Graduate Texts in Mathematics

Polynomials and polynomial Inequalities

多项式和多项式不等式

Peter Borwein
Tamás Erdélyi

Springer-Verlag
世界图书出版公司

Peter Borwein Tamás Erdélyi

Polynomials and Polynomial Inequalities

Sl
世界图书出版公司
北京・广州・上海・西安

书　　名 Polynomials and Polynomial Inequalities
作　　者: P. Borwein, T. Erdélyi
中 译 名: 多项式和多项式不等式
出 版 者: 世界图书出版公司北京公司
印 刷 者: 北京中西印刷厂
发　　行: 世界图书出版公司北京公司 (北京朝内大街 137 号　100010)
开　　本: 大 32　　印张: 15
版　　次: 1997 年 9 月第 1 版　　1997 年 9 月第 1 次印刷
书　　号: ISBN 7-5062-3302-9/O·182
版权登记: 图字 01-97-0459
定　　价: 72.00 元

Graduate Texts in Mathematics **161**

Springer
New York
Berlin
Heidelberg
Barcelona
Budapest
Hong Kong
London
Milan
Paris
Santa Clara
Singapore
Tokyo

For Theresa, Sophie, Alexandra, Jennifer, and Erika

Preface

Polynomials pervade mathematics, and much that is beautiful in mathematics is related to polynomials. Virtually every branch of mathematics, from algebraic number theory and algebraic geometry to applied analysis, Fourier analysis, and computer science, has its corpus of theory arising from the study of polynomials. Historically, questions relating to polynomials, for example, the solution of polynomial equations, gave rise to some of the most important problems of the day. The subject is now much too large to attempt an encyclopedic coverage.

The body of material we choose to explore concerns primarily polynomials as they arise in analysis, and the techniques of the book are primarily analytic. While the connecting thread is the polynomial, this is an analysis book. The polynomials and rational functions we are concerned with are almost exclusively of a single variable.

We assume at most a senior undergraduate familiarity with real and complex analysis (indeed in most places much less is required). However, the material is often tersely presented, with much mathematics explored in the exercises, some of which are quite hard, many of which are supplied with copious hints, some with complete proofs. Well over half the material in the book is presented in the exercises. The reader is encouraged to at least browse through these. We have been much influenced by Pólya and Szegő's classic "Problems and Theorems in Analysis" in our approach to the exercises. (Though unlike Pólya and Szegő we chose to incorporate the hints with the exercises.)

The book is mostly self-contained. The text, without the exercises, provides an introduction to the material, but much of the richness is reserved for the exercises. We have attempted to highlight the parts of the theory and the techniques we find most attractive. So, for example, Müntz's lovely characterization of when the span of a set of monomials is dense is explored in some detail. This result epitomizes the best of the subject: an attractive and nontrivial result with several attractive and nontrivial proofs.

There are excellent books on orthogonal polynomials, Chebyshev polynomials, Chebyshev systems, and the geometry of polynomials, to name but a few of the topics we cover, and it is not our intent to rewrite any of these. Of necessity and taste, some of this material is presented, and we have attempted to provide some access to these bodies of mathematics. Much of the material in the later chapters is recent and cannot be found in book form elsewhere.

Students who wish to study from this book are encouraged to sample widely from the exercises. This is definitely "hands on" material. There is too much material for a single semester graduate course, though such a course may be based on Sections 1.1 through 5.1, plus a selection from later sections and appendices. Most of the material after Section 5.1 may be read independently.

Not all objects labeled with "E" are exercises. Some are examples. Sometimes no question is asked because none is intended. Occasionally exercises include a statement like, "for a proof see ..."; this is usually an indication that the reader is not expected to provide a proof.

Some of the exercises are long because they present a body of material. Examples of this include E.11 of Section 2.1 on the transfinite diameter of a set and E.11 of Section 2.3 on the solvability of the moment problem. Some of the exercises are quite technical. Some of the technical exercises, like E.4 of Section 2.4, are included, in detail, because they present results that are hard to access elsewhere.

Acknowledgments

We would like to thank Dick Askey, Weiyu Chen, Carl de Boor, Karl Dilcher, Jens Happe, András Kroó, Doron Lubinsky, Gua-Hua Min, Paul Nevai, Allan Pinkus, József Szabados, Vilmos Totik, and Richard Varga. Their thoughtful and helpful comments made this a better book. We would also like to thank Judith Borwein, Maria Fe Elder, and Chiara Veronesi for their expert assistance with the preparation of the manuscript.

Contents

1

Introduction and Basic Properties

Overview

The most basic and important theorem concerning polynomials is the Fundamental Theorem of Algebra. This theorem, which tells us that every polynomial factors completely over the complex numbers, is the starting point for this book. Some of the intricate relationships between the location of the zeros of a polynomial and its coefficients are explored in Section 2. The equally intricate relationships between the zeros of a polynomial and the zeros of its derivative or integral are the subject of Section 1.3. This chapter serves as a general introduction to the body of theory known as the geometry of polynomials. Highlights of this chapter include the Fundamental Theorem of Algebra, the Eneström-Kakeya theorem, Lucas' theorem, and Walsh's two-circle theorem.

1.1 Polynomials and Rational Functions

The focus for this book is the polynomial of a single variable. This is an extended notion of the polynomial, as we will see later, but the most important examples are the algebraic and trigonometric polynomials, which we now define. The complex $(n+1)$-dimensional vector space of algebraic polynomials of degree at most n with complex coefficients is denoted by $\mathcal{P}_n^c$.

If $\mathbb{C}$ denotes the set of complex numbers, then

$$\mathcal{P}_n^c := \left\{ p : p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{C} \right\}. \tag{1.1.1}$$

When we restrict our attention to polynomials with real coefficients we will use the notation

$$\mathcal{P}_n := \left\{ p : p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{R} \right\}. \tag{1.1.2}$$

where $\mathbb{R}$ is the set of real numbers. Rational functions of type (m, n) with complex coefficients are then defined by

$$\mathcal{R}_{m,n}^c := \left\{ \frac{p}{q} : p \in \mathcal{P}_m^c, q \in \mathcal{P}_n^c \right\}, \tag{1.1.3}$$

while their real cousins are denoted by

$$\mathcal{R}_{m,n} := \left\{ \frac{p}{q} : p \in \mathcal{P}_m, q \in \mathcal{P}_n \right\}. \tag{1.1.4}$$

The distinction between the real and complex cases is particularly important for rational functions (see E.4).

The set of trigonometric polynomials $\mathcal{T}_n^c$ is defined by

$$\mathcal{T}_n^c := \left\{ t : t(\theta) := \sum_{k=-n}^{n} a_k e^{ik\theta}, \quad a_k \in \mathbb{C} \right\}. \tag{1.1.5}$$

A real trigonometric polynomial of degree at most n is an element of $\mathcal{T}_n^c$ taking only real values on the real line. We denote by $\mathcal{T}_n$ the set of all real trigonometric polynomials of degree at most n. Other characterizations of $\mathcal{T}_n$ are given in E.9. Note that if $z := e^{i\theta}$, then an arbitrary element of $\mathcal{T}_n^c$ is of the form

$$z^{-n} \sum_{k=0}^{2n} b_k z^k, \qquad b_k \in \mathbb{C} \tag{1.1.6}$$

and so many properties of trigonometric polynomials reduce to the study of algebraic polynomials of twice the degree on the unit circle in $\mathbb{C}$.

The most basic theorem of this book, and arguably the most basic nonelementary theorem of mathematics, is the Fundamental Theorem of Algebra. It says that a polynomial of exact degree n (that is, an element of $\mathcal{P}_n^c \backslash \mathcal{P}_{n-1}^c$) has exactly n complex zeros counted according to their multiplicities.

Theorem 1.1.1 (Fundamental Theorem of Algebra). *If*

$$p(z) := \sum_{i=0}^{n} a_i z^i, \qquad a_i \in \mathbb{C}, \ a_n \neq 0,$$

then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ *such that*

$$p(z) = a_n \prod_{i=1}^{n} (z - \alpha_i).$$

Here the multiplicity of the zero at α_i is the number of times it is repeated. So, for example,

$$(z-1)^3(z+i)^2$$

is a polynomial of degree 5 with a zero of multiplicity 3 at 1 and with a zero of multiplicity 2 at $-i$. The polynomial

$$p(z) := \sum_{i=0}^{n} a_i z^i, \qquad a_i \in \mathbb{C}, \ a_n \neq 0$$

is called *monic* if its *leading coefficient* a_n equals 1. There are many proofs of the Fundamental Theorem of Algebra based on elementary properties of complex functions (see Theorem 1.2.1 and E.4 of Section 1.2). We will explore this theorem more substantially in the next section of this chapter.

Comments, Exercises, and Examples.

The importance of the solution of polynomial equations in the history of mathematics is hard to overestimate. The Greeks of the classical period understood quadratic equations (at least when both roots were positive) but could not solve cubics. The explicit solutions of the cubic and quartic equations in the sixteenth century were due to Niccolo Tartaglia (ca 1500–1557), Ludovico Ferrari (1522–1565), and Scipione del Ferro (ca 1465–1526) and were popularized by the publication in 1545 of the "Ars Magna" of Girolamo Cardano (1501–1576). The exact priorities are not entirely clear, but del Ferro probably has the strongest claim on the solution of the cubic. These discoveries gave western mathematics an enormous boost in part because they represented one of the first really major improvements on Greek mathematics. The impossibility of finding the zeros of a polynomial of degree at least 5, in general, by a formula containing additions, subtractions, multiplications, divisions, and radicals would await Niels Henrik Abel (1802–1829) and his 1824 publication of "On the Algebraic Resolution of Equations." Indeed, so much algebra, including Galois theory, analysis, and particularly complex analysis, is born out of these ideas that it is hard to imagine how the flow of mathematics might have proceeded without these issues being raised. For further history, see Boyer [68].

E.1 Explicit Solutions.

a] **Quadratic Equations.** Verify that the quadratic polynomial x^2+bx+c has zeros at

$$\frac{-b-\sqrt{b^2-4c}}{2}, \qquad \frac{-b+\sqrt{b^2-4c}}{2}.$$

b] **Cubic Equations.** Verify that the cubic polynomial x^3+bx+c has zeros at

$$\alpha+\beta, \quad -\left(\frac{\alpha+\beta}{2}\right)+i\sqrt{3}\left(\frac{\alpha-\beta}{2}\right), \quad -\left(\frac{\alpha+\beta}{2}\right)-i\sqrt{3}\left(\frac{\alpha-\beta}{2}\right),$$

where

$$\alpha=\sqrt[3]{\frac{-c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}}$$

and

$$\beta=\sqrt[3]{\frac{-c}{2}-\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}}.$$

c] Show that an arbitrary cubic polynomial, x^3+ax^2+bx+c, can be transformed into a cubic polynomial as in part b] by a transformation $x\mapsto ex+f$.

d] Observe that if the polynomial x^3+bx+c has three distinct real zeros, then α and β are necessarily nonreal and hence $4b^3+27c^2$ is negative. So, in this simplest of cases one is forced to deal with complex numbers (which was a serious technical problem in the sixteenth century).

e] **Quartic Equations.** The quartic polynomial $x^4+ax^3+bx^2+cx+d$ has zeros at

$$-\frac{a}{4}+\frac{R}{2}\pm\frac{\alpha}{2}, \qquad -\frac{a}{4}+\frac{R}{2}\pm\frac{\beta}{2},$$

where

$$R=\sqrt{\frac{a^2}{4}-b+y},$$

y is any root of the resolvent cubic

$$y^3-by^2+(ac+4d)y-a^2d+4bd-c^2,$$

and

$$\alpha,\beta=\sqrt{\frac{3a^2}{4}-R^2-2b\pm\frac{4ab-8c-a^3}{4R}}, \qquad R\neq 0,$$

while

$$\alpha,\beta=\sqrt{\frac{3a^2}{4}-2b\pm 2\sqrt{y^2-4d}}, \qquad R=0.$$

These unwieldy equations are quite useful in conjunction with any symbolic manipulation package.

E.2 Newton's Identities. Write

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = x^n - c_1x^{n-1} + c_2x^{n-2} - \cdots + (-1)^n c_n \,.$$

The coefficients c_k are, by definition, the *elementary symmetric functions* in the variables $\alpha_1, \dots, \alpha_n$.

a] For positive integers k, let

$$s_k := \alpha_1^k + \alpha_2^k + \cdots + \alpha_n^k \,.$$

Prove that

$$s_k = (-1)^{k+1}kc_k + (-1)^k \sum_{j=1}^{k-1} (-1)^j c_{k-j}s_j \,, \qquad k \le n$$

and

$$s_k = (-1)^{k+1} \sum_{j=k-n}^{k-1} (-1)^j c_{k-j}s_j \,, \qquad k > n \,.$$

Here, and in what follows, an empty sum is understood to be 0.

A *polynomial of n variables* is a function that is a polynomial in each of its variables. A *symmetric polynomial of n variables* is a polynomial of n variables that is invariant under any permutation of the variables.

b] Show by induction that any symmetric polynomial in n variables (with integer coefficients) may be written uniquely as a polynomial (with integer coefficients) in the elementary symmetric functions $f_1, f_2, \dots, f_n$.

Hint: For a symmetric polynomial f in n variables, let

$$\sigma(f) := (\nu_1, \nu_2, \dots, \nu_n) \,, \qquad \nu_1 \ge \nu_2 \ge \cdots \ge \nu_n \ge 0$$

if

$$f(x_1, x_2, \dots, x_n) = \sum_{\alpha_1=0}^{\nu_1} \sum_{\alpha_2=0}^{\nu_2} \cdots \sum_{\alpha_n=0}^{\nu_n} c_{\alpha_1,\alpha_2,\dots,\alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

and $c_{\nu_1,\nu_2,\dots,\nu_n} \neq 0$. If

$$\sigma(f) = (\nu_1, \nu_2, \dots, \nu_n) \quad \text{and} \quad \sigma(g) = (\widetilde{\nu}_1, \widetilde{\nu}_2, \dots, \widetilde{\nu}_n) \,,$$

then let $\sigma(f) < \sigma(g)$ if $\nu_j \le \widetilde{\nu}_j$ for each j with a strict inequality for at least one index. This gives a (partial) well ordering of symmetric polynomials in n variables, that is, every set of symmetric polynomials in n variables has a minimal element. Now use induction on $\sigma(f)$. □

c] Show that

$$\left(\frac{1+\sqrt{5}}{2}\right)^k \to 0 \pmod 1.$$

(By convergence to zero (mod 1) we mean that the quantity approaches integral values.)

Hint: Consider the integers

$$s_k := \alpha_1^k + \alpha_2^k,$$

where $\alpha_1 := \frac{1}{2}(1+\sqrt{5})$ and $\alpha_2 := \frac{1}{2}(1-\sqrt{5})$. □

d] Find another algebraic integer α with the property that

$$\alpha^k \to 0 \pmod 1.$$

Such numbers are called *Salem numbers* (see Salem [63]). It is an open problem whether any nonalgebraic numbers $\alpha > 1$ satisfy $\alpha^k \to 0 \pmod 1$.

E.3 Norms on $\mathcal{P}_n$. $\mathcal{P}_n$ is a vector space of dimension $n+1$ over $\mathbb{R}$. Hence $\mathcal{P}_n$ equipped with any norm is isomorphic to the Euclidean vector space $\mathbb{R}^{n+1}$, and these norms are equivalent to each other. Similarly, $\mathcal{P}_n^c$ is a vector space of dimension $n+1$ over $\mathbb{C}$. Hence $\mathcal{P}_n^c$ equipped with any norm is isomorphic to the Euclidean vector space $\mathbb{C}^{n+1}$, so these norms are also equivalent to each other. Let

$$p_n(x) := \sum_{k=0}^{n} a_k x^k, \qquad a_k \in \mathbb{R}.$$

Some common norms on $\mathcal{P}_n$ and $\mathcal{P}_n^c$ are

$$\|p\|_A := \sup_{x\in A} |p(x)| \qquad \textit{supremum norm}$$

$$:= \|p\|_{L_\infty(A)} \qquad L_\infty \textit{ norm}$$

$$\|p\|_{L_p(A)} := \left(\int_A |p(t)|^p\,dt\right)^{1/p} \qquad L_p \textit{ norm}, p \ge 1$$

$$\|p\|_{l_\infty} := \max_k\{|a_k|\} \qquad l_\infty \textit{ norm}$$

$$\|p\|_{l_p} := \left(\sum_{k=0}^{n} |a_k|^p\right)^{1/p} \qquad l_p \textit{ norm}, p \ge 1.$$

In the first case A must contain $n+1$ distinct points. In the second case A must have positive measure.

a] Conclude that there exist constants C_1, C_2, and C_3 depending only on n so that

$$\|p'\|_{[-1,1]} \leq C_1 \|p\|_{[-1,1]},$$

$$\sum_{i=0}^{n} |a_i| \leq C_2 \|p\|_{[-1,1]},$$

$$\|p\|_{[-1,1]} \leq C_3 \|p\|_{L_2[-1,1]}$$

for every $p \in \mathcal{P}_n^c$, and, in particular, for every $p \in \mathcal{P}_n$.

These inequalities will be revisited in detail in later chapters, where precise estimates are given in terms of n.

b] Show that there exist extremal polynomials for each of the above inequalities. That is, for example,

$$\sup_{0 \neq p \in \mathcal{P}_n} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}}$$

is achieved.

E.4 On $\mathcal{R}_{n,m}$.

a] $\mathcal{R}_{n,m}$ is not a vector space because it is not closed under addition.

b] **Partial Fraction Decomposition.** Let $r_{n,m} \in \mathcal{R}_{n,m}^c$ be of the form

$$\frac{p(x)}{\prod_{k=1}^{m'} (x - \alpha_k)^{m_k}}, \qquad p \in \mathcal{P}_n^c, \quad \alpha_k \text{ distinct}, \quad p(\alpha_k) \neq 0.$$

Then there is a unique representation of the form

$$r_{n,m}(x) = q(x) + \sum_{k=1}^{m'} \sum_{j=1}^{m_k} \frac{a_{k,j}}{(x - \alpha_k)^j}, \qquad q \in \mathcal{P}_{n-m}^c, \quad a_{k,j} \in \mathbb{C}$$

(if $m > n$, then $\mathcal{P}_{n-m}^c$ is meant to be $\{0\}$).

Hint: Consider the type and dimension of expressions of the above form. □

c] Show that if

$$r_{n,m} \in \mathcal{R}_{n,m}^c,$$

then

$$\mathrm{Re}(r_{n,m}(\cdot)) \in \mathcal{R}_{n+m,2m}.$$

This is an important observation because in some problems a rational function in $\mathcal{R}_{n,n}^c$ can behave more like an element of $\mathcal{R}_{2n,2n}$ than $\mathcal{R}_{n,n}$.

E.5 Horner's Rule.

a] We have

$$\sum_{i=0}^{n} a_i x^i = (\cdots((a_n x + a_{n-1})x + a_{n-2})x + \cdots + a_1)x + a_0 .$$

So every polynomial of degree n can be evaluated by using at most n additions and n multiplications. (The converse is clearly not true; consider x^{2^n}.)

b] Show that every rational function of type $(n-1, n)$ can be put in a form so that it can be evaluated by using n divisions and n additions.

E.6 Lagrange Interpolation. Let z_i and y_i be arbitrary complex numbers except that the z_i must be distinct ($z_i \neq z_j$, for $i \neq j$). Let

$$l_k(z) := \frac{\prod_{i=0.i\neq k}^{n}(z - z_i)}{\prod_{i=0.i\neq k}^{n}(z_k - z_i)}, \qquad k = 0, 1, \ldots, n .$$

a] Show that there exists a unique $p \in \mathcal{P}_n^c$ that takes $n+1$ specified values at $n+1$ specified points, that is,

$$p(z_i) = y_i , \qquad i = 0, 1, \ldots, n .$$

This $p \in \mathcal{P}_n^c$ is of the form

$$p(z) = \sum_{k=0}^{n} y_k l_k(z)$$

and is called the *Lagrange interpolation polynomial.*

If all the z_i and y_i are real, then this unique interpolation polynomial is in $\mathcal{P}_n$.

b] Let

$$\omega(z) := \prod_{i=0}^{n} (z - z_i) .$$

Show that l_k is of the form

$$l_k(z) = \frac{\omega(z)}{(z - z_k)\omega'(z_k)}$$

and

$$p(z) = \sum_{k=0}^{n} \frac{y_k \omega(z)}{(z - z_k)\omega'(z_k)} .$$