CAMBRIDGE

金融数学丛书

影印版

Financial Engineering and Computation:

Principles, Mathematics, Algorithms

金融工程和计算

——原理・数学・算法

Yuh-Dauh Lyuu 著

高等教育出版社
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金融工程和计算

—— 原理 · 数学 · 算法

过去十年来, 许多复杂的数学和计算技巧为分析金融市场而发展起来. 有意在任何金融领域工作的学生和业界人士不仅必须掌握高级的概念和数学模型, 并且还必须学会怎样在计算上实现这些模型. 本书全面讨论了金融工程背后的理论和数学, 并强调了在当今资本市场中金融工程实际应用的计算.

与大多数有关投资学、金融工程或衍生证券的书不同的是, 本书从金融学的基本观念出发, 逐步构建理论. 在现代金融学中所需要的高级数学概念以一种可接受的层次来阐释. 这样, 它就为金融方面的 MBA、有志于从事金融业的理工科学生、计算金融的研究工作者、系统分析师和金融工程师在这一主题上提供了全面的基础.

构建理论的同时, 作者介绍了在定价、风险管理和证券组合管理方面的计算技巧的算法, 并且对它们的效率进行了分析. 对金融证券和衍生证券的定价是本书的中心论题. 各种各样的金融工具都得到讨论: 债券、期权、期货、远期、利率衍生品、有抵押支持的证券、嵌入期权的债券, 以及诸如此类的其他工具. 为便于参考使用, 每种金融工具都以简短而自成体系的一章来论述.

许多算法都用 Java 来编程, 并在本书的主页

`www.csie.ntu.edu.tw/~lyuu/Capitals/capitals.htm`

上可获取. 这些程序可在 Windows, MacOS 或 Unix 平台上执行.

吕育道 (Yuh-Dauh Lyuu) 教授在哈佛大学获得计算机科学专业的博士学位. 他过去的职位包括贝尔实验室的技术人员、NEC 研究所 (普林斯顿) 的研究员以及花旗证券 (纽约) 的助理副总裁. 他目前是台湾大学的计算机科学与信息工程学教授和金融学教授. 他的前一本著作是《信息散布和并行计算》(*Information Dispersal and Parallel Computation*).

吕教授在计算机科学和金融两方面都出版过著作, 他也持有美国专利, 并曾因指导优秀研究生论文多次获奖.

Preface

[A book] is a node within a network.
Michel Foucault (1926–1984), *The Archaeology of Knowledge*

Intended Audience

As the title of this book suggests, a modern book on financial engineering has to cover investment theory, financial mathematics, and computer science evenly. This interdisciplinary emphasis is tuned more to the capital markets wherever quantitative analysis is being practiced. After all, even economics has moved away from a time when "the bulk of [Alfred Marshall's] potential readers were both unable and unwilling to read economics in mathematical form" according to Viner (1892–1970) [860] toward the new standard of which Markowitz wrote in 1987, "more than half my students cannot write down the formal definition of [the limit of a sequence]" [642].

This text is written mainly for students of engineering and the natural sciences who want to study quantitative finance for academic or professional reasons. No background in finance is assumed. Years of teaching students of business administration convince me that technically oriented MBA students will benefit from the book's emphasis on computation. With a sizable bibliography, the book can serve as a reference for researchers.

This text is also written for practitioners. System analysts will find many compact and useful algorithms. Portfolio managers and traders can obtain the quantitative underpinnings for their daily activities. This work also serves financial engineers in their design of financial instruments by expounding the underlying principles and the computational means to pricing them.

The marketplace has already offered several excellent books on derivatives (e.g., [236, 470, 514, 746, 878]), financial engineering (e.g., [369, 646, 647]), financial theory (e.g., [290, 492]), econometrics (e.g., [147]), numerical techniques (e.g., [62, 215]), and financial mathematics (e.g., [59, 575, 692, 725]). There are, however, few books that come near to integrating the wide-ranging disciplines. I hope this text succeeds at least partially in that direction and, as a result, one no longer has to buy four or five books to get good coverage of the topics.

Presentation

This book is self-contained. Technically sophisticated undergraduates and graduates should be able to read it on their own. Mathematical materials are added where they are needed. In many instances, they provide the coupling between earlier chapters and upcoming themes. Applications to finance are generally added to set the stage. Numerical techniques are presented algorithmically and clearly; programming them should therefore be straightforward. The underlying financial theory is adequately covered, as understanding the theory underlying the calculations is critical to financial innovations.

The large number of exercises is an integral part of the text. Exercises are placed right after the relevant materials. Hints are provided for the more challenging ones. There are also numerous programming assignments. Those readers who aspire to become software developers can learn a lot by implementing the programming assignments. Thoroughly test your programs. The famous adage of Hamming (1916–1998), "The purpose of computing is insight, not numbers," does not apply to erroneous codes. Answers to all nontrivial exercises and some programming assignments can be found near the end of the book.

Most of the graphics were produced with *Mathematica* [882]. The programs that generate the data for the plots have been written in various languages, including C, C++, Java, JavaScript, Basic, and Visual Basic. It is a remarkable fact that most – if not all – of the programming works could have been done with spreadsheet software [221, 708]. Some computing platforms admit the integration of the spreadsheet's familiar graphical user interface and programs written in more efficient high-level programming languages [265]. Although such a level of integration requires certain sophistication, it is a common industry practice. Freehand graphics were created with Canvas and Visio.

The manuscript was typeset in LaTeX [580], which is ideal for a work of this size and complexity. I thank Knuth and Lamport for their gifts to technical writers.

Software

Many algorithms in the book have been programmed. However, instead of being bundled with the book in disk, my software is Web-centric and platform-independent [412]. Any machine running a World Wide Web browser can serve as a host for those programs on *The Capitals* page at

www.csie.ntu.edu.tw/~lyuu/capitals.html

There is no more need for the (rare) author to mail the upgraded software to the reader because the one on the Web page is always up to date. This new way of software development and distribution, made possible by the Web, has turned software into an Internet service.

Organization

Here is a grand tour of the book:

Chapter 1 sets the stage and surveys the evolution of computer technology.

Chapter 2 introduces algorithm analysis and measures of complexity. My convention for expressing algorithms is outlined here.

Chapter 3 contains a relatively complete treatment of standard financial mathematics, starting from the time value of money.

Chapter 4 covers the important concepts of duration and convexity.

Chapter 5 goes over the static term structure of interest rates. The coverage of classic, static finance theory ends here.

Chapter 6 marks the transition to stochastic models with coverage of statistical inference.

Chapters 7–12 are about options and derivatives. Chapter 7 presents options and basic strategies with options. Chapter 8 introduces the arbitrage argument and derives general pricing relations. Chapter 9 is a key chapter. It covers option pricing under the discrete-time binomial option pricing model. The celebrated Black–Scholes formulas are derived here, and algorithms for pricing basic options are presented. Chapter 10 presents sensitivity measures for options. Chapter 11 covers the diverse applications and kinds of options. Additional derivative securities such as forwards and futures are treated in Chap. 12.

Chapters 13–15 introduce the essential ideas in continuous-time financial mathematics. Chapter 13 covers martingale pricing and Brownian motion, and Chap. 14 moves on to stochastic integration and the Ito process. Together they give a fairly complete treatment of the subjects at an accessible level. From time to time, we go back to discrete-time models and establish the linkage. Chapter 15 focuses on the partial differential equations that derivative securities obey.

Chapter 16 covers hedging by use of derivatives.

Chapters 17–20 probe deeper into various technical issues. Chapter 17 investigates binomial and trinomial trees. One of the motives here is to demonstrate the use of combinatorics in designing highly efficient algorithms. Chapter 18 covers numerical methods for partial differential equations, Monte Carlo simulation, and quasi–Monte Carlo methods. Chapter 19 treats computational linear algebra, least-squares problems, and splines. Factor models are presented as an application. Chapter 20 introduces financial time series analysis as well as popular time-series models.

Chapters 21–27 are related to interest-rate-sensitive securities. Chapter 21 surveys the wide varieties of interest rate derivatives. Chapter 22 discusses yield curve fitting. Chapter 23 introduces interest rate modeling and derivative pricing with the elementary, yet important, binomial interest rate tree. Chapter 24 lays the mathematical foundations for interest rate models, and Chaps. 25 and 26 sample models from the literature. Finally, Chap. 27 covers fixed-income securities, particularly those with embedded options.

Chapters 28–30 are concerned with mortgage-backed securities. Chapter 28 introduces the basic ideas, institutions, and challenging issues. Chapter 29 investigates the difficult problem of prepayment and pricing. Chapter 30 surveys collateralized mortgage obligations.

Chapter 31 discusses the theory and practice of portfolio management. In particular, it presents modern portfolio theory, the Capital Asset Pricing Model, the Arbitrage Pricing Theory, and value at risk.

Chapter 32 documents the Web software developed for this book.

Chapter 33 contains answers or pointers to all nontrivial exercises.

This book ends with an extensive index. There are two guiding principles behind its structure. First, related concepts should be grouped together. Second, the index should facilitate search. An entry containing parentheses indicates that the term within should be consulted instead, first at the current level and, if not found, at the outermost level.

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This book benefited greatly from the comments of several anonymous reviewers. As the first readers of the book, their critical eyes made a lasting impact on its evolution. As with my first book with Cambridge University Press, the editors at the Press were invaluable. In particular, I would like to thank Lauren Cowles, João da Costa, Caitlin Doggart, Scott Parris, Eleanor Umali, and the anonymous copy editor.

I want to thank my wife Chih-Lan and my son Raymond for their support during the project, which started in January 1995. This book, I hope, finally puts to rest their dreadful question, "When are you going to finish it?"

Useful Abbreviations

Acronyms

APT	Arbitrage Pricing Theory
AR	autoregressive (process)
ARCH	autoregressive conditional heteroskedastic (process)
ARM	adjustable-rate mortgage
ARMA	autoregressive moving average (process)
BDT	Black–Derman–Toy (model)
BEY	bond-equivalent yield
BOPM	binomial option pricing model
BPV	basis-point value
CAPM	Capital Asset Pricing Model
CB	convertible bond
CBOE	Chicago Board of Exchange
CBT	Chicago Board of Trade
CD	certificate of deposit
CIR	Cox–Ingersoll–Ross
CME	Chicago Mercantile Exchange
CMO	collateralized mortgage obligation
CMT	constant-maturity Treasury (rate)
COFI	Cost of Funds Index
CPR	conditional prepayment rate
DEM	German mark
DJIA	Dow Jones Industrial Average
FHA	Federal Housing Administration
FHLMC	Federal Home Loan Mortgage Corporation ("Freddie Mac")
FNMA	Federal National Mortgage Association ("Fannie Mae")
forex	foreign exchange
FRA	forward rate agreement
FV	future value

GARCH	generalized autoregressive conditional heteroskedastic
GLS	generalized least-squares
GMM	generalized method of moments
GNMA	Government National Mortgage Association ("Ginnie Mae")
HJM	Heath–Jarrow–Morton
HPR	holding period return
IAS	index-amortizing swap
IMM	International Monetary Market
IO	interest-only
IRR	internal rate of return
JPY	Japanese yen
LIBOR	London Interbank Offered Rate
LTCM	Long-Term Capital Management
MA	moving average
MBS	mortgage-backed security
MD	Macauley duration
ML	maximum likelihood
MPTS	mortgage pass-through security
MVP	minimum-variance point
NPV	net present value
NYSE	New York Stock Exchange
OAC	option-adjusted convexity
OAD	option-adjusted duration
OAS	option-adjusted spread
OLS	ordinary least-squares
PAC	Planned Amortization Class (bond)
P&I	principal and interest
PC	participation certificate
PO	principal-only
PSA	Public Securities Association
PV	present value
REMIC	Real Estate Mortgage Investment Conduit
RHS	Rural Housing Service
RS	Ritchken–Sankarasubramanian
S&P 500	Standard and Poor's 500 Index
SMBS	stripped mortgage-backed security
SMM	single monthly mortality
SSE	error sum of squares

SSR	regression sum of squares
SST	total sum of squares
SVD	singular value decomposition
TAC	Target Amortization Class (bond)
VA	Department of Veterans Affairs
VaR	value at risk
WAC	weighted average coupon
WAL	weighted average life
WAM	weighted average maturity
WWW	World Wide Web

Ticker Symbols

DJ	Dow Jones Industrial Average
IRX	thirteen-week T-bill
NDX	Nasdaq 100
NYA	New York Stock Exchange Composite Index
OEX	S&P 100
RUT	Russell 200
SPX	S&P 500
TYX	thirty-year T-bond
VLE	Value Line Index
WSX	Wilshire S-C
XMI	Major Market Index

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CHAPTER ONE

Introduction

But the age of chivalry is gone. That of sophisters, oeconomists, and calculators, has succeeded; and the glory of Europe is extinguished for ever.

Edmund Burke (1729–1797), *Reflections on the Revolution in France*

1.1 Modern Finance: A Brief History

Modern finance began in the 1950s [659, 666]. The breakthroughs of Markowitz, Treynor, Sharpe, Lintner (1916–1984), and Mossin led to the Capital Asset Pricing Model in the 1960s, which became the quantitative model for measuring risk. Another important influence of research on investment practice in the 1960s was the Samuelson–Fama efficient markets hypothesis, which roughly says that security prices reflect information fully and immediately. The most important development in terms of practical impact, however, was the Black–Scholes model for option pricing in the 1970s. This theoretical framework was instantly adopted by practitioners. Option pricing theory is one of the pillars of finance and has wide-ranging applications [622, 658]. The theory of option pricing can be traced to Louis Bachelier's Ph.D. thesis in 1900, "Mathematical Theory of Speculation." Bachelier (1870–1946) developed much of the mathematics underlying modern economic theories on efficient markets, random-walk models, Brownian motion [ahead of Einstein (1879–1955) by 5 years], and martingales [277, 342, 658, 776].[1]

1.2 Financial Engineering and Computation

Today, the wide varieties of financial instruments dazzle even the knowledgeable. Individuals and corporations can trade, in addition to stocks and bonds, options, futures, stock index options, and countless others. When it comes to diversification, one has thousands of mutual funds and **exchange-traded funds** to choose from. Corporations and local governments increasingly use complex derivative securities to manage their financial risks or even to speculate. **Derivative securities** are financial instruments whose values depend on those of other assets. All are the fruits of **financial engineering**, which means structuring financial instruments to target investor preferences or to take advantage of arbitrage opportunities [646].