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COLLEGE TRIGONOMETRY

College Trigonometry

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Preface

This text provides a comprehensive and mathematically sound treatment of the topics considered essential for a college trigonometry course. It is intended for the student who has successfully completed an intermediate algebra course.

To help the student master the concepts in this text, we have tried to maintain a balance among theory, application, and drill. Each definition and theorem is precisely stated and many theorems are proved. Carefully developed mathematics is complemented by abundant, creative applications that are both contemporary and representative of a wide range of disciplines.

Extensive exercise sets ranging from routine exercises to thought-provoking problems are provided at the end of each section. An ample selection of review exercises can be found at the end of each chapter.

Features

Interactive Presentation *College Trigonometry* is written in a style that encourages the student to interact with the textbook. At various places throughout the text, a question in the form of (Why?) is asked of the reader. This question encourages the reader to pause and think about the current discussion and to answer the question. To make sure the student does not miss important information, the answer to the question is provided as a footnote on the same page.

Each section contains a variety of worked examples. Each example is given a name so that a student can see at a glance the type of problem being illustrated. Each example is accompanied by annotations that assist the student in moving from step to step. Following the worked example is a suggested exercise from the exercise set for the student to work. The exercises are color coded by number in the exercise set and the complete solution of that exercise can be found in an appendix in the text.

Extensive Exercise Sets The exercise sets of *College Trigonometry* were carefully developed to provide the student with a variety of exercises. The exercises range from drill and practice to interesting challenges and were chosen to illustrate the many facets of topics discussed in the text. Besides the regular exercise sets, there is a set of supplemental problems that includes material from previous chapters, present extensions of topics, or are of the form "prove or disprove."

Applications One way to motivate a student to an interest in mathematics is through applications. The applications in *College Trigonometry* have been taken from agriculture, architecture, biology, business, chemistry, earth science, economics, engineering, medicine, and physics. Besides providing motivation to study mathematics, the applications provide an avenue to problem solving. The applications problems require the student to organize and implement a problem solving scheme.

Supplements for the Student

Two computerized study aids, the Computer Tutor and the Math Assistant, accompany this text.

The Computer Tutor The Computer Tutor is an interactive instructional microcomputer program for student use. Each section in the text is supported by a lesson on the Computer Tutor. Lessons on the tutor provide additional instruction and practice and can be used in several ways: (1) to cover material the student missed because of absence from class; (2) to reinforce instruction on a concept that the student has not yet mastered; (3) to review material in preparation for examinations. This tutorial is available for the IBM PC and compatible microcomputers.

The Math Assistant The Math Assistant is a collection of programs that can be used by both the instructor and the student. Some programs are instructional and allow the student to practice a skill like finding the inverse of a matrix. Other programs are computational routines that perform numerical calculations. In addition, there is a function grapher that graphs elementary functions and polar equations. The Math Assistant is available for the IBM PC and compatible microcomputers.

Supplements for the Instructor

College Trigonometry has an unusually complete set of teaching aids for the instructor.

Solutions Manual The Solutions Manual contains worked-out solutions for all end-of-section, supplemental, challenge and review exercises.

Instructor's Manual with Testing Program The Instructor's Manual contains the printed testing program, which is the first of three sources of testing material available to the user. Four printed tests (in two formats—free response and multiple choice) are provided for each chapter. In addition, the Instructor's Manual includes documentation of all the software ancillaries—the Math Assistant, the Computer Tutor, and the Computerized Test Generator. Finally, it contains answers to all the even-numbered exercises in the text.

Computerized Test Generator The Computerized Test Generator is the second source of testing material. The data base contains more than 1500 test items. These questions are unique to the test generator and do not repeat items provided in the Instructor's Manual testing program. The Test Generator is designed to produce an unlimited number of tests for each chapter of the text, including cumulative tests and final exams. It is available for the IBM PC and compatible microcomputers, the Macintosh microcomputer and the Apple II family of microcomputers.

Printed Test Bank The Printed Test Bank, the third component of the testing material, is a printout of all items in the Computerized Test Generator. Instructors using the Test Generator can use the test bank to select specific items from the data base. Instructors who do not have access to a computer can use the test bank to select items to be included on a test being prepared by hand.

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1

Introduction to Functions

So that we may say the door is now opened, for the first time, to a new method fraught with numerous and wonderful results which in future years will command the attention of other minds.

GALILEO GALILEI (1564–1642)

TRAP-DOOR FUNCTIONS, CRYPTOGRAPHY, AND SMART CARDS

Some activities are naturally easier to *reverse* than others. Activities that are not easy to reverse include squeezing the toothpaste out of the tube and cooking fresh eggs to make an omelet. These activities can be thought of as *one-way activities.*

The activity of using your car to pull a trailer is a *trap-door activity.* It is said to be a trap-door activity, because although you can drive either forward or backward, the activity of backing up is very difficult until you have mastered the necessary skills.

There are mathematical functions called *trap-door functions*. For example, the function of cubing a number is a trap-door function. It is easy to cube a number, but more difficult to discover the method of determining what number has been cubed, by examining the cube.

Ideally, a mathematical trap-door function would be such that it and its inverse can be evaluated in a matter of seconds. However, just to discover how to evaluate the inverse function requires several years of intense effort.

Mathematicians have found interesting and useful applications of trap-door functions. One of these applications is the art of writing and deciphering secret codes. This art, called cryptography, is vital to the military and commerce of every large nation.

Adi Shamir, Amos Fiat, and Uriel Feige, of the Weizmann Institute have recently used trap-door functions to develop the concept of zero-knowledge proofs. This has lead to the development of credit cards called "smart" cards. These credit cards contain electronic chips that use zero-knowledge proofs. The cards can identify the user of the card as the owner of the card, without giving a sales clerk the card number or any other information that could be used to make unauthorized purchases.

1.1 Preliminaries

The real numbers are used extensively in mathematics. The set of real numbers is quite comprehensive and contains several unique sets of numbers.

The **integers** is the set of numbers

$$\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

Recall that the brace symbols, { }, are used to identify a set. The positive integers are called **natural numbers.**

The **rational numbers** is the set of numbers of the form a/b, where a and b are integers, and $b \neq 0$. Thus, the rational numbers include $\frac{-3}{4}$ and $\frac{5}{2}$. Because each integer can be expressed in the form a/b with denominator $b = 1$, the integers are included in the set of rational numbers. Every rational number can be written as either a terminating or repeating decimal.

A number written in decimal form that does not repeat or terminate is called an **irrational number.** Some examples of irrational numbers are $0.141141114\ldots$, $\sqrt{2}$ and π. These numbers cannot be expressed as quotients of integers. The set of real numbers is the union of the sets of rational and irrational numbers.

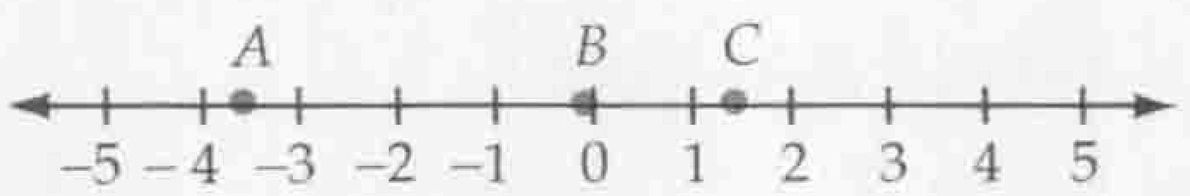


Figure 1.1
A coordinate line

A real number can be represented geometrically on a **coordinate line.** Each point on this line is associated with a real number called the **coordinate** of the point. Conversely, each real number can be associated with a point on a coordinate line. In Figure 1.1, the coordinate of A is $-\frac{7}{2}$, the coordinate of B is 0, and the coordinate of C is $\sqrt{2}$.

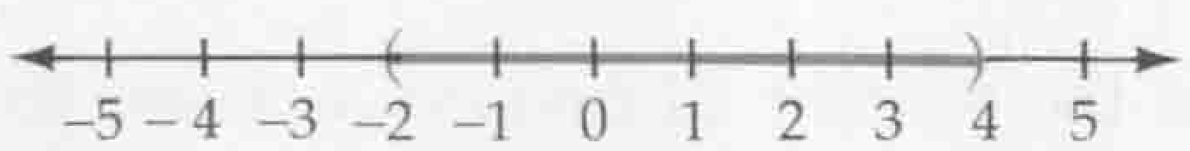
Given any two real numbers a and b, we say that a is **less than** b, denoted by $a < b$, if $a - b$ is a negative number. Similarly, we say that a is **greater than** b, denoted by $a > b$, if $a - b$ is a positive number. When a **equals** b, $a - b$ is zero. The symbols $<$ and $>$ are called **inequality symbols.** Two other inequality symbols, $\leq$ (less than or equal) and $\geq$ (greater than or equal) are also used. The inequality symbols can be used to designate sets of real numbers.

If $a < b$, the notation (a, b) is used to indicate the set of real numbers between a and b. This set of numbers can be described using **set builder notation:**

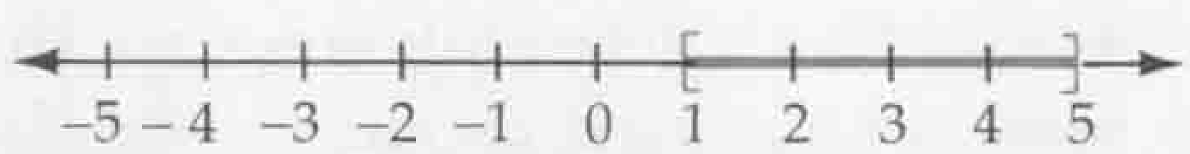
$$(a, b) = \{x \mid a < x < b\}.$$

When reading a set written in set builder notation, read "$\{x \mid$" as "the set of x such that." The expression that follows the vertical bar designates the elements in the set.

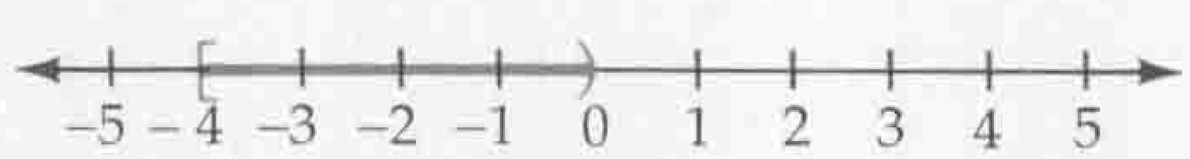
The set (a, b) is called a **open interval.** The graph of the open interval consists of all the points on the coordinate line between a and b, not including a and b. A **closed interval,** denoted by $[a, b]$, consists of all points between a and b including a and b. We can also discuss **half-open intervals.** An example of each of these is shown in Figure 1.2.



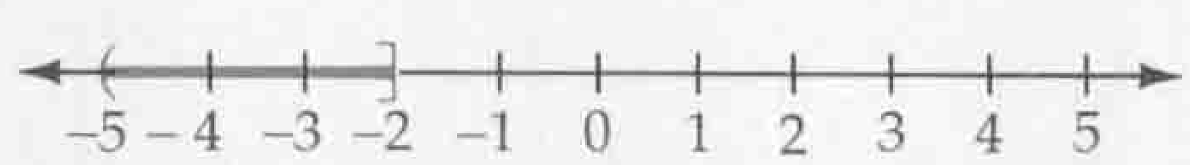
The open interval (–2, 4)



The closed interval [1, 5]



The half-open interval [–4, 0)



The half-open interval (–5, –2]

Figure 1.2

$$(-2, 4) = \{x \mid -2 < x < 4\} \quad \text{An open interval}$$

$$[1, 5] = \{x \mid 1 \leq x \leq 5\} \quad \text{A closed interval}$$

$$[-4, 0) = \{x \mid -4 \leq x < 0\} \quad \text{A half-open interval}$$

$$(-6, -2] = \{x \mid -6 < x \leq -2\} \quad \text{A half-open interval}$$

The *absolute value* of a number is a measure of the distance from zero to the point associated with the number on a coordinate line. Therefore, the absolute value of a number is always positive or zero. We now give a more formal definition of absolute value.

Absolute Value

For a real number a, the **absolute value** of a, denoted by $|a|$, is

$$|a| = \begin{cases} a & \text{if } a \geq 0. \\ -a & \text{if } a < 0. \end{cases}$$

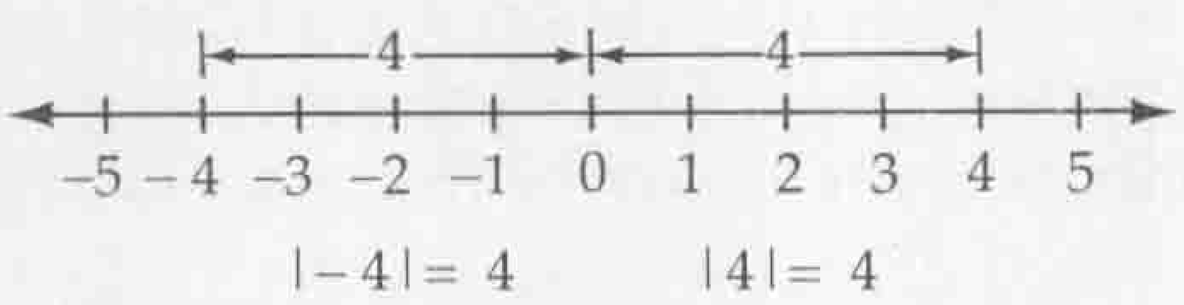


Figure 1.3

The distance d between the points with coordinates -3 and 2 on a coordinate line is is the absolute value of the difference between the coordinates.

$$d = |2 - (-3)| = 5$$

Because the absolute value is used, we could also write

$$d = |(-3) - 2| = 5$$

In general, we define the distance between any two points A and B on a coordinate line as the absolute value of the difference between the coordinates of the points.

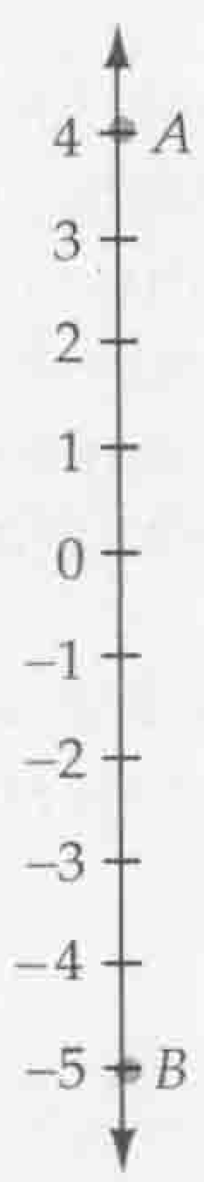


Figure 1.4

Distance Between Two Points on a Coordinate Line

Let a and b be the coordinates of the points A and B, respectively, on a coordinate line. Then the distance between A and B, denoted $d(A, B)$ is

$$d(A, B) = |a - b|.$$

Remark This formula applies to any coordinate line. Thus, it can be used to find the distance between two points on a vertical coordinate line as shown in Figure 1.4.

An **equation** is a statement about the equality of two expressions. Examples of equations are

$$7 = 2 + 5, \qquad x^2 = 4x + 5, \qquad 3x - 2 = 2(x + 1) + 3$$

The values of the variable that make an equation a true statement are the **roots** or **solutions** of the equation. To **solve** an equation means to find the solutions of the equation. The **solution set** of an equation is the set of all solutions of the equation.

First Degree Equation

A **first degree** or **linear equation** in one variable is an equation of the form $ax + b = c$, where $a \neq 0$.

To solve a first degree equation, isolate the variable on one side of the equal sign.

EXAMPLE 1 Solve a First Degree Equation

Solve the equation $3x - 5 = 2$.

Solution

$$3x - 5 = 2$$

$$3x - 5 + 5 = 2 + 5 \quad \text{Add 5 to each side of the equation.}$$

$$3x = 7$$

$$\frac{3x}{3} = \frac{7}{3} \quad \text{Divide each side of the equation by 3.}$$

$$x = \frac{7}{3}$$

The solution is $\frac{7}{3}$.

■ *Try Exercise* **6**, *page 9.*

Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ is a **quadratic equation** or **second degree equation** in one variable.

A quadratic equation can be solved using the quadratic formula.

Quadratic Formula

The solution of the quadratic equation $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 2 Solve a Quadratic Equation

Solve the equation $2x^2 - 4x + 1 = 0$.

Solution We have $a = 2$, $b = -4$, and $c = 1$.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)} = \frac{4 \pm \sqrt{16 - 8}}{4}$$

$$= \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = \frac{2 \pm \sqrt{2}}{2}$$

The solutions are $\frac{2 + \sqrt{2}}{2}$ and $\frac{2 - \sqrt{2}}{2}$.

■ *Try Exercise* **16**, *page 9.*

Although every quadratic equation can be solved using the quadratic formula, sometimes it is easier to use another method. Factoring and using the Principle of Zero Products, is one such method.

Principle of Zero Products

> If a and b are real numbers and $ab = 0$, then $a = 0$ or $b = 0$.

For example, to solve $2x^2 + x - 6 = 0$, first factor the polynomial.

$$2x^2 + x - 6 = 0$$
$$(2x - 3)(x + 2) = 0$$

Now let each factor equal zero and solve for x.

$$2x - 3 = 0 \qquad x + 2 = 0$$
$$x = \frac{3}{2} \qquad x = -2.$$

The solutions are $\frac{3}{2}$ and -2.

Rectangular Coordinate System

Each point on a flat two-dimensional surface, called a **plane**, can be associated with a pair of numbers. To do this, we use a coordinate system.

A **rectangular coordinate system** in a plane is formed by two coordinate lines, one horizontal and one vertical, that intersect at the origin of each line. The two coordinate lines are called **coordinate axes** or simply **axes**, and the point of intersection is called the **origin** of the coordinate system. The horizontal axis is usually referred to as the ***x*-axis**, the vertical axis as the ***y*-axis**, and the plane as the ***xy*-plane**. The coordinate axes divide a plane into four regions called **quadrants**. The quadrants are numbered counterclockwise from I to IV. See Figure 1.5.

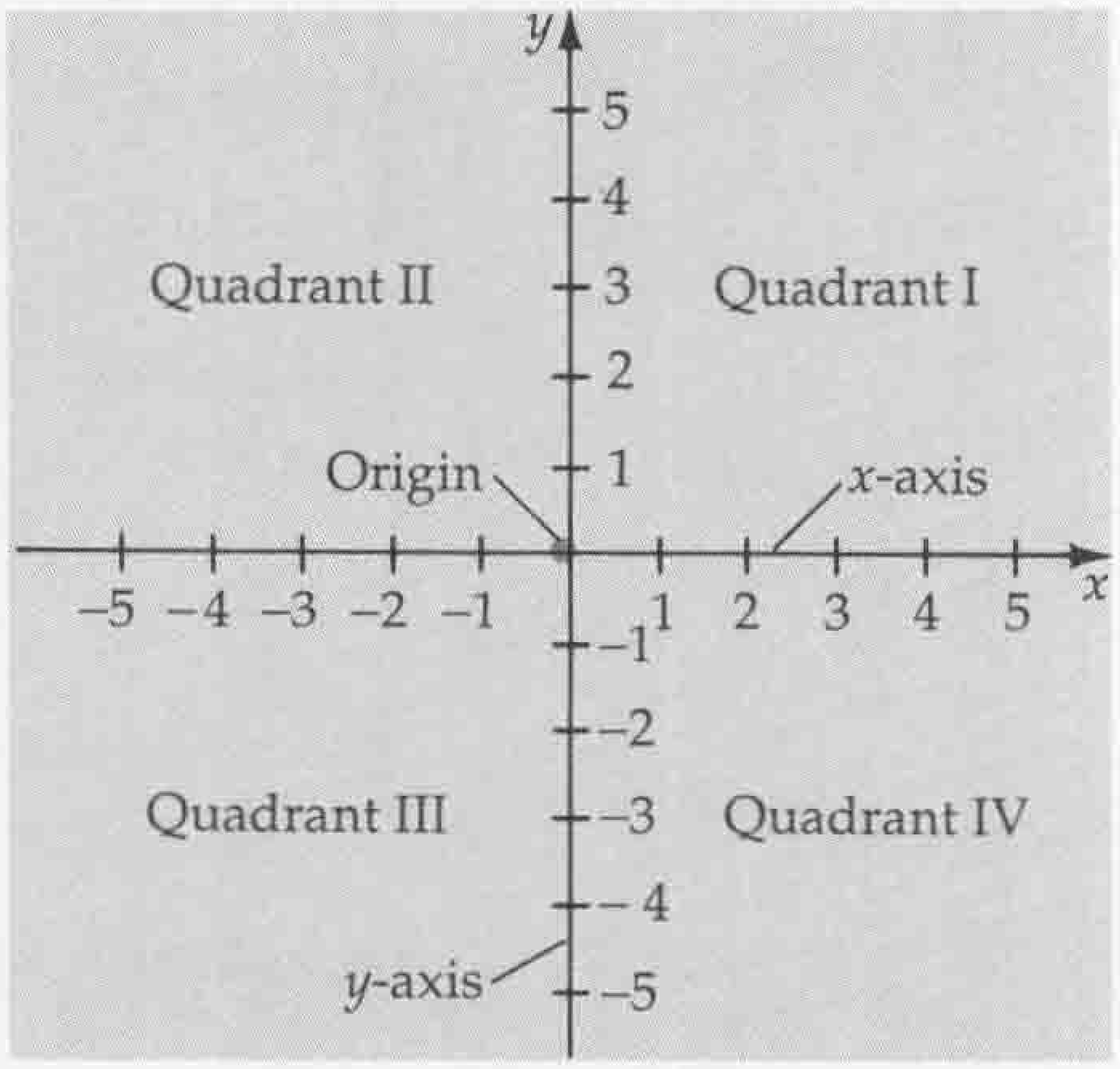


Figure 1.5

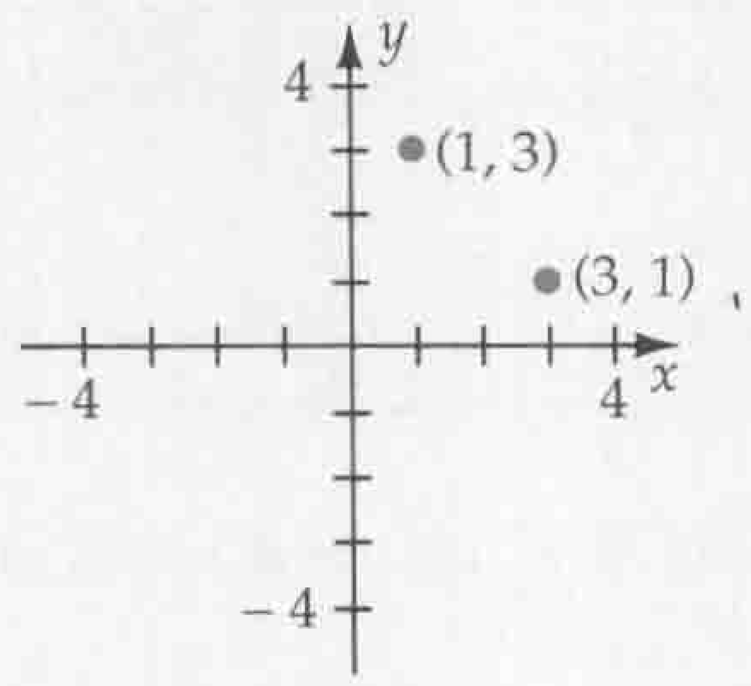


Figure 1.6

In the xy-plane, each point P can be associated with an **ordered pair** of numbers called **coordinates** of the point. Ordered pairs are denoted by (a, b), where a is the **x-coordinate** or **abscissa** and b is the **y-coordinate** or **ordinate**. The first number a of the pair locates a point on the line parallel to the y-axis passing through a on the x-axis. The second number b of the pair locates a point on the line parallel to the x-axis passing through b on the y-axis. To **plot** a point means to place a dot at the coordinates of the point.

The order in which the coordinates of an ordered pair are listed is important. Figure 1.6 shows that $(1, 3)$ and $(3, 1)$ do not denote the same point.

Equality of Ordered Pairs

Two ordered pairs (a, b) and (c, d) are equal if and only if

$$a = c \quad \text{and} \quad b = d.$$

The combination of algebra and geometry with a coordinate system is the branch of mathematics called **analytic geometry.** This branch of mathematics extends the techniques available to us to analyze not only problems in mathematics but also problems in such areas as science, engineering, and business.

The distance between any two points in a coordinate plane can be found by using the Pythagorean Theorem. Recall that a right triangle contains one 90° angle called a **right angle.** The side opposite the 90° angle is the hypotenuse and the two other sides are the legs of the triangle.

Pythagorean Theorem

If a and b denote the lengths of the legs of a right triangle and c is the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$

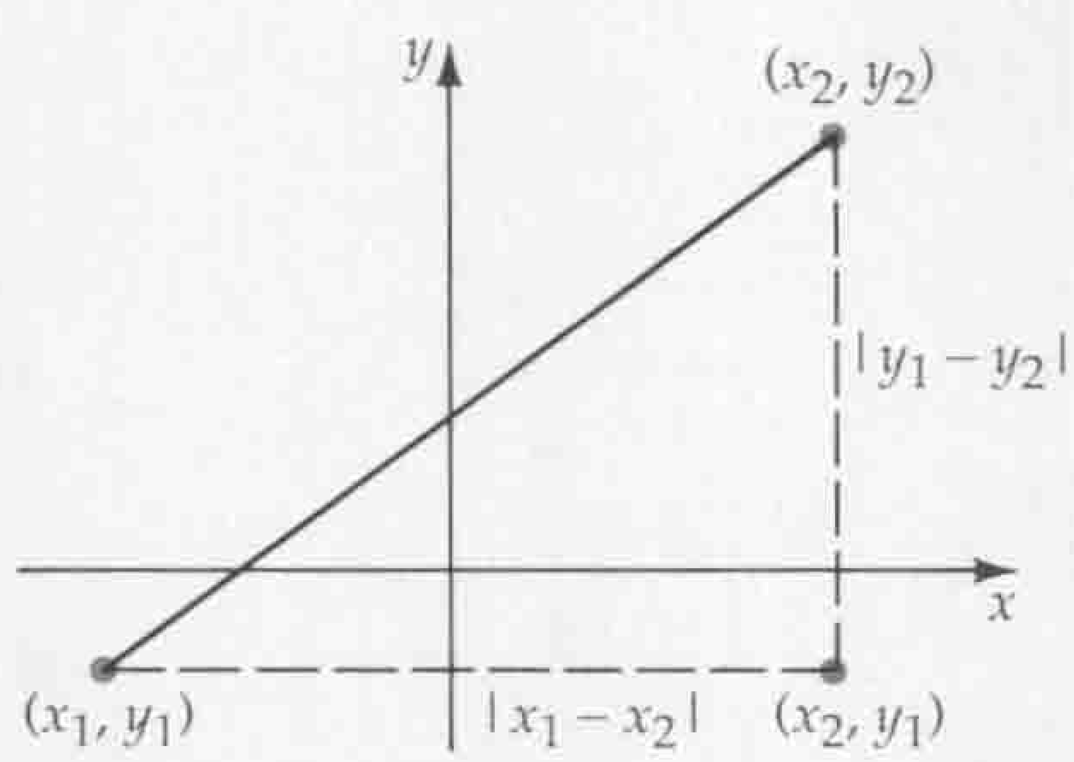


Figure 1.7

Let (x_1, y_1) and (x_2, y_2) be two points in an xy-plane. Notice in Figure 1.7 that the distance between the two points is the hypotenuse of a right triangle whose legs are horizontal and vertical line segments. The length of the horizontal segment is $|x_1 - x_2|$, and the length of the vertical segment is $|y_1 - y_2|$. Applying the Pythagorean Theorem to the triangle and using the fact that $|a|^2 = a^2$, we have

$$d^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2$$

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Distance Formula

Let (x_1, y_1) and (x_2, y_2) be two points in a plane. The distance d between the two points is given by

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Remark This formula also is valid for points that lie on a vertical or horizontal line.

EXAMPLE 3 Find the Distance Between Two Points

Find the distance d between the points

a. $(-4, 3)$ and $(2, 7)$ b. $(5, -1)$ and $(-2, -4)$

Solution

a. $d = \sqrt{[(-4) - 2)]^2 + [3 - 7]^2} = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$

b. $d = \sqrt{[5 - (-2)]^2 + [(-1) - (-4)]^2} = \sqrt{49 + 9} = \sqrt{58}$

■ *Try Exercise* **28**, *page 9.*

The midpoint of the line segment with endpoints $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is the point on the line equidistant from P_1 and P_2. The coordinates of this point are given by the *Midpoint Formula.*

Midpoint Formula

The midpoint M of the line segment joining P_1 and P_2 is given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

To prove this formula, let $M(x_3, y_3)$ be the midpoint of the line segment joining P_1 and P_2 where $x_1 < x_2$. Now consider the two triangles P_1QP_2 and P_1RM in Figure 1.8. Because these are similar triangles, the ratios of corresponding sides are equal. Thus,

$$\frac{d(P_1, R)}{d(P_1, Q)} = \frac{d(P_1, M)}{d(P_1, P_2)}$$

$$\frac{x_3 - x_1}{x_2 - x_1} = \frac{1}{2}$$

$$x_3 - x_1 = \frac{1}{2}(x_2 - x_1)$$

Solving for x_3, we have

$$x_3 = \frac{x_1 + x_2}{2}.$$

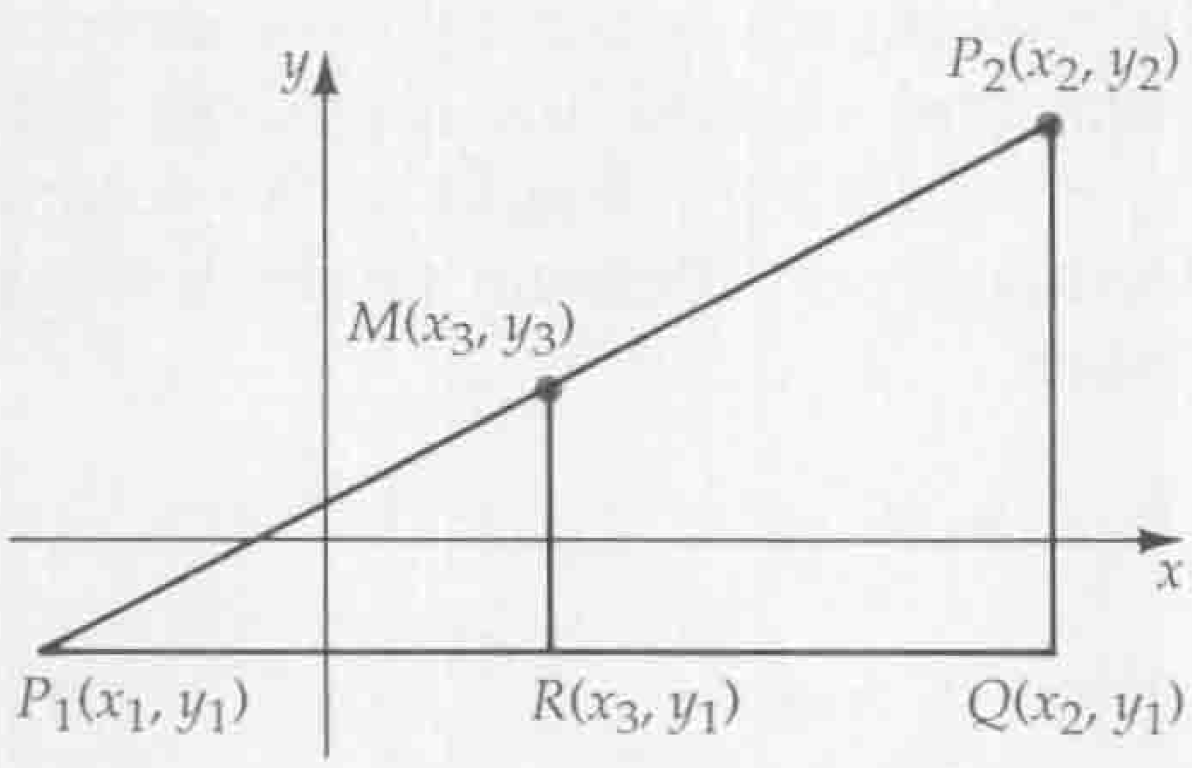


Figure 1.8

We have proved the equation of the x-coordinate of the midpoint. In a similar manner, we can verify the formula for the y-coordinate.

EXAMPLE 4 Find the Midpoint of a Line Segment

Find the midpoint of the line segment connecting
a. $(6, -2)$ and $(1, 3)$ b. $(3, -2)$ and $(-5, -4)$

Solution

a. $$\left(\frac{6+1}{2}, \frac{(-2)+3}{2}\right) = \left(\frac{7}{2}, \frac{1}{2}\right)$$

b. $$\left(\frac{3+(-5)}{2}, \frac{-2+(-4)}{2}\right) = (-1, -3)$$

■ *Try Exercise* **38**, *page 9.*

EXERCISE SET 1.1

In Exercises 1 to 10, solve the following first degree equations.

1. $2x - 1 = 7$ **2.** $3x + 2 = 11$

3. $4 - 5x = 19$ **4.** $1 - 7x = 15$

5. $2x + 4 = x - 5$ **6.** $5x - 2 = 3x + 4$

7. $2 - 3(2x + 1) = 5$

8. $3x - 2(2x - 3) = 4(2x + 1)$

9. $5x - 3(1 - 2x) = 2 - 5(2x + 1)$

10. $4 - 3(2 - 2x) = 3x + 2(3 - 2x)$

In Exercises 11 to 20, solve the following quadratic equations. Solve by factoring when possible.

11. $x^2 - x - 6 = 0$ **12.** $2x^2 - x - 1 = 0$

13. $x^2 - 3x = 10$ **14.** $6x^2 - 5x = 6$

15. $4x^2 - 4x + 1 = 0$ **16.** $x^2 - 6x - 4 = 0$

17. $x^2 + 6x = 2$ **18.** $2x^2 - x = 3$

19. $3x^2 = 2x + 1$ **20.** $4x^2 = 7x - 1$

In Exercises 21 to 30, find the distance between the given points.

21. $(2, 3)$ and $(-3, 4)$ **22.** $(4, 0)$ and $(8, -2)$

23. $(0, -4)$ and $(6, 3)$ **24.** $(-2, -4)$ and $(5, 4)$

25. $(-5, 6)$ and $(5, -6)$ **26.** $(7, 0)$ and $(3, -3)$

27. $(1, -5)$ and $(-4, -1)$ **28.** $(5, -2)$ and $(2, -4)$

29. $(5, -2)$ and $(4, -2)$ **30.** $(-1, 5)$ and $(-1, 3)$

In Exercises 31 to 40, find the midpoint of the line segment connecting the two points.

31. $(4, 1)$ and $(-2, 3)$ **32.** $(3, 2)$ and $(6, -4)$

33. $(5, -2)$ and $(0, 4)$ **34.** $(-1, -3)$ and $(2, 6)$

35. $(6, -6)$ and $(-4, 5)$ **36.** $(0, 3)$ and $(2, -3)$

37. $(-1, 4)$ and $(-3, -5)$ **38.** $(1, -4)$ and $(3, -5)$

39. $(9, -4)$ and $(5, -4)$ **40.** $(-8, 1)$ and $(-8, 3)$

41. **a.** Plot the points $A(1, 2)$, $B(5, 6)$, $C(10, 1)$, and $D(6, -3)$.
b. Draw line segments AB, BC, CD, and AD. Show that $d(A, B) = d(C, D)$ and $d(B, C) = d(A, D)$.
c. Show that $d(B, D) = d(A, C)$ for the diagonals BD and AC.
d. Name the figure $ABCD$.

42. **a.** Plot the points $A(-1, -1)$, $B(5, 2)$, $C(10, -3)$, and $D(4, -6)$.
b. Draw line segments AB, BC, CD, and AD. Show that $d(A, B) = d(C, D)$ and $d(B, C) = d(A, D)$.
c. Show that for $d(B, D) \neq d(A, C)$ the diagonals BD and AC.
d. Name the figure $ABCD$.

43. The points $A(0, 2)$, $B(2, 6)$, $C(10, 2)$, and $D(8, -2)$ are the vertices of a rectangle. Find the area of the rectangle.