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MATHEMATICS

ON COMPOUND BOUNDARY PROBLEMS*

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ABSTRACT

In this paper, the so-called Riemann-Hilbert compound problems are considered. Such problems may be formulated as follows. Let D be a Liapounoff region with the boundary L, and Γ be a finite set of non-intersecting smooth contours in D. Find a sectionally holomorphic function in D satisfying a Riemann boundary condition on Γ and a Hilbert boundary condition on L. The cases in which Γ consists of open arcs and the corresponding vector problems are also considered. The method of solution is to eliminate the "Riemann part" of the problem at first. The problem then reduces to a certain Hilbert one.

I. INTRODUCTION

In [1], we have given a brief sketch of the so-called compound boundary problems and the process of solution by the method of elimination. Here we shall give these results in detail. We retain the notations in [2], and the Riemann problems and the Hilbert problems will be denoted briefly by the R-problems and the H-problems respectively, while the compound boundary problems, by the RH-problems.

II. THE RH-PROBLEMS FOR SIMPLY CONNECTED REGIONS

Let D be the region bounded by a Liapounoff curve L (i.e., a curve, the inclination of the tangent of which, considered as the function of its arc length, satisfies the Hölder condition), and $\Gamma_1, \cdots, \Gamma_n$ be a set of non-intersecting and mutually exclusive contours in D. Denote $\Gamma = \Gamma_1 + \cdots + \Gamma_n$. Take the counter-clockwise sense along L as its positive sense, while along Γ_j, we take the clockwise sense as positive. By D_j^- we denote the inner region bounded by Γ_j, and by D_0^+, the region bounded by L and all the Γ_j.

The RH-problem for D may be formulated as follows. Find a sectionally holomorphic function $\Phi(z)$ in D (i.e., regular everywhere in D except the points on Γ and continuous to Γ from both sides of it) subjected to the following two conditions: 1) in the neighbourhood of Γ, $\Phi(z)$ should satisfy the boundary condition

$$\Phi^+(\tau) = G(\tau)\Phi^-(\tau) + g(\tau), \quad \tau \in \Gamma, \tag{2.1}$$

where $G(\tau)$, $g(\tau)$ are given on Γ, satisfying the Hölder conditions and $G(\tau) \neq 0$; 2)

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$\Phi(z)$ should be continuous to L from the interior of D and satisfy the boundary condition

$$\operatorname{Re}[\overline{\lambda(t)}\Phi(t)] = c(t), \quad t \in L, \tag{2.2}$$

where $\lambda(t)$, $c(t)$ are given on L, also satisfying the Hölder conditions and $\lambda(t) \neq 0$ ($c(t)$ being real of course).

Denote

$$\operatorname{Ind}_{\Gamma_j} G(\tau) = \frac{1}{2\pi}[\arg G(\tau)]_{\Gamma_j} = \kappa_j, \quad \operatorname{Ind}_\Gamma G(\tau) = \kappa = \sum_{j=1}^{n} \kappa_j,$$

$$\operatorname{Ind}_L \lambda(t) = k.$$

We shall call

$$K = \kappa + k$$

the *index* of the above RH-problem.

In each D_j^-, take an arbitrary point z_j and denote

$$\Pi(z) = \prod_{j=1}^{n} (z - z_j)^{\kappa_j}.$$

We shall find first of all a function $\Phi_1(z)$, sectionally holomorphic in D and continuous to L, satisfying the condition (2.1), with no regard to (2.2) temporarily. The general solution of this problem (called the corresponding R-problem of the previous problem) is

$$\Phi_1(z) = X(z)[\Psi(z) + F(z)], \tag{2.3}$$

where $X(z)$ is its canonical function:

$$X(z) = \begin{cases} X^+(z) = \Pi(z)^{-1} e^{\Gamma^+(z)} & \text{when } z \in D_0^+, \\ X^-(z) = e^{\Gamma^-(z)}, & \text{when } z \in \sum_j D_j^-, \end{cases}$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\ln G(\tau)\Pi(\tau)}{\tau - z} d\tau$$

(any branch of the logarithm may be chosen); $\Psi(z)$ may be taken as

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z},$$

while $F(z)$ is an arbitrary function, holomorphic in D and continuous on $\bar{D}$. As we deal with $\bar{D}$ only in the original RH-problem, so the corresponding R-problem is always solvable, and the arbitrary function $F(z)$ is now in place of the polynomial presented in the general solution for the ordinary R-problem in the entire plane.

Note that, when $g(\tau) = 0$, $X(z)$ itself is also a solution of (2.1):

$$X^+(\tau) = G(\tau)X^-(\tau), \quad \tau \in \Gamma. \tag{2.4}$$

Transform the unknown function $\Phi(z)$ to another one, $\Phi_0(z)$, by means of the expression

$$\Phi(z) = \Phi_1(z) + X(z)\Phi_0(z); \tag{2.5}$$

then it is evident that $\Phi_0(z)$ is also sectionally holomorphic in D and continuous to L. Besides, when $\tau \in \Gamma$, noting that $\Phi_1(z)$ satisfies (2.1), we obtain, by (2.4),

$$\begin{aligned}\Phi^+(\tau) &= \Phi_1^+(\tau) + X^+(\tau)\Phi_0^+(\tau) = \\ &= G(\tau)\Phi_1^-(\tau) + g(\tau) + G(\tau)X^-(\tau)\Phi_0^+(\tau) = \\ &= G(\tau)[\Phi_1^-(\tau) + X^-(\tau)\Phi_0^+(\tau)] + g(\tau).\end{aligned}$$

If $\Phi(z)$ is the required solution, then it should satisfy (2.1) also, i.e.,

$$\Phi^+(\tau) = G(\tau)\Phi^-(\tau) + g(\tau) = G(\tau)[\Phi_1^-(\tau) + X^-(\tau)\Phi_0^-(\tau)] + g(\tau).$$

Comparing these and remembering that $X^-(\tau) \neq 0$, immediately we obtain

$$\Phi_0^+(\tau) = \Phi_0^-(\tau), \quad \tau \in \Gamma,$$

hence $\Phi_0(z)$ is holomorphic in D and continuous on $\bar{D}$.

Conversely, if $\Phi_0(z)$ is holomorphic in D and continuous on $\bar{D}$, then it is easy to prove that the sectionally holomorphic function $\Phi(z)$ determined by (2.5) will satisfy (2.1) and will also be continuous to L.

Therefore, the proposed problem has been transformed to such one: to find a function $\Phi_0(z)$ holomorphic in D and continuous on $\bar{D}$ such that it should satisfy the corresponding condition transformed from (2.2). Substituting (2.5) into (2.2), we get readily this condition:

$$\mathrm{Re}\,[\overline{\lambda(t)}X(t)\Phi_0(t)] = c^*(t), \quad t \in L, \tag{2.6}$$

where

$$c^*(t) = c(t) - \mathrm{Re}\,[\overline{\lambda(t)}\Phi_1(t)]. \tag{2.7}$$

Thus, the original problem has been reduced to the H-problem (2.6) for D, while the condition (2.1) and all the Γ_j have been eliminated. That is why the name "method of elimination" is suggested.

The choice of the arbitrary function $F(z)$ in (2.3) is indifferent. For, if a function $f(z)$ is added to $F(z)$, then a term $X(z)f(z)$ will be added to $\Phi_1(z)$, and from (2.6), $\Phi(z)$ will remain invariant if we subtract $f(z)$ from the required function $\Phi_0(z)$; this is equivalent to a subtraction of the term $\mathrm{Re}[\bar{\lambda}Xf]$ from both sides of (2.6), so that $\Phi_0(z)$ is invariant also. Hence, without loss of generality, we may always take $F(z) \equiv 0$ in (2.3).

For the H-problem (2.6) satisfied by $\Phi_0(z)$, the index is

$$\mathrm{Ind}_L\,[\lambda(t)\overline{X(t)}] = k - \mathrm{Ind}_L\,X(t). \tag{2.8}$$

Noting that, when $t \in L$,

$$X(t) = X^+(t) = \prod_{j=1}^{n}(t - z_j)^{-\kappa_j}e^{\Gamma^+(t)},$$

we have

$$\mathrm{Ind}_L X(t) = -\kappa + \mathrm{Ind}_L e^{\Gamma^+(t)}. \tag{2.9}$$

Since $\Gamma^+(t)$ is single-valued and continuous on L, so

$$\mathrm{Ind}_L e^{\Gamma^+(t)} = \frac{1}{2\pi i}[\ln e^{\Gamma^+(t)}]_L = \frac{1}{2\pi i}[\Gamma^+(t)]_L = 0;$$

substituting in (2.8) and (2.9), we finally get

$$\mathrm{Ind}_L [\lambda(t)\overline{X(t)}] = k + \kappa = K. \tag{2.10}$$

That is to say, the index of the transformed H-problem is just that of the original RH-problem.

Therefore, applying the usual theory about the number of solutions and the conditions of solubility of H-problems, we can discuss RH-problems similarly:

1) Suppose $g \equiv 0$, $c \equiv 0$ (the homogeneous case). If $K \geqslant 0$, the original problem has $2K + 1$ linearly independent (with respect to real coefficients) solutions:

$$\Phi(z) = X(z) \sum_{s=1}^{2K+1} c_s \Phi_{0s}(z), \tag{2.11}$$

where $\Phi_{0s}(z)$ $(s = 1, \cdots, 2K + 1)$ are the complete system of solutions of the corresponding H-problem (2.6) (now $c^* \equiv 0$), and c_s are arbitrary real constants (now $\Psi(z) \equiv 0$).

If $K < 0$, the corresponding H-problem has the trivial solution $\Phi_0(z) \equiv 0$ only, consequently the original problem has also the trivial solution only.

2) Suppose $g \equiv 0$, $c \not\equiv 0$. Now we may take $\Phi_1(z) \equiv 0$, so that $\Phi(z) = X(z)\Phi_0(z)$, and $\Phi_0(z)$ is the solution of the non-homogeneous H-problem on L:

$$\mathrm{Re}\,[\overline{\lambda(t)}X(t)\Phi_0(t)] = c(t), \quad t \in L.$$

In this case, if $K \geqslant 0$, the problem is solvable. Let $\Phi_{00}(z)$ be a particular solution; then the general solution of the original problem is

$$\Phi(z) = X(z)\left[\Phi_{00}(z) + \sum_{s=1}^{2K+1} c_s \Phi_{0s}(z)\right]. \tag{2.12}$$

If $K < 0$, the problem has a unique solution when and only when $c(t)$ satisfies $-2K - 1$ conditions.

3) Suppose $g \not\equiv 0$; then $\Phi_1(z) \not\equiv 0$. We now have a classical H-problem by noting that $c^*(t)$ also satisfies the Hölder condition on L. Two subcases are possible now:

(i) If $c^*(t) \not\equiv 0$, i.e., $c(t) \not\equiv \mathrm{Re}[\overline{\lambda(t)}\Phi_1(t)]$, then it is the same as the previous case 2). When $K \geqslant 0$, to the right-hand side of the general solution (2.12), a term $\Phi_1(z)$ must be added; when $K < 0$, then there is a unique solution when and only when $-2K - 1$ conditions are satisfied among c, λ, G, g.

(ii) If $c^*(t) \equiv 0$, i.e., $c(t) \equiv \mathrm{Re}[\overline{\lambda(t)}\Phi_1(t)]$, then the problem will be transformed to a homogeneous H-problem again. When $K \geqslant 0$, the general solution is the same as (2.11), but a term $\Phi_1(z)$ must be added to the right-hand side; when $K < 0$, the H-problem has the trivial solution $\Phi_0(z) \equiv 0$ only, hence the original problem has the unique (non-trivial) solution $\Phi(z) = \Phi_1(z)$.

We shall call the last case the *quasi-homogeneous* RH-problem, while the cases 2) and (i) of 3) the *proper* non-homogeneous problems.

The condition of the quasi-homogeneity of the problem is $c(t) = \text{Re}[\overline{\lambda(t)}\Phi_1(t)]$, where $\Phi_1(t) = X(t)\Psi(t)$. If we take the particular solution of the corresponding R-problem as (2.3) at the very beginning, where $F(z)$ is a certain function holomorphic in D and continuous on $\bar{D}$ (satisfying the Hölder condition on L), then the quasi-homogeneous RH-problem may be reduced to a non-homogeneous H-problem for $\Phi_0(z)$:

$$\text{Re}[\overline{\lambda(t)}X(t)\Phi_0(t)] = -\text{Re}[\overline{\lambda(t)}X(t)F(t)].$$

When $K \geqslant 0$, the general solution of the problem is

$$\Phi(z) = \Phi_1(z) + X(z)\left[\Phi_{00}(z) + \sum_{s=1}^{2K+1} c_s\Phi_{0s}(z)\right],$$

where $\Phi_{00}(z)$ is a particular solution of this H-problem. When $K < 0$, the H-problem possibly has one solution at most, but $\Phi_0(z) = -F(z)$ is its solution evidently, hence $\Phi(z) = \Phi_1(z) - F(z) = \Phi^*(z)$ is the unique (non-trivial) solution of the original problem. This result is exactly the same as that mentioned above.

Therefore, the general condition for an RH-problem to be quasi-homogeneous is (besides $g \not\equiv 0$): there exists a function $\Phi_1(z)$ (expressed as (2.3)), sectionally holomorphic in D, continuous to L, and satisfying the condition (2.1), such that

$$\text{Re}[\overline{\lambda(t)}\Phi_1(t)] = c(t), \quad t \in L. \tag{2.13}$$

From the above discussion, we conclude:

Theorem 1. *For simply connected regions, when* $K = \kappa + k > 0$, *in the general solution of the* RH-*problem there exist* $2K+1$ *arbitrary real constants; when* $K < 0$, 1) *there is only the trivial solution for the homogeneous case* ($c \equiv 0, g \equiv 0$), 2) *there is a unique non-trivial solution for the quasi-homogeneous case,* 3) *there is a unique solution if and only if the given coefficients of the problem satisfy* $-2K-1$ *conditions for the proper non-homogeneous case.*

When $K < 0$, there is only one solution (trivial or non-trivial) for the homogeneous or quasi-homogeneous case.

The quasi-homogeneous problem may be considered as a particular case of the proper non-homogeneous problem. When $K \geqslant 0$, this conclusion has been brought out in Theorem 1. Suppose now $K < 0$. Since there is only one solution for the quasi-homogeneous problem, its coefficients should also satisfy $-2K-1$ conditions, similar to those of the non-homogeneous case. Or, we may obtain them alternatively: we know that (cf. [2], §§ 28, 29), if $p(s)$ is the factor of regularization for $\lambda(s)$, and if we put

$$e^{-\omega_1(s)} = p(s)|\lambda(s)|,$$

then these $-2K-1$ conditions may be written as

$$\int_0^{2\pi} e^{\omega_1(s)}c^*(s)e^{-ihs}ds = 0 \quad (h = 1, \cdots, -2K-1).$$

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In the quasi-homogeneous case, we may choose $\Phi_1(z)$ suitably in order that (2.13) is satisfied; from (2.7), $c^*(s) \equiv 0$, therefore these conditions are certainly satisfied.

If we consider the problem stated at the beginning for the exterior region bounded by L (Γ_j lying also in this exterior region) and supplementarily require $\Phi(\infty)$ to be bounded, it is obvious that we may obtain the results analogous to Theorem 1. Or, by inversion, it may be transformed to a problem for an inner region and then treated as before. We shall not go to details.

We have shown in [1] that the RH-problems for simply connected regions may be solved by using N. I. Muskhelishville's method (cf. [3], § 42), but this method is unavailable for multiply connected regions, with which we shall deal in the next section.

III. The RH-Problems for Multiply Connected Regions

Let now D be a multiply connected region bounded by a set of Liapounoff curves $L_0, L_1, \cdots, L_m$, in which L_0 surrounds all the others (L_0 may be absent). Denote $L = \sum_{p=0}^{m} L_p$[1]. In the interior of D, there are a set of curves $\Gamma_1, \cdots, \Gamma_n$ as before, but some of Γ_j may surround some of L_p. For L_0, we take the counter-clockwise sense as its positive sense, and for the others, the clockwise senses.

At this time, the RH-problem for D may be stated as follows. Find a function $\Phi(z)$, sectionally holomorphic in D, continuous to L, satisfying the conditions (2.1) and (2.2). The conditions subjected to all the given coefficients remain unchanged. If L_0 is absent, then we supplementarily require $\Phi(\infty)$ to be bounded.

Note that κ and k are defined as before, but now

$$k = \sum_{p=0}^{m} k_p, \quad k_p = \mathrm{Ind}_{L_p} \lambda(t);$$

$K = \kappa + k$ is called the index of the RH-problem again.

The method of solution is similar to that in the case of simply connected region. Take an arbitrary z_j in the interior of Γ_j. When Γ_j surrounds some L_p, z_j may be taken interior to, exterior to, or on L_p. Find at first the solution (2.3) for the R-part of the problem (we may again take $F(z) \equiv 0$), then through (2.5), the unknown function is transformed to $\Phi_0(z)$, and the problem is reduced to H-problem (2.6) for D. The index of the latter is

$$\mathrm{Ind}_L[\lambda(t)\overline{X(t)}] = \mathrm{Ind}_L \lambda(t) - \mathrm{Ind}_L X(t) = k - \sum_{p=0}^{m} \mathrm{Ind}_{L_p} X(t).$$

For L_0,

$$\mathrm{Ind}_{L_0} X(t) = \mathrm{Ind}_{L_0} X^+(t) = -\kappa,$$

which is the same as before. For $L_p (p \geqslant 1)$, if no Γ_j surrounds it, we then have

$$\mathrm{Ind}_{L_p} X(t) = \mathrm{Ind}_{L_p} X^+(t) = \mathrm{Ind}_{L_p} \Pi(t) e^{\Gamma^+(t)} = 0$$

---

1) In case of the absense of L_0, the summation begins from $p = 1$. This will be always the same in the following.

since all the z_i are situated in the exterior of L_p. Hence $\mathrm{Ind}_{L_p}\Pi(t) = 0$, and we have already seen that $\mathrm{Ind}_{L_p} e^{\Gamma^+(t)} = 0$. If L_p is surrounded by some Γ_i, then

$$\mathrm{Ind}_{L_p} X(t) = \mathrm{Ind}_{L_p} X^-(t) = \mathrm{Ind}_{L_p} e^{\Gamma^-(t)} = 0.$$

Therefore, in any case, the index of the transformed H-problem is $K = \kappa + k$ too.

Thus, using the known results about the H-problems for multiply connected regions, we obtain immediately

Theorem 2. *For the* RH-*problem for the* $(m+1)$-*ply connected regions, if its index* $K > m-1$, *then it is always solvable and there are* $2K - m + 1$ *independent constants in its general solution; if* $K < 0$, *then the homogeneous problem has the trivial solution only, the quasi-homogeneous problem has a unique solution (non-trivial), and the proper non-homogeneous problem has a unique solution when and only when the given coefficients satisfy* $-2K + m - 1$ *conditions.*

By the quasi-homogeneous problem, we mean here that the condition (2.13) is fulfilled.

For the singular cases $0 \leqslant K \leqslant m-1$, we shall have results analogous to those due to F. D. Gahov[4] and B. V. Boiarsky[5], which we shall omit here.

IV. The RH-Problems for Open Arcs

Let D be as before (simply or multiply connected), and let $\Gamma_1, \cdots, \Gamma_n$ be open smooth Jordan arcs: $\Gamma_j = \widehat{a_j b_j} (j = 1, \cdots, n)$. Take the positive sense of Γ_j as that from a_j to b_j[1]. The formulation of the RH-problem now is the same as before, the conditions subjected to the coefficients of (2.1) and (2.2) remain unchanged, but in the neighbourhoods of the end-points of all the Γ_j, we shall require $\Phi(z)$ to be bounded or integrably unbounded. We shall observe that, in the neighbourhoods of some of them, we can only require $\Phi(z)$ to be almost bounded (i.e., for any $\varepsilon > 0$, $|z-c|^\varepsilon \Phi(z) \to 0$ when $z \to c$). Such end-points will be called *special* ones for the RH-problem. It will be proved in the following that they are just the special end-points $c_{r+1}, \cdots, c_{2n}$ of the corresponding R-problem (2.1). (For the definition of them, see, for example, [2], § 42.) For the remaining non-special end-points, we may, for example, require $\Phi(z)$ to be bounded near $c_1, \cdots, c_q$, to be integrably unbounded at most near $c_{q+1}, \cdots, c_r$. According to the notations in [3], such solutions are said to belong to the class $h(c_1, \cdots, c_q)$.

Here, the previous method of elimination remains valid. We show this briefly as follows.

On each Γ_j, we take a definite branch of $\ln G(\tau)$ such that

$$\alpha_j = \mathrm{Re}\,\gamma_j = \mathrm{Re}\left[-\frac{\ln G(a_j)}{2\pi i}\right]$$

satisfies the condition:

$$\begin{cases} 0 \leqslant \alpha_j < 1 & \text{when } a_j \in \{c_1, \cdots, c_q, c_{r+1}, \cdots, c_{2n}\}, \\ -1 < \alpha_j < 0 & \text{when } a_j \in \{c_{q+1}, \cdots, c_r\}, \end{cases}$$

1) If some of the Γ_j are closed and the others are open, the discussion may be similarly carried out.

when $a_j \in \{c_{r+1}, \cdots, c_{2n}\}$, we have precisely $\alpha_j = 0$. On putting

$$\alpha'_j = \operatorname{Re} \gamma'_j = \operatorname{Re} \frac{\ln G(b_j)}{2\pi i},$$

we choose integers κ_j such that

$$\begin{cases} 0 \leqslant \alpha'_j - \kappa_j < 1 & \text{when } b_j \in \{c_1, \cdots, c_q, c_{r+1}, \cdots, c_{2n}\}, \\ -1 < \alpha'_j - \kappa_j < 0 & \text{when } b_j \in \{c_{q+1}, \cdots, c_r\}, \end{cases}$$

when $b_j \in \{c_{r+1}, \cdots, c_{2n}\}$, we then have precisely $\alpha'_j - \kappa_j = 0$. Hence

$$\kappa = \sum_{j=1}^{n} \kappa_j$$

is the index of the corresponding R-problem with respect to the class $h(c_1, \cdots, c_q)$. Denoting also $k = \operatorname{Ind}_L \lambda(t)$, we call $K = \kappa + k$ the *index* of the RH-problem *with respect to the class* $h(c_1, \cdots, c_q)$.

The canonical function of the corresponding R-problem is

$$X(z) = \prod_{j=1}^{n} (z - b_j)^{-\kappa_j} e^{\Gamma(z)}, \tag{4.1}$$

where

$$\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\ln G(\tau)}{\tau - z} d\tau. \tag{4.2}$$

Solve first the corresponding R-problem for D and find out a particular solution (the arbitrary function $F(z)$ may be taken to be identical with zero):

$$\Phi_1(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{g(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z}. \tag{4.3}$$

Using the method of elimination, we put

$$\Phi(z) = \Phi_1(z) + X(z)\Phi_0(z), \tag{4.4}$$

where $\Phi_0(z)$ is the new unknown function.

Similar to § II, it is easy to prove that, at the inner points of each Γ_j, $\Phi_0^+(\tau) = \Phi_0^-(\tau)$, so that $\Phi(z)$ is single-valued, analytic in D, possibly except all the end-points a_j, b_j. We shall discuss the properties of $\Phi_0(z)$ near all these points.

Before going on, we recollect some properties of $X(z)$. From (4.2), we have

$$\Gamma(z) = \begin{cases} \gamma_j \ln(z - a_j) + \Gamma_j(z) & \text{when } z \text{ lies near } a_j, \\ \gamma'_j \ln(z - b_j) + \Gamma'_j(z) & \text{when } z \text{ lies near } b_j, \end{cases}$$

where $\Gamma_j(z)$ and $\Gamma'_j(z)$ are bounded near a_j and b_j respectively[1]. Therefore,

$$e^{\Gamma(z)} = \begin{cases} (z - a_j)^{\gamma_j} e^{\Gamma_j(z)} & \text{when } z \text{ lies near } a_j, \\ (z - b_j)^{\gamma'_j} e^{\Gamma'_j(z)} & \text{when } z \text{ lies near } b_j. \end{cases}$$

1) The logarithm is to be understood as follows: after cutting the plane along Γ'_j and extending the cut to infinity, we take an arbitrary branch of it.

We then have

$$X(z)=\begin{cases}(z-a_j)^{r_j}e^{\Gamma_j^*(z)} & \text{when } z \text{ lies near } a_j,\\ (z-b_j)^{r_j'-\kappa_j}e^{\Gamma_j^{*\prime}(z)} & \text{when } z \text{ lies near } b_j,\end{cases}$$

or, near any end-point $z=c$, we may write

$$X(z)=(z-c)^{\gamma_c}e^{\Gamma_c(z)}, \tag{4.5}$$

where $\Gamma_c(z)$ is bounded near c, and

$$\begin{cases}0\leq \operatorname{Re}\gamma_c<1 & \text{when } c\in\{c_1,\cdots,c_q,c_{r+1},\cdots,c_{2n}\},\\ -1<\operatorname{Re}\gamma_c<0 & \text{when } c\in\{c_{q+1},\cdots,c_r\},\end{cases}$$

where c is a special end-point, $\operatorname{Re}\gamma_c=0$.

According to the well-known properties of integrals of the Cauchy type, from (4.2), we know that the particular solution $\Phi_1(z)$ of the R-problem belongs to the class $h(c_1, \cdots, c_q)$. From the requirements subjected to the unknown function $\Phi(z)$, we know that $X(z)\Phi_0(z)$ must be bounded near the end-points $c_1, \cdots, c_q$ and integrably unbounded at most near the remaining end-points.

Let c be a special end-point. In order to insure that $\Phi(z)$ is integrably unbounded near c, $X(z)\Phi_0(z)$ must follow suit. But now since $\operatorname{Re}\gamma_c=0$, so $X(z)$ is bounded near $z=c$ and $\neq 0$; since c is an isolated singular point of $\Phi_0(z)$ and $X(z)\Phi_0(z)$ is integrable, so c is actually an ordinary point of $\Phi_0(z)$ and then $X(z)\Phi_0(z)$ is bounded near c. Eventually $\Phi(z)$ is almost bounded near c. Thus, near the special end-points, $\Phi(z)$ must be almost bounded.

If $c\in\{c_1,\cdots,c_q\}$, then, in order that $X(z)\Phi_0(z)$ is bounded near c, $\Phi_0(z)$ must take c as an ordinary point since $0<\operatorname{Re}\gamma_c<1$. If $c\in\{c_{q+1},\cdots,c_r\}$, then, in order that $X(z)\Phi_0(z)$ is integrably unbounded near c, c can only be an ordinary point of $\Phi_0(z)$ also since $-1<\operatorname{Re}\gamma_c<0$.

In a word, whatever end-point c may be, it must be an ordinary point of $\Phi_0(z)$. Hence $\Phi_0(z)$ is holomorphic in D. Since both $\Phi_1(z)$ and $X(z)$ are continuous to L and $X(z)\neq 0$ in the neighbourhood of L, so $\Phi_0(z)$ must be continuous to L also. Now, the condition to be satisfied by $\Phi_0(z)$ is the same as that in (2.6). Thus, the problem again reduces to an H-problem for $\Phi_0(z)$. From (4.1), we get readily

$$\operatorname{Ind}_L X(t)=-\sum_{j=1}^{n}\kappa_j=-\kappa;$$

therefore the index of this H-problem again is

$$\operatorname{Ind}_L[\lambda(t)\overline{X(t)}]=k+\kappa=K.$$

The remaining discussion is then similar to that in § II or § III.

Evidently, for the cases of discontinuous coefficients, we may treat the problem similarly.

V. The RH-Problems for Several Unknown Functions

Notations L, Γ are the same as those in § II. Let the required $\Phi(z)$ be an N-dimensional vector, holomorphic in D, continuous to L, and satisfying (2.1) and (2.2), where $G(\tau)(\tau \in \Gamma)$, $\lambda(t)(t \in L)$ are given non-degenerate matrices of order N, $g(\tau)$, $c(t)$ are given N-dimensional vectors, and all of them satisfy the Hölder conditions. Thus, we have an RH-problem for N unknown functions or an N-dimensional vector.

This problem may be solved again by the method of elimination described in § II. Let

$$\operatorname{Ind}_\Gamma \det G(\tau) = \kappa, \quad \operatorname{Ind}_L \det \lambda(t) = k,$$

and call $K = \kappa + k$ the *total index* of the previous RH-problem.

On solving it, we first erect the canonical matrix of the corresponding R-problem (2.1) (the definition and the general method of construction of it may be referred to [6]), then find out a particular solution of this R-problem:

$$\Phi_1(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{[X^+(\tau)]^{-1} g(\tau)}{\tau - z} d\tau,$$

and at last, according to (2.5), change the unknown vector $\Phi(z)$ to $\Phi_0(z)$. Noting that, in the present case, the reasoning from (2.5) on remains valid, we obtain finally $\Phi_0^+(\tau) = \Phi_0^-(\tau)$, i.e., $\Phi_0(z)$ is an N-dimensional vector, holomorphic in D and continuous on $\bar{D}$. Thus, the problem is reduced to the H-problem (2.6) once again.

The total index of this H-problem may be shown to be exactly equal to K. (According to the definition of N. P. Vekua, it is $2K$, cf. [7], p. 179.) To this end, we first prove that

$$\operatorname{Ind}_L \det X(t) = -\kappa. \tag{5.1}$$

It is well known that (cf. [7], p. 41)

$$\operatorname{Ind}_\Gamma \det X^+(\tau) = \operatorname{Ind}_\Gamma \det G(\tau) = \kappa,$$

but between Γ and L the canonical function $\det X(z) = \det X^+(z)$ is holomorphic and has no zero point, so that

$$\operatorname{Ind}_L \det X(t) = \operatorname{Ind}_L \det X^+(t) = -\operatorname{Ind}_\Gamma \det X^+(\tau)$$

(the presence of the minus sign on the right-hand side is due to that the positive sense of L is its counter-clockwise sense, whereas that of Γ is its clockwise sense). Thus (5.1) is true. From this, we readily obtain that the total index of the reduced H-problem is

$$\operatorname{Ind}_L \det [\lambda(t)\overline{X(t)}] = \operatorname{Ind}_L \det \lambda(t) + \operatorname{Ind}_L \overline{\det X(t)} = k + \kappa = K.$$

If we can calculate $X(z)$ effectively, then the coefficients of this H-problem are known. Therefore the original RH-problem can be reduced to a given definite H-problem, and we may solve it forwards (cf. [7], Chap. III, Part 1). For instance, this is the case when $G(\tau)$ may be written as the product of two factors, which are the boundary values of two meromorphic matrices in the interior and exterior of Γ respectively (cf. [6], § 5).

For the cases when Γ consists of open arcs or when the coefficients have discontinuities, using the results of N. P. Vekua (cf. [7], Chap. II) and the method described in § IV, we may solve it analogously.

At last, we note that, as we have pointed out in [1], the method of elimination proposed here may be applied generally to solve an R-problem compounded with a boundary problem of some other type, and may also be employed to retreat in a simple manner the R-problems for open arcs, with discontinuous coefficients, or for complex boundaries.

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