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郭柏灵 著



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序

今年是恩师郭柏灵院士 70 寿辰,华南理工大学出版社决定出版《郭柏灵论文集》。郭老师的弟子,也就是我的师兄弟,推举我为文集作序。这使我深感荣幸。我于 1985 年考入北京应用物理与计算数学研究所,师从郭柏灵院士和周毓麟院士。研究生毕业后我留在研究所工作,继续跟随郭老师学习和研究偏微分方程理论。老师严谨的治学作风和对后学的精心培养与殷切期望,给我留下了深刻的印象,同时老师在科研上的刻苦精神也一直深深地印在我的脑海中。

郭老师 1936 年生于福建省龙岩市新罗区龙门镇,1953 年从福建省龙岩市第一中学考入复旦大学数学系,毕业后留校工作。1963 年,郭老师服从祖国的需要,从复旦大学调入北京应用物理与计算数学研究所,从事核武器研制中有关的数学、流体力学问题及其数值方法研究和数值计算工作。他全力以赴地做好了这项工作,为我国核武器的发展做出了积极的贡献。1978 年改革开放以后,他又在非线性发展方程数学理论及其数值方法领域开展研究工作,现为该所研究员、博士生导师,中国科学院院士。迄今他共发表学术论文 300 余篇、专著 9 部,1987 年获国家自然科学三等奖,1994 年和 1998 年两度获得国防科工委科技进步一等奖,为我国的国防建设与人才培养做出了巨大贡献。

郭老师的研究方向涉及数学的多个领域,其中包括非线性发展方程的数学理论及其数值解、孤立子理论、无穷维动力系统,其研究工作的主要特点是紧密联系数学物理中提出的各种重要问题。他对力学及物理学等应用学科中出现的许多重要的非线性发展方程进行了系统深入的研究,其中对 Landau-Lifshitz 方程和 Benjamin-Ono 方程的大初值的整体可解性、解的唯一性、正则性、渐近行为以及爆破现象等建立了系统而深刻的数学理论。在无穷维动力系统方面,郭老师研究了一批重要的无穷维动力系统,建立了有关整体吸引子、惯性流形和近似惯性流形的存在性和分形维数精细估计等理论,提出了一种证明强紧吸引子的新方法,并利用离散化等方法进行理论分析和数值计算,展示了吸引子的结构和图像。下面我从这几个方面介绍郭老师的一些学术成就。

Landau-Lifshitz 方程(又称铁磁链方程)由于其结构的复杂性,特别是强耦合性和不带阻尼时的强退化性,在 20 世纪 80 年代之前国内外几乎没有从数学上进行理论研究的成果出现。最先进行研究的,当属周毓麟院士和郭老师,他们在 1982 年到 1986 年间,采用 Leray-Schauder 不动点定理、离散方法、Galerkin 方法证明了从一维到高维的各种边值问题整体弱解的存在性,比国外在 1992 年才出现的同类结果早了将近 10 年。

20 世纪 90 年代初期,周毓麟、郭柏灵和谭绍滨,郭柏灵和洪敏纯得到了两个在国内外至今影响很大的经典结果。第一,通过差分方法结合粘性消去法,利用十分巧妙的先验估计,证明了一维 Landau-Lifshitz 方程光滑解的存在唯一

性,对于一维问题给出了完整的答案,解决了长期悬而未决的难题。第二,系统分析了带阻尼的二维 Landau - Lifshitz 方程弱解的奇性,发现了 Landau - Lifshitz 方程与调和映照热流的联系,其弱解具有与调和映照热流完全相同的奇性。现在,国内外这方面的文章基本上都引用这个结果。调和映照的 Landau - Lifshitz 流的概念,即是源于此项结果。

20 世纪 90 年代中期,郭老师对于 Landau - Lifshitz 方程的长时间性态、Landau - Lifshitz 方程耦合 Maxwell 方程的弱解及光滑解的存在性问题进行了深入的研究,得到了一系列的成果。铁磁链方程的退化性以及缺少相应的线性化方程解的表达式,对研究解的长时间性态带来很大困难。郭老师的一系列成果克服了这些困难,证明了近似惯性流形的存在性、吸引子的存在性,给出了其 Hausdorff 和分形维数的上、下界的精细估计。此外,我们知道,与调和映照热流比较,高维铁磁链方程的研究至今还很不完善。其中最重要的是部分正则性问题,其难点在于单调不等式不成立,导致能量衰减估计方面的困难。另外一个问题是 Blow-up 解的存在性问题,至今没有解决;而对于调和映照热流来说,这样的问题的研究是比较成熟的。

对于高维问题,20 世纪 90 年代后期至今,郭老师和陈韵梅、丁时进、韩永前、杨千山一道,得出了许多成果,大大地推动了该领域的研究。首先,证明了二维问题的能量有限弱解的几乎光滑性及唯一性,这个结果类似于 Freire 关于调和映照热流的结果。第二,得到了高维 Landau - Lifshitz 方程初边值问题的奇点集合的 Hausdorff 维数和测度的估计。第三,得到了三维 Landau - Lifshitz - Maxwell 方程的奇点集合的 Hausdorff 维数和测度的估计。第四,得到了一些高维轴对称问题的整体光滑解和奇性解的精确表达式。郭老师还开创了一些新的研究领域。例如,关于一维非均匀铁磁链方程光滑解的存在唯一性结果后来被其他数学家引用并推广到一般流形上。其次,率先讨论了可压缩铁磁链方程测度值解的存在性。最近,在 Landau - Lifshitz 方程耦合非线性 Maxwell 方程方面,也取得了许多新的进展。

多年来,郭老师还对一大批非线性发展方程解的整体存在唯一性、有限时刻的爆破性、解的渐近性态等开展了广泛而深入的研究,受到国内外同行的广泛关注。研究的模型源于数学物理、水波、流体力学、相变等领域,如含磁场的 Schrödinger 方程组、Zakharov 方程、Schrödinger - Boussinesq 方程组、Schrödinger - KdV 方程组、长短波方程组、Maxwell 方程组、Davey - Stewartson 方程组、Klein - Gordon - Schrödinger 方程组、波动方程、广义 KdV 方程、Kadomtsev - Petviashvili(KP)方程、Benjamin - Ono 方程、Newton - Boussinesq 方程、Cahn - Hilliard 方程、Ginzburg - Landau 方程等。其中不少耦合方程组都是郭老师得到了第一个结果,开创了研究的先河,对国内外同行的研究产生了深远的影响。

郭老师在无穷维动力系统方面也开展了广泛的研究,取得了丰硕的成果。

对耗散非线性发展方程所决定的无穷维动力系统,研究了整体吸引子的存在性、分形维数估计、惯性流形、近似惯性流形、指数吸引子等问题。特别是在研究无界域上耗散非线性发展方程的强紧整体吸引子存在性时所提出的化弱紧吸引子成为强紧吸引子的重要方法和技巧,颇受同行关注并广为利用。对五次非线性 Ginzburg - Landau 方程,郭老师利用空间离散化方法将无限维问题化为有限维问题,证明了该问题离散吸引子的存在性,并考虑五次 Ginzburg - Landau 方程的定态解、慢周期解、异宿轨道等的结构。利用有限维动力系统的理论和方法,结合数值计算得到具体的分形维数(不超过 4)和结构以及走向混沌、湍流的具体过程和图像,这是一种寻求整体吸引子细微结构新的探索和尝试,对其他方程的研究也是富有启发性的。1999 年以来,郭老师集中于近可积耗散的和 Hamilton 无穷维动力系统的结构性研究,利用孤立子理论、奇异摄动理论、Fenichel 纤维理论和无穷维 Melnikov 函数,对于具有小耗散的三次到五次非线性 Schrödinger 方程,证明了同宿轨道的不变性,并在有限维截断下证明了 Smale 马蹄的存在性,目前,正把这一方法应用于具小扰动的 Hamilton 系统的研究上。他对于非牛顿流无穷维动力系统也进行了系统深入的研究,建立了有关的数学理论,并把有关结果写成了专著。以上这些工作得到国际同行们的高度评价,被称为“有重大的国际影响”、“对无穷维动力系统理论有重要持久的贡献”。最近,郭老师及其合作者又证明了具耗散的 KdV 方程 L^2 整体吸引子的存在性,该结果也是引人注目的。

郭老师不仅自己辛勤地搞科研,还尽心尽力培养了大批的研究生(硕士生、博士生、博士后),据不完全统计,有 40 多人。他根据每个人不同的学习基础和特点,给予启发式的具体指导,其中的不少人已成为了该领域的学科带头人,有些人虽然开始时基础较差,经过培养,也得到了很大提高,成为了该方向的业务骨干。

《郭柏灵论文集》按照郭老师在不同时期所从事的研究领域,分成多卷出版。文集中所搜集的都是郭老师正式发表过的学术成果。把这些成果整理成集出版,不仅系统地反映了他的科研成就,更重要的是对于从事这方面学习、研究的学者无疑大有裨益。这本文集的出版得到了多方面的帮助与支持,特别要感谢华南理工大学校长李元元教授、华南理工大学出版社范家巧社长和华南理工大学数学科学学院吴敏院长的支持。还要特别感谢华南理工大学的李用声教授、华南师范大学的丁时进教授、北京应用物理与计算数学研究所的苗长兴研究员等人在论文的搜集、选择与校对等工作中付出了辛勤的劳动。感谢华南理工大学出版社的编辑对文集的精心编排工作。

谭绍滨

2005 年 8 月于厦门大学

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The Global Solution for Some Systems of Nonlinear Schrödinger Equations*

Guo Boling(郭柏灵)

§ 1. Introduction

One class of the system of nonlinear Schrödinger equations(SNSE)

$$\begin{aligned}iu_{1t} &= u_{1xx} + 2u_1(\sigma_{11}|u_1|^2 + \sigma_{12}|u_2|^2) \\iu_{2t} &= u_{2xx} + 2u_2(\sigma_{11}|u_1|^2 + \sigma_{12}|u_2|^2)\end{aligned}\quad (1.1)$$

has been proposed in [1,2], and the scattering inverse method for the system (1.1) has been studied in [2]. The existence of the global solution for the nonlinear Schrödinger equation has been studied, see [3,4]. In this paper, we shall examine the existence and uniqueness of the global solution for some broad systems of the nonlinear Schrödinger equations (including system (1.1)), and consider the problems for some systems of multi-dimensional nonlinear Schrödinger equations that concern more dimensional solitons. To obtain a global solution in time, we use the method of perturbation and the Galerkin method. We shall use the standard notations; let $\Omega \in \mathbf{R}^n$ be a smooth bounded open subset and $\partial\Omega$ its boundary. For the definitions of the spaces $C^m(\Omega)$, $L_p(\Omega)$, $L_\infty(\Omega)$, $L(0, T; H^s)$, $\dots$, see [2,5].

§ 2. The Initial Value Problem and the Periodic Initial Value Problem of Some SNSE

We consider the following initial value problem

$$u_{1t} - iu_{1xx} + \beta q(|u_1|^2 + |u_2|^2)u_1 + k_1(x, t)u_1 = f_1(x, t); \quad (2.1)$$

$$u_{2t} - iu_{2xx} + \beta q(|u_1|^2 + |u_2|^2)u_2 + k_2(x, t)u_2 = f_2(x, t); \quad (2.2)$$

$$u_1|_{t=0} = u_1^0(x); \quad u_2|_{t=0} = u_2^0(x), \quad -\infty < x < \infty, \quad (2.3)$$

and its periodic initial value problem, i.e., the periodic functions $u_j(x, t)$ (the variable x with period 2π) satisfy (2.1)–(2.3), $j=1, 2$, where $\beta > 0$, $i = \sqrt{-1}$, unknown functions u_1, u_2 are complex valued functions; $q(s)$ is a real valued function in $[0, \infty)$, and $q(s) \geq 0$; $k_j(x, t), f_j(x, t)$ and $u_j^0(x)$ ($j=1, 2$) are known complex valued functions. We require that u_1, u_2 and some of their derivatives with respect to x tend to zero as $|x| \rightarrow \infty$ for the

* Proc. of DD-1 Symposium, 1980(3):1227~1246.

problem (2.1)–(2.3).

Consider the corresponding regularized problem for the problem(2.1)–(2.3)

$$u_{1t} - i u_{1xx} + \beta q(|u_1|^2 + |u_2|^2) u_1 - \epsilon u_{1xx} + k_1(x, t) u_1 = f_1(x, t); \quad (2.4)$$

$$u_{2t} - i u_{2xx} + \beta q(|u_1|^2 + |u_2|^2) u_2 - \epsilon u_{2xx} + k_2(x, t) u_2 = f_2(x, t); \quad (2.5)$$

$$u_1|_{t=0} = u_0^1(x); \quad u_2|_{t=0} = u_0^2(x), \quad -\infty < x < \infty, \epsilon > 0. \quad (2.6)$$

Lemma 1. Let u_1, u_2 be a solution of the problem (2.4)–(2.6). Assume that

(i) $|k_j| \leq M, \|f_j\|_{L_2 \times L_\infty}^2 \leq M_1 \quad (j=1,2)$, where M, M_1 are constants;

(ii) $u_0^j(x) \in L_2 \quad (j=1,2)$.

Then we have the following estimates:

$$\begin{aligned} \sum_{j=1}^2 \|u_j\|_{L_2 \times L_\infty} &\leq E_0; \\ \beta \int_0^T \left(\sum_{j=1}^2 \|q|u_j|^2\|_{L_1} \right) dt + \epsilon \int_0^T \left(\sum_{j=1}^2 \|u_{jx}\|_{L_2}^2 \right) dt &\leq C_3, \end{aligned} \quad (2.7)$$

where the constants E_0, C_3 are independent of ϵ .

Lemma 2. Suppose that the conditions of Lemma 1 are satisfied and that

(i) $q'(s) \geq 0$ in $[0, \infty)$;

(ii) $|k_j| \leq M, |k_{jx}| \leq M, \sum_{j=1}^2 \|f_{jx}\|_{L_2}^2 \leq M_1 \quad (j=1,2)$;

(iii) $\sum_{j=1}^2 \|u_{jx}(0)\|_{L_2}^2 \leq M_2$.

Then we have the estimates

$$\sum_{j=1}^2 \|u_{jx}\|_{L_2 \times L_\infty}^2 \leq E_1, \quad \epsilon \int_0^T \left(\sum_{j=1}^2 \|u_{jxx}\|_{L_2}^2 \right) dt \leq C_3, \quad (2.8)$$

where

$$E_1 = (M_2 + 2C_2 T) e^{C_1 T}, \quad C_1 = 3M + 1, \quad C_2 = ME_0 + M_1,$$

$$C_3 = \left(\frac{C_1}{2} + 1 \right) E_1 + \frac{C_2}{2} T.$$

Proof. We differentiate (2.4) and (2.5) with respect to x , multiply them by $\bar{u}_{1x}$ and $\bar{u}_{2x}$, respectively, integrate over $(-\infty, \infty)$, take the real part. Noticing that

$$\beta \sum_{j=1}^2 \operatorname{Re}((qu_j)_x, u_{jx}) = \beta \sum_{j=1}^2 \int_{-\infty}^{\infty} q |u_{jx}|^2 dx + \beta \int_{-\infty}^{\infty} q' \left(\frac{\partial}{\partial x} (|u_1|^2 + |u_2|^2) \right) dx \geq 0,$$

we obtain(2.8).

Lemma 3. Suppose that the conditions of Lemma 2 are satisfied and that

(i) $|k_j| \leq M, |k_{jxx}| \leq M; \sum_{j=1}^2 \|f_{jxx}\|_{L_2 \times L_\infty}^2 \leq M_1$;

(ii) $q(s) \in C^2$;

(iii) $\sum_{j=1}^2 \|u_{0xx}^j\|_{L_2}^2 \leq M_2$.

Then we have

$$\sum_{j=1}^2 \|u_{jxx}\|_{L_2 \times L_\infty}^2 \leq E_2, \quad \epsilon \int_0^T \left(\sum_{j=1}^2 \|u_{jxxx}\|_{L_2}^2 \right) dt \leq C_4, \quad (2.9)$$

where the constants E_2, C_4 are independent of ϵ .

Proof. We differentiate (2.4), (2.5) with respect to x two times, multiply them by $\bar{u}_{1xx}$ and $\bar{u}_{2xx}$ respectively, and integrate over $(-\infty, \infty)$. Using Sobolev's inequality, Hölder's inequality, Gronwall's inequality and the results of Lemma 1 and Lemma 2, we can obtain (2.9).

Differentiating (2.4), (2.5) with respect to x (or t) many times, using the Sobolev's inequality and the preceding results repeatedly, and noticing the order of the integral estimates, similarly we can derive that the norms

$$\sum_{j=1}^2 \|u_{jxxx}\|_{L_2 \times L_\infty}, \quad \sum_{j=1}^2 \|u_{jxt}\|_{L_2 \times L_\infty}, \quad \sum_{j=1}^2 \|u_{jtt}\|_{L_2 \times L_\infty},$$

and

$$\sum_{j=1}^2 \|u_{jxxt}\|_{L_2 \times L_\infty}, \quad \sum_{j=1}^2 \|u_{jtxx}\|_{L_2 \times L_\infty}$$

are uniformly bounded independent of ϵ .

We can prove that the sequences $\{u_j^\epsilon\}$, $\{u_{jxx}^\epsilon\}$ of the global smooth solution for the problem (2.4)—(2.6) are uniformly bounded and equicontinuous. Using the Arzela Ascoli selection theorem, and let $\epsilon \rightarrow 0$, we obtain a subsequence converging to the solution.

Theorem 1. Suppose that there exists a local smooth solution of the problem (2.4)—(2.6), for which not only the derivatives which appear in the system (2.4), (2.5), but also the derivatives u_{jtt} , u_{jxt} , u_{jxxx} , u_{jxxt} and u_{jtxx} exist ($j=1, 2$). Assume that

- (i) the complex functions $k_j(x, t) \in C^3$ ($j=1, 2$), and all their derivatives up to order three are bounded;
- (ii) the complex functions $f_j(\cdot, t) \in H^3$, and all f_{jt} , f_{jtt} , f_{jxt} , f_{jxxt} , $f_{jtxx} \in L_2$ ($j=1, 2$);
- (iii) the real-valued functions $q(s) \geq 0$, $q'(s) \geq 0$ and $q(s) \in C^3$ in $[0, \infty)$;
- (iv) the initial data $u_0^j(x) \in H^6$ ($j=1, 2$).

Then there exists the global classical solution for the problem (2.1)—(2.3). For this classical solution, the partial derivatives which appear in the system (2.1), (2.2) exist and are continuous.

We can extend the above results to the following class of SNSE

$$u_t - i u_{xx} + \beta q(|u|^2) u + k u = f, \quad (2.10)$$

where $u = (u_1, u_2, \dots, u_N)^T$, functional matrix $k = (k_{ij}(x, t))$ ($i, j=1, 2, \dots, N$);

$$f = (f_1, f_2, \dots, f_N)^T.$$

Theorem 2. Suppose that the conditions of theorem 1 are satisfied, in which we substitute $k_{ij}(x, t)$ for $k_j(x, t)$. Then there exists a global classical solution for the initial value problem and the periodic initial value problem of (2.10).

Consider the following initial value problem with the higher space

$$u_{jt} - i \Delta u_j + \beta(|u_1|^2 + |u_2|^2) u_j + k_j(x, t) u_j = f_j(x, t) \quad (j=1, 2), \quad (2.11)$$

$$u_j|_{t=0} = u_0^j(x) \quad (-\infty < x < \infty; j = 1, 2), \quad (2.12)$$

where

$$x = (x_1, x_2, \dots, x_n), \quad \Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}.$$

For the sequence of the solution of the corresponding regularized problem of (2.11), (2.12), and by analogical proof, we can derive that the norms

$$\sum_{j=1}^2 \|u_j^\varepsilon\|_{L_2 \times L_\infty}, \quad \sum_{j=1}^2 \|\nabla u_j^\varepsilon\|_{L_2 \times L_\infty} \quad \text{and} \quad \sum_{j=1}^2 \|u_{jt}^\varepsilon\|_{L_2 \times L_\infty}$$

are uniformly bounded. Hence the norms

$$\sum_{j=1}^2 \|u_j^\varepsilon\|_{L_2}, \quad \sum_{j=1}^2 \|u_j^\varepsilon\|_{L_\infty}$$

are uniformly bounded also ($n \leq 3$).

Theorem 3. Assume that the conditions of theorem 1 are satisfied, where $q(s) \equiv s$, $\beta \geq \beta_0 > 0$. Then there exists a global classical solution for multi-dimensional ($n \leq 3$) initial value problem (2.11), (2.12) and its corresponding periodic initial value problem.

For the problem (2.4)–(2.6), let the sequence $\{u_j^{(n)}(x, t)\}$ ($n = 1, 2, \dots$) be defined by the (linearized) problem

$$u_{jt}^{(n)} - i u_{jxx}^{(n)} - \varepsilon u_{jxx}^{(n)} = -\beta q(|v_1^{(n-1)}|^2 + |v_2^{(n-1)}|^2) v_j^{(n-1)} - k_j v_j^{(n-1)} + f_j \quad (j = 1, 2), \quad (2.13)$$

where

$$u_j^0 = 0, \quad v_j^{(n-1)} = \phi^j + u_j^{(n-1)}.$$

Let ϕ^j be a solution to the problem

$$\begin{aligned} \phi_t^j - i \phi_{xx}^j - \varepsilon \phi_{xx}^j &= 0 \quad (\varepsilon > 0; j = 1, 2); \\ \phi^j|_{t=0} &= u_0^j(x) \quad (-\infty < x < \infty; j = 1, 2). \end{aligned} \quad (2.14)$$

We can prove that $\|u_j^{(n)}(t)\|_m$ ($m = 2, 3, \dots, s$) are uniformly bounded independent of n , and

$$\sup_{0 \leq \tau \leq T^*} \sum_{j=1}^2 \|u_j^{(n+1)}(\tau) - u_j^{(n)}(\tau)\|_s \leq \rho \sup_{0 \leq \tau \leq T^*} \sum_{j=1}^2 \|u_j^{(n)}(\tau) - u_j^{(n-1)}(\tau)\|_s, \quad (0 < \rho < 1).$$

We have:

Theorem 4. Let $u_0^j(x) \in H^s$, $f_j(\cdot, t) \in H^{s-1}$ ($s \geq 2$), $q(z) \in C^{s-1}$, and assume that the derivatives $D^l k_j$ ($l \leq s-1$) are bounded. Then for any real number β , the problem (2.4)–(2.6) has a local solution $u_j(x, t) \in L^\infty(0, t_0; H^s)$ for some $t_0 = T^*$ ($j = 1, 2$).

For the multi-dimensional problem (2.11), (2.12), we have:

Theorem 5. Suppose that the conditions of theorem 3 are satisfied and that $s \geq \left[\frac{n}{2}\right] + 1$.

Then there exists the local solution $u^j(x, t) \in L^\infty(0, t_0; H^s)$ ($j = 1, 2$) for the problem (2.11), (2.12) and its corresponding periodic initial value problem.

Theorem 6. Let $q(z) \in C^1$ and assume that $k_j(x, t)$ ($j = 1, 2$) are bounded. Then the bounded solution for the problem (2.1)–(2.3) is unique.

Proof. Using ordinary method of the energetical estimates, we easily obtain this conclusion.

§ 3. The Periodic Initial Value Problem for Another Class of SNSE

We consider the following problem

$$\begin{aligned} i u_{jt} - u_{jxx} + \beta(x) q(\sigma_{11} |u_1|^2 + \sigma_{12} |u_2|^2) u_j + k_j(x) u_j &= 0; \\ u_j|_{t=0} &= u_0^j(x) \quad (0 \leq x \leq 2\pi; j = 1, 2); \end{aligned} \quad (3.1)$$

$$u_j(x, t) = u_j(x + 2\pi, t) \quad (\forall x, t > 0). \quad (3.2)$$

Let a basis $\{w_j(x)\} \in H_{2\pi}^1$ (where $H_{2\pi}^1$ denotes the space H^1 in which the functions possess period 2π). We construct an approximate solution of (3.1), (3.2) by Galerkin method. Let

$$u_m^l(t) = \sum_{j=1}^m c_{jm}^l(t) w_j(x) \quad (l = 1, 2),$$

where $c_{jm}^l(t)$ are complex functions, such that $u_m^l(t)$ satisfy the following system

$$\begin{aligned} i(u_{mt}^l, w_j) + (u_{mx}^l, w_{jx}) + (\beta u_m^l q(\sigma_{11} |u_m^l|^2 + \sigma_{12} |u_m^l|^2), w_j) + \\ (k_l u_m^l, w_j) = 0 \quad (l = 1, 2; j = 1, 2, \dots, m), \end{aligned} \quad (3.3)$$

and the initial conditions

$$(u_m^l(0), w_j(x)) = (u_0^l(0), w_j(x)); \quad (l = 1, 2; j = 1, 2, \dots, m). \quad (3.4)$$

Using the method of the integral estimates, we can obtain that the norms $\|u_m^l(t)\|_{L_2}$, $\|u_{mx}^l\|_{L_2}$ and $\|u_{mt}^l\|_{L_2}$ ($l = 1, 2$) are uniformly bounded independent of m . Now define the generalized solution (I) of the problem (3.1), (3.2). "The generalized solution (I)": unknown functions

$$u^l(x, t) \in L^\infty(0, T; H_{2\pi}^1); \quad u_t^l(x, t) \in L^\infty(0, T; L_2) \quad (l = 1, 2),$$

which satisfy the integral identity

$$\begin{aligned} i(u_t^l, v_l) + (u_x^l, v_{lx}) + (\beta u^l q(\sigma_{11} |u^l|^2 + \sigma_{12} |u^l|^2), v_l) + \\ (k_l u^l, v_l) = 0 \quad (\forall v_l \in H_{2\pi}^1; t > 0; l = 1, 2); \end{aligned} \quad (3.5)$$

$$(u^l|_{t=0}, v^l) = (u_0^l(x), v_l) \quad (l = 1, 2). \quad (3.6)$$

Theorem 7. Assume that

- (i) $\sigma_{11} > 0, \sigma_{12} > 0$, and the periodic real-valued function $\beta(x) \geq 0$;
- (ii) the real-valued function $q(s) \in C^1, q(s) \geq 0$ in $[0, \infty)$;
- (iii) the periodic real-valued functions $k_j(x)$ are bounded;
- (iv) $u_0^j(x) \in H_{2\pi}^2$ ($j = 1, 2$).

Then there exists the generalized solution (I) of the problem (3.1), (3.2).

Theorem 8. Let $q(s) \in C^1$, and assume that the functions $k_j(x)$ are bounded. Then the generalized solution (I) of the problem (3.1), (3.2) is unique.

Remark 1. For the initial-boundary value problem of (3.1) (specifically, $u^j|_{x=0} = u^j|_{x=1} = 0, j = 1, 2$), the conclusions of theorem 7 and theorem 8 are still true. In fact, by choosing the basis $\{w_j\} \in H_0^1$, we can obtain these results directly.

Consider the following problem:

$$i u_{jt} - u_{jxx} + \beta(x) (|u_1|^2 + |u_2|^2) u_j + k_j(x) u_j = 0 \quad (j = 1, 2); \quad (3.7)$$

$$u_j|_{t=0} = u_0^j, 0 \leq x \leq 2\pi; \quad u_j(x, t) = u_j(x + 2\pi, t) \quad (\forall x, t \geq 0), \quad (3.8)$$

where $\beta(x) \leq 0$. Using the Galerkin method and the Sobolev estimates, we can also obtain that the norms $\|u_{nx}^j\|_{L_2}$ and $\|u_m^j\|_{L_\infty}$ are uniformly bounded. We have:

Theorem 9. If

- (i) $\sigma_{11} > 0, \sigma_{12} > 0$;
- (ii) $\beta(x)$ and $k_j(x)$ are periodic real-valued functions with period 2π ;
- (iii) $u_0^j(x) \in H_{2\pi}^2$ ($j = 1, 2$).

Then there exists a generalized solution (I) of the problem (3.7), (3.8) and it is unique.

Now consider the problem

$$iu_{jt} - \Delta u_j + \beta(x)q(\sigma_{11}|u_1|^2 + \sigma_{12}|u_2|^2)u_j + k_j u_j = 0 \quad (j = 1, 2); \quad (3.9)$$

$$u_j|_{\partial\Omega} = 0; \quad u_j|_{t=0} = u_0^j(x) \quad (j = 1, 2). \quad (3.10)$$

Lemma 4. If

- (i) $\sigma_{11} > 0, \sigma_{12} > 0, \beta(x) > 0$;
- (ii) $q(s) \geq 0, k_j(x)$ are bounded real-valued functions;
- (iii) $u_0^j(x) \in H_0^1$ ($j = 1, 2$); $\beta(x)q(\sigma_{11}|u_0^1|^2 + \sigma_{12}|u_0^2|^2) \in L_1$.

Then we have

$$E_1(t) = \sigma_{11} \|\nabla u_1\|_{L_2}^2 + \sigma_{12} \|\nabla u_2\|_{L_2}^2 + \int_{\Omega} \beta(x)q(\sigma_{11}|u_1|^2 + \sigma_{12}|u_2|^2)dx + \int_{\Omega} (k_1\sigma_{11}|u_1|^2 + k_2\sigma_{12}|u_2|^2)dx = E(0), \quad (3.11)$$

where

$$q(s) = \int_0^s q(z)dz.$$

Lemma 5. If the conditions of Lemma 4 are satisfied, then

$$\|\nabla u_1\|_{L_2}^2 + \|\nabla u_2\|_{L_2}^2 \leq C_2, \quad (3.12)$$

where C_2 is a constant.

Now we define the generalized solution (II), which is more general broad than the generalized solution (I). "The generalized solution (II)": unknown functions $u_j(x, t) \in L^\infty(0, T; H^1)$, which satisfy the integral identity

$$\int_0^T (iu_j, \Phi_{jt})dt - \int_0^T \int_{\Omega} (\nabla u_j \cdot \nabla \Phi_j)dxdt - \int_0^T (\beta q(\sigma_{11}|u_1|^2 + \sigma_{12}|u_2|^2)u_j, \Phi_j)dt - \int_0^1 (k_j u_j, \Phi_j)dt + i(u_0^j, \Phi_j(0)) = 0 \quad (j = 1, 2), \quad (3.13)$$

for any functions

$$\Phi_j|_{t=T} = 0, \quad \Phi_j|_{\partial\Omega} = 0, \quad \Phi_j \in C^1 \quad (j = 1, 2).$$

Theorem 10. If

- (i) $\sigma_{11} > 0, \sigma_{12} > 0, \beta(x) > 0$;
- (ii) the real-valued functions $k_j(x)$ are bounded;
- (iii) $u_0^j(x) \in H^1$, $\int_{\Omega} \beta(x)q(\sigma_{11}|u_0^1|^2 + \sigma_{12}|u_0^2|^2)dx < \infty$.

Then there exists the generalized solution (II) of the problem (3.9), (3.10).

For the problem (3.1), (3.2), moreover, we choose the basis $\{w_k\} \in H_{2\pi}^6 \subset H_{2\pi}^1$, as

periodic eigenfunctions satisfying

$$-\omega_{kxx} = \mu_k \omega_k \quad (k=1,2)\cdots,$$

and let

$$u_m^j(0) = \sum_{k=1}^m c_{km}^j(0) \omega_k(x) \rightarrow u_0^j(x) \text{ in } H^6.$$

Differentiate (3.1) with respect to t many times, again set $t=0$, then take the orthogonal projector. We can derive that the norms $\|u_{mt}^j(0)\|_{L_2}$, $\|u_{mtt}^j(0)\|_{L_2}$, $\|u_{mtxx}^j(0)\|_{L_2}$, $\|u_{mttx}^j(0)\|_{L_2}$ and $\|u_{mttt}^j(0)\|_{L_2}$ are uniformly bounded and independent of m .

Using the method of the integral estimates, we can obtain the norms $\|u_{mtt}^j(T)\|_{L_2}$, $\|u_{mtx}^j(T)\|_{L_2}$, $\|u_{mttt}^j\|_{L_2}$, $\|u_{mttx}^j\|_{L_2}$ are bounded and independent of m . We have:

Theorem 11. Under the conditions of theorem 7, and if that

- (i) $q(s) \in C^3$ and all derivatives until up to order three of $k_j(x)$ are bounded ($j=1,2$);
- (ii) the periodic functions $u_0^j(x) \in H^6$ ($j=1,2$).

Then there exists a classical solution of the problem (3.1), (3.2).

Remark 2. The existence of a generalized solution and a classical solution for the initial value problem of (3.1) can be obtained by the following method.

Consider the problem

$$i u_{jt} - u_{jxx} + \beta q(\sigma_{11} |u_1|^2 + \sigma_{12} |u_2|^2) u_j + k_j(x) u_j = 0 \\ (j=1,2; |x| < m+1; t > 0); \quad (3.14)$$

$$u_j|_{t=0} = u_0^j \zeta_m(x), \quad u_j|_{|x|=m+1} = 0 \quad (j=1,2), \quad (3.15)$$

where $\zeta_m(x) \in C^2$, $0 \leq \zeta_m(x) \leq 1$, $\zeta_m(x) = \begin{cases} 1 & |x| < m \\ 0 & |x| \geq m+1 \end{cases}$.

From remark 1, there exists the generalized solution (I) u_{jm} of the problem (3.14), (3.15). From the uniform estimates of the functions $u_0^j \zeta_m(x)$ and their some derivatives independent of m , one can conclude that as $m \rightarrow \infty$, $u_{jm}(\cdot, t) \rightarrow u_j(\cdot, t)$, strongly in L_2 ; $u_{jmt}(\cdot, t) \rightarrow u_{jt}(\cdot, t)$, weakly in L_2 . Hence we have proved the existence of the generalized solution (I) of the initial value problem of (3.1). If the coefficients of the equations and the initial data are smoother, then we can show the existence of the classical solution.

§ 4. The Initial-boundary Valued Problem for A Class of Multi-dimensional SNSE

We consider the following initial-boundary value problem

$$u_{tt} - \gamma \sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u_l}{\partial x_j} + \beta q(\sigma_{11} |u_1|^2 + \sigma_{12} |u_2|^2) u_l + \lambda u_l \\ = f_l(x, t), \quad (4.1)$$

with initial-boundary conditions

$$u_l|_{t=0} = u_0^l(x) \quad (x \in \Omega); \quad u_l|_{\partial\Omega} = 0 \quad (l=1,2); \quad (4.2)$$

where $\gamma = \gamma_0 + i\gamma_1$ ($i = \sqrt{-1}$); $\operatorname{Re}\gamma = \gamma_0 \geq 0$; $\sigma_{11} \geq 0$; $\sigma_{12} \geq 0$.

Let the real-valued functions

$$a_{ij}(x) = a_{ji}(x) \quad (i, j = 1, 2, \dots, n)$$

and satisfy

$$\sum_{i,j=1}^n a_{ij} \xi_i \bar{\xi}_j \geq \alpha \sum_{i=1}^n |\xi_i|^2, \quad \alpha > 0.$$

Let

$$a(u, v) = \sum_{i,j=1}^n \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j} dx,$$

choose a basis $\{w_k\}$, which are eigenfunctions, i. e., satisfy

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij} \frac{\partial w_k}{\partial x_j} = \lambda_k w_k, \quad w_k|_{\partial\Omega} = 0. \quad (4.3)$$

It is well known that if $a_{ij} \in C^2, \Omega \in C^2$, then there is such a special basis $\{w_k\} \subset H_0^2(\Omega) \subset$

$H_0^1(\Omega)$, which is dense in H_0^1 . Let $u_m^j = \sum_{k=1}^m c_{km}^j(t) w_k(x)$ and satisfy

$$\begin{aligned} (u_m^j, w_k) + \gamma a(u_m^j, w_k) + \beta(u_m^j q(\sigma_{11}|u_m^1|^2 + \sigma_{12}|u_m^2|^2), w_k) + (\chi_j u_m^j, w_k) \\ = (f_j, w_k) \quad (j = 1, 2), \end{aligned} \quad (4.4)$$

$$(u_m^j(0), w_k) = (u_0^j(x), w_k) \quad (j = 1, 2, \dots, m).$$

Obviously, there exists a local solution $u_m^j(x)$ of the problem (4.4), (4.5) in $[0, t_0]$, by a priori estimates of u_m^j . From the following lemma we can obtain the existence of the global solution u_m^j of the problem (4.4), (4.5) in $[0, T]$.

Lemma 6. Assume that

$$(i) \quad |\chi_j(x, t)| \leq M, \quad \|f_j\|_{L_2 \times L_\infty}^2 \leq M_1 \quad (j = 1, 2);$$

$$(ii) \quad \operatorname{Re}\gamma = \gamma_0 \geq 0, \quad \beta \geq 0;$$

$$(iii) \quad u_0^j(x) \in L_2.$$

Then we have

$$\sum_{j=1}^2 \|u_m^j\|_{L_2 \times L_\infty} \leq E_0, \quad \sum_{j=1}^2 \int_0^T \int_{\Omega} q(\sigma_{11}|u_m^1|^2 + \sigma_{12}|u_m^2|^2) |u_m^j|^2 dx dt \leq C_2, \quad (4.6)$$

$$\int_0^T \left(\sum_{j=1}^2 \|u_m^j\|_1^2 \right) dt \leq C_3, \quad (4.7)$$

where the constants E_0, C_2 and C_3 are independent of m .

Lemma 7. Assume that

$$(i) \quad \chi_j, f_j, a_{ij} \in C^1, \quad \left| \frac{\partial \chi_j}{\partial x_i} \right| < M, \quad |a_{ij}| < M, \quad \sum_{k=1}^2 \|\nabla f_k\|_{L_2}^2 \leq M_1 \quad (i, j = 1, 2);$$

$$(ii) \quad \beta > 0, \sigma_{11} > 0, \sigma_{12} > 0, q(s) \geq 0, q'(s) \geq 0;$$

$$(iii) \quad u_0^j(x) \in H_0^1(\Omega) \quad (j = 1, 2).$$

Let a_{ij} satisfy the condition of the uniform positive definiteness. Then we have

$$\sum_{j=1}^2 \|u_m^j(T)\|_1^2 \leq E_1, \quad \int_0^T (\|Au_m^1\|_{L_2}^2 + \|Au_m^2\|_{L_2}^2) dt \leq C_{10}, \quad (4.8)$$

where $-Au_m^k = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u_m^k}{\partial x_j}$ ($k=1,2$) and the constants E_1, C_{10} are independent of m .

Proof. From (4.4), (4.5), we derive

$$\begin{aligned} & (u_{mt}^j, -Au_m^j) + (-Au_m^j, -Au_m^j) + \beta(u_m^j q(\sigma_{11}|u_m^1|^2 + \sigma_{12}|u_m^2|^2), -Au_m^j) + \\ & (\chi_j u_m^j, -Au_m^j) = (f_j, -Au_m^j) \quad (j=1,2). \end{aligned} \quad (4.9)$$

since
$$\operatorname{Re}(u_{mt}^l, -Au_m^l) = \frac{1}{2} \frac{d}{dt} a(u_m^l, u_m^l),$$

$$\operatorname{Re} \gamma (-Au_m^l, -Au_m^l) = \gamma_0 \|Au_m^l\|_{L_2}^2 \geq 0 \quad (l=1,2),$$

by the conditions of this lemma, we have

$$|(\chi_l u_m^l, -Au_m^l)| \leq 2M^2 n \|\nabla u_m^l\|_{L_2}^2 + n^2 m^2 \|u_m^l\|_{L_2}^2 \quad (l=1,2),$$

$$|(f_l, -Au_m^l)| \leq \frac{1}{2} Mn (\|\nabla f_l\|_{L_2}^2 + \|\nabla u_m^l\|_{L_2}^2) \quad (l=1,2),$$

and notice that

$$\operatorname{Re} \sum_{l=1}^2 (u_m^l q(\sigma_{11}|u_m^1|^2 + \sigma_{12}|u_m^2|^2), -\sigma_{1l} Au_m^l) \geq 0,$$

it is not difficult to obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{l=1}^2 \sigma_{1l} a(u_m^l, u_m^l) \leq C_5 \left(\sum_{l=1}^2 \sigma_{1l} \|\nabla u_m^l\|_{L_2}^2 \right) + C_6.$$

Integrate the above inequality with respect to t , and then by Gronwall's Lemma we obtain (4.8).

Lemma 8. Assume that

- (i) $|\chi_{jt}| \leq M, |\chi_j| \leq M, \sup_{0 \leq t \leq T} \|f_{jt}\|_{L_2} \leq M_1 \quad (j=1,2);$
- (ii) $\operatorname{Re} \gamma = \gamma_0 \geq 0, \beta > 0, \sigma_{11} > 0, \sigma_{12} > 0;$
- (iii) $q(s) \in C^1, q(s) \geq 0, q'(s) \geq 0$ in $[0, \infty);$
- (iv) $\|u_{mt}^j(0)\|_{L_2} \leq K \quad (j=1,2).$

Then we have

$$\sum_{l=1}^2 \|u_{mt}^l(T)\|_{L_2}^2 \leq C_{11}, \quad (4.10)$$

where the constant C_{11} is independent of m .

Proof. Differentiating (4.4) with respect to t , and noticing

$$\beta \operatorname{Re} \sum_{l=1}^2 \left(\frac{d}{dt} u_m^l q(\sigma_{11}|u_m^1|^2 + \sigma_{12}|u_m^2|^2), \sigma_{1l} u_{mt}^l \right) \geq 0,$$

we have

$$\sum_{l=1}^2 \sigma_{1l} \|u_{mt}^l(T)\|_{L_2}^2 \leq \sum_{l=1}^2 (\sigma_{1l} \cdot K^2 + \sigma_{1l} M E_0 + \sigma M_1^2) e^{(3M+1)T} = C'_{11}.$$