the isoparametric co-ordinates, and the formulation will be simplified and the resulting element stiffness matrix will be less sensitive to the reference Cartesian co-ordinate system. For shell elements, the new formulation can always avoid the coupling between the membrane stresses and moment stresses in the H matrix: hence, it will lead to simpler computing procedures.

References

[1] TANG L M. CHEN W J, LIU Y X. Quasi-conforming element and generalized

variational illegalities. paper presented at CMES CSTAM CMS Symp. on Finite Element Methods, 1981, 5: 19-23.

[2] CHEN W J, TANG L M. Isoparametric quasi-conforming element (in Chinese)[J]. Da-Lian Inst Technol, 1981, 20 (1): 63-74.

[3] de VEUBEKE B M F. Displacement and equilibrium models in finite element method. Stress Analysts, Eds. Wiley, 1965: 145-197.

[4] WILSON E L, TAYLOR R L, DOHERTY W, et al. Incompatible displacement models. Numerical and Computer Methods in Structural Mechanics Academic Press, 1973, 43-57.

[5] TAYLOR R L, BERESFORD P J, WILSON E L. A non-conforming element for stress analysis. Int J Num Meth Engng, 1976, 10: 1 211-1 219.

[6] WACHSPRESS E L. Incompatible quadrilateral basis functions. Int J Num Meth Engng, 1978, 12: 589-595.

[7] TONG P. A family of hybrid plate elements. Int J Num Meth Engng, 1982, 18: 1 455-1 468.

[8] TONG P. On the construction of three dimensional hybrid stress elements, to be published.

[9] HORRIGMOE G. Finite element instability analysis of free-form shells. Norwegian Inst. of Technology, Div. of Structural Mechanics, Report No. 77-2, 1977.

[10] BATOZ J L, BATHE K J, HO L W. A study of three-node triangular plate bending elements. Int J Num Meth Engng, 1980, 15: 1 771-1 812.

[11] PIAN T H H. Derivation of element stiffness matrices by assumed stress distributions. AIAA J, 1964, 2: 1 333-1 336.

[12] EROTER M, NILSSON L, SAMUELSSON A. The rectangular plane stress element by Turner. Pian and Wilson. Int J Num Meth Engng, 1974, 8: 433-437.

[13] CHEN W J, A high accuracy eight-node hexagonal element (in Chinese). Da-Lian Inst of Technology, Engineering Mechanics Research Lab. Report, 1981.

[14] LIU Y X, LU H X, TANG L M. Quasi-conforming cylindrical shell element (in Chinese). Da-Lian Inst. of Tehonology, Engineering Mechanics Research Lab. Report, 1980.

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A New Formulation of Hybrid/mixed Finite Element

【Abstract】 A new formulation of finite element method is accomplished by the Hellinger-Reissner principle for which the stress equilibrium conditions are not introduced initially but are brought-in through the use of additional internal displacement parameters. The method can lead to the same result as the assumed stress hybrid model. However, it is more general and more flexible. The use of natural coordinates for stress assumptions leads to elements which are less sensitive to the choice of reference coordinates. Numerical solutions by 3-D solid element indicate that more efficient elements can be constructed by assumed stresses which only partially satisfy the equilibrium conditions.

0 Introduction

Since its initial introduction[1], the assumed stress hybrid method has been extended to different types of elements and for both linear and nonlinear problems[2-7]. It was also recognized that the method can be formulated using the Hellinger-Reissner principle[8] except that for the assumed stressed the equilibrium conditions are initially satisfied. The method, thus, may be named hybrid/mixed finite element method. Alternative formulations for the assumed stress method have been suggested[9] using the Hellinger-Reissner principle and the Hu-Washizu principle in which the stress equilibrium conditions are not considered initially but are brought-in as constraint equations by using additional internal displacements as Lagrange multipliers.

This paper is to present the new formulation by the Hellinger-Reissner principle and to describe its generality and flexibility using examples in 3-D solids.

1 Formulation by Hellinger-Reissner Principle

In using the Hellinger-Reissner principle to formulate the element stiffness matrix the following functional π_R for an element should be stationary,

$$\pi_R = \int_V \left[-\frac{1}{2}\sigma^T S\sigma + \sigma^T(\mathrm{D}u) \right] \mathrm{d}V - \int_{\partial V} T^T(u-\bar{u})\mathrm{d}S \tag{1}$$

where σ = stresses; u = element displacements; $\bar{u}$ = element boundary displacements in terms of nodal displacements; S = elastic compliance matrix; T = boundary traction = $\nu\sigma$; ν = directional cosine of surface normal.

Here the equation

$$\varepsilon = \mathrm{D}u \tag{2}$$

in the strain displacement relation and the corresponding homogeneous equilibrium condition is

$$\mathrm{D}^T\sigma = 0 \tag{3}$$

When the element displacements u are compatible, i.e. $u=\bar{u}$ on ∂V, and when the assumed stresses satisfy equation (3), an element stiffness matrix k derived by equation (1) is the same as that by the original hybrid stress model[8]. In this case, the various components in the assumed stresses are coupled and hence the complete flexibility matrix H of $m \times m$, where m is the number of stress parameters, must be inverted. Also due to the equilibrium constraints the assumed stresses are expressed in Cartesian coordinates.

In the new formulation, the stress equilibrium conditions equation (3) are not satisfied but the element displacements u are divided into two parts

$$u = u_q + u_\lambda \tag{4}$$

where

$$u_q = Nq \tag{5}$$

are compatible displacements in terms of nodal displacements q and

$$u_\lambda = M\lambda \quad (6)$$

where λ are additional internal displacement parameters. M may be incompatible or may be bubble functions.

By integrating the terms with u_λ and by recognizing, that

$$u_\lambda = u - \overline{u} \quad \partial V$$

one obtains from equation (1)

$$\pi_{\mathrm{R}} = \int_V \left[-\frac{1}{2}\sigma^{\mathrm{T}} S\sigma + \sigma^{\mathrm{T}}(\mathrm{D}u_q) - (\mathrm{D}^{\mathrm{T}}\sigma)^{\mathrm{T}} u_\lambda \right] \mathrm{d}V \quad (7)$$

It is seen that the last term in equation (7) contains the stress equilibrium condition with u_λ as Lagrange multipliers. Thus, in the finite element formulation the stresses need not initially satisfy the equilibrium condition, since the introduction of u_λ will enforce this condition. By expressing stresses in terms of stress parameters β

$$\sigma = P\beta \quad (8)$$

π_{R} can be written as

$$\pi_{\mathrm{R}} = -\frac{1}{2}\beta^{\mathrm{T}} H\beta + \beta^{\mathrm{T}} Gq - \beta^{\mathrm{T}} R\lambda \quad (9)$$

where

$$H = \int_V P^{\mathrm{T}} SP\mathrm{d}V \quad (10)$$

$$G = \int_V P^{\mathrm{T}}(DN)\,\mathrm{d}V \quad (11)$$

and

$$R = \int_V (D^{\mathrm{T}} P)^{\mathrm{T}} M\mathrm{d}V \quad (12)$$

The stationary condition of π_{R} then yields

$$\beta = H^{-1}(Gq - R\lambda) \quad (13)$$

and

$$R^{\mathrm{T}}\beta = 0. \quad (14)$$

Eliminating λ and substituting β into the strain energy expression one obtains the element stiffness matrix

$$k = \overline{G}^{\mathrm{T}} H^{-1}\overline{G} \quad (15)$$

where

$$\overline{G} = G - R(R^{\mathrm{T}} H^{-1} R)^{-1} R^{\mathrm{T}} H^{-1} G \quad (16)$$

Equation (14) represents the equilibrium constraints for the stress parameter β. If equation (14) is applied such that the equilibrium conditions are