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Random Walk:

A Modern Introduction

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Random Walk: A Modern Introduction

Random walks are stochastic processes formed by successive summation of independent, identically distributed random variables and are one of the most studied topics in probability theory. This contemporary introduction evolved from courses taught at Cornell University and the University of Chicago by the first author, who is one of the most highly regarded researchers in the field of stochastic processes. This text meets the need for a modern reference to the detailed properties of an important class of random walks on the integer lattice.

It is suitable for probabilists, mathematicians working in related fields, and for researchers in other disciplines who use random walks in modeling.

Gregory F. Lawler is Professor of Mathematics and Statistics at the University of Chicago. He received the George Pólya Prize in 2006 for his work with Oded Schramm and Wendelin Werner.

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Preface

Random walk – the stochastic process formed by successive summation of independent, identically distributed random variables – is one of the most basic and well-studied topics in probability theory. For random walks on the integer lattice $\mathbb{Z}^d$, the main reference is the classic book by Spitzer (1976). This text considers only a subset of such walks, namely those corresponding to increment distributions with zero mean and finite variance. In this case, one can summarize the main result very quickly: the central limit theorem implies that under appropriate rescaling the limiting distribution is normal, and the functional central limit theorem implies that the distribution of the corresponding path-valued process (after standard rescaling of time and space) approaches that of Brownian motion.

Researchers who work with perturbations of random walks, or with particle systems and other models that use random walks as a basic ingredient, often need more precise information on random walk behavior than that provided by the central limit theorems. In particular, it is important to understand the size of the error resulting from the approximation of random walk by Brownian motion. For this reason, there is need for more detailed analysis. This book is an introduction to the random walk theory with an emphasis on the error estimates. Although "mean zero, finite variance" assumption is both necessary and sufficient for normal convergence, one typically needs to make stronger assumptions on the increments of the walk in order to obtain good bounds on the error terms.

This project was embarked upon with an idea of writing a book on the simple, nearest neighbor random walk. Symmetric, finite range random walks gradually became the central model of the text. This class of walks, while being rich enough to require analysis by general techniques, can be studied without much additional difficulty. In addition, for some of the results, in particular, the local central limit theorem and the Green's function estimates, we have extended the discussion to include other mean zero, finite variance walks, while indicating the way in which moment conditions influence the form of the error.

The first chapter is introductory and sets up the notation. In particular, there are three main classes of irreducible walk in the integer lattice $\mathbb{Z}^d$ — $\mathcal{P}_d$ (symmetric, finite range), $\mathcal{P}'_d$ (aperiodic, mean zero, finite second moment), and $\mathcal{P}^*_d$ (aperiodic with no other assumptions). Symmetric random walks on other integer lattices such as the triangular lattice can also be considered by taking a linear transformation of the lattice onto $\mathbb{Z}^d$.

The local central limit theorem (LCLT) is the topic for Chapter 2. Its proof, like the proof of the usual central limit theorem, is done by using Fourier analysis to express the probability of interest in terms of an integral, and then estimating the integral. The error estimates depend strongly on the number of finite moments of the corresponding increment distribution. Some important corollaries are proved in Section 2.4; in particular, the fact that aperiodic random walks starting at different points can be coupled so that with probability $1 - O(n^{-1/2})$ they agree for all times greater than n is true for any aperiodic walk, without any finite moment assumptions. The chapter ends by a more classical, combinatorial derivation of LCLT for simple random walk using Stirling's formula, while again keeping track of error terms.

Brownian motion is introduced in Chapter 3. Although we would expect a typical reader to be familiar already with Brownian motion, we give the construction via the dyadic splitting method. The estimates for the modulus of continuity are also given. We then describe the Skorokhod method of coupling a random walk and a Brownian motion on the same probability space, and give error estimates. The dyadic construction of Brownian motion is also important for the dyadic coupling algorithm of Chapter 7.

Green's function and its analog in the recurrent setting, the potential kernel, are studied in Chapter 4. One of the main tools in the potential theory of random walk is the analysis of martingales derived from these functions. Sharp asymptotics at infinity for Green's function are needed to take full advantage of the martingale technique. We use the sharp LCLT estimates of Chapter 2 to obtain the Green's function estimates. We also discuss the number of finite moments needed for various error asymptotics.

Chapter 5 may seem somewhat out of place. It concerns a well-known estimate for one-dimensional walks called the gambler's ruin estimate. Our motivation for providing a complete self-contained argument is twofold. First, in order to apply this result to all one-dimensional projections of a higher dimensional walk simultaneously, it is important to show that this estimate holds for non-lattice walks uniformly in few parameters of the distribution (variance, probability of making an order 1 positive step). In addition, the argument introduces the reader to a fairly general technique for obtaining the overshoot estimates. The final two sections of this chapter concern variations of one-dimensional walk that arise naturally in the arguments for estimating probabilities of hitting (or avoiding) some special sets, for example, the half-line.

In Chapter 6, the classical potential theory of the random walk is covered in the spirit of Spitzer (1976) and Lawler (1996) (and a number of other sources). The difference equations of our discrete space setting (that in turn become matrix equations on finite sets) are analogous to the standard linear partial differential equations of (continuous) potential theory. The closed form of the solutions is important, but we emphasize here the estimates on hitting probabilities that one can obtain using them. The martingales derived from Green's function are very important in this analysis, and again special care is given to error terms. For notational ease, the discussion is restricted here to symmetric walks. In fact, most of the results of this chapter hold for nonsymmetric walks, but in this case one must distinguish between the "original" walk and the "reversed" walk, i.e. between an operator and its adjoint. An implicit exercise for a dedicated student would be to redo this entire chapter for nonsymmetric walks, changing the statements of the propositions as necessary. It would be more work to relax the finite range assumption, and the moment conditions would become a crucial component of the analysis in this general setting. Perhaps this will be a topic of some future book.

Chapter 7 discusses a tight coupling of a random walk (that has a finite exponential moment) and a Brownian motion, called the dyadic coupling or KMT or Hungarian coupling, originated in Kómlos *et al.* (1975a, b). The idea of the coupling is very natural (once explained), but hard work is needed to prove the strong error estimate. The sharp LCLT estimates from Chapter 2 are one of the key points for this analysis.

In bounded rectangles with sides parallel to the coordinate directions, the rate of convergence of simple random walk to Brownian motion is very fast. Moreover, in this case, exact expressions are available in terms of finite Fourier sums. Several of these calculations are done in Chapter 8.

Chapter 9 is different from the rest of this book. It covers an area that includes both classical combinatorial ideas and topics of current research. As has been gradually discovered by a number of researchers in various disciplines (combinatorics, probability, statistical physics) several objects inherent to a graph or network are closely related: the number of spanning trees, the determinant of the Laplacian, various measures on loops on the trees, Gaussian free field, and loop-erased walks. We give an introduction to this theory, using an approach that is focused on the (unrooted) random walk loop measure, and that uses Wilson's algorithm (1996) for generating spanning trees.

The original outline of this book put much more emphasis on the path-intersection probabilities and the loop-erased walks. The final version offers only a general introduction to some of the main ideas, in the last two chapters. On the one hand, these topics were already discussed in more detail in Lawler (1996), and on the other, discussing the more recent developments in the area would require familiarity with Schramm–Loewner evolution, and explaining this would take us too far from the main topic.

Most of the content of this text (the first eight chapters in particular) are well-known classical results. It would be very difficult, if not impossible, to give a detailed and complete list of references. In many cases, the results were obtained in several places at different occasions, as auxiliary (technical) lemmas needed for understanding some other model of interest, and were therefore not particularly noticed by the community. Attempting to give even a reasonably fair account of the development of this subject would have inhibited the conclusion of this project. The bibliography is therefore restricted to a few references that were used in the writing of this book. We refer the reader to Spitzer (1976) for an extensive bibliography on random walk, and to Lawler (1996) for some additional references.

This book is intended for researchers and graduate students alike, and a considerable number of exercises is included for their benefit. The appendix consists of various results from probability theory that are used in the first eleven chapters but are, however, not really linked to random walk behavior. It is assumed that the reader is familiar with the basics of measure-theoretic probability theory.

♣ The book contains quite a few remarks that are separated from the rest of the text by this typeface. They are intended to be helpful heuristics for the reader, but are not used in the actual arguments.

A number of people have made useful comments on various drafts of this book including students at Cornell University and the University of Chicago. We thank Christian Beneš, Juliana Freire, Michael Kozdron, José Truillijo Ferreras, Robert Masson, Robin Pemantle, Mohammad Abbas Rezaei, Nicolas de Saxcé, Joel Spencer, Rongfeng Sun, John Thacker, Brigitta Vermesi, and Xinghua Zheng. The research of Greg Lawler is supported by the National Science Foundation.

Contents

1
Introduction

1.1 Basic definitions

We will define the random walks that we consider in this book. We focus our attention on random walks in $\mathbb{Z}^d$ that have bounded symmetric increment distributions, although we occasionally discuss results for wider classes of walk. We also impose an irreducibility criterion to guarantee that all points in the lattice $\mathbb{Z}^d$ can be reached.

We start by setting some basic notation. We use x, y, z to denote points in the integer lattice $\mathbb{Z}^d = \{(x^1, \ldots, x^d) : x^j \in \mathbb{Z}\}$. We use superscripts to denote components and we use subscripts to enumerate elements. For example, $x_1, x_2, \ldots$ represents a sequence of points in $\mathbb{Z}^d$, and the point x_j can be written in component form $x_j = (x_j^1, \ldots, x_j^d)$. We write $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_d = (0, \ldots, 0, 1)$ for the standard basis of unit vectors in $\mathbb{Z}^d$. The prototypical example is (discrete time) simple random walk starting at $x \in \mathbb{Z}^d$. This process can be considered either as a sum of a sequence of independent, identically distributed random variables

$$S_n = x + X_1 + \cdots + X_n$$

where $\mathbb{P}\{X_j = \mathbf{e}_k\} = \mathbb{P}\{X_j = -\mathbf{e}_k\} = 1/(2d), k = 1, \ldots, d$, or it can be considered as a Markov chain with state space $\mathbb{Z}^d$ and transition probabilities

$$\mathbb{P}\{S_{n+1} = z \mid S_n = y\} = \frac{1}{2d}, \quad z - y \in \{\pm\mathbf{e}_1, \ldots \pm \mathbf{e}_d\}.$$

We call $V = \{x_1, \ldots, x_l\} \subset \mathbb{Z}^d \setminus \{0\}$ a *(finite) generating set* if each $y \in \mathbb{Z}^d$ can be written as $k_1 x_1 + \cdots + k_l x_l$ for some $k_1, \ldots, k_l \in \mathbb{Z}$. We let $\mathcal{G}$ denote the collection of generating sets V with the property that if $x = (x^1, \ldots, x^d) \in V$, then the first nonzero component of x is positive. An example of such a set is

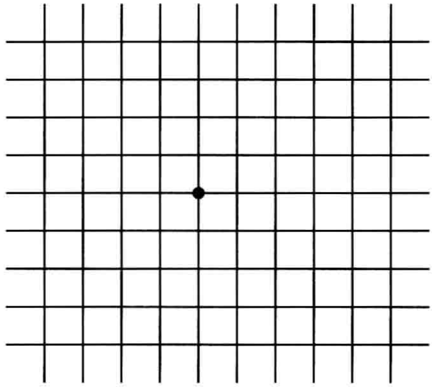

Figure 1.1 The square lattice $\mathbb{Z}^2$

$\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$. A *(finite range, symmetric, irreducible) random walk* is given by specifying a $V = \{x_1, \ldots, x_l\} \in \mathcal{G}$ and a function $\kappa : V \to (0, 1]$ with $\kappa(x_1) + \cdots + \kappa(x_l) \leq 1$. Associated to this is the symmetric probability distribution on $\mathbb{Z}^d$

$$p(x_k) = p(-x_k) = \frac{1}{2}\,\kappa(x_k), \qquad p(0) = 1 - \sum_{x \in V} \kappa(x).$$

We let $\mathcal{P}_d$ denote the set of such distributions p on $\mathbb{Z}^d$ and $\mathcal{P} = \cup_{d \geq 1} \mathcal{P}_d$. Given p, the corresponding random walk S_n can be considered as the time-homogeneous Markov chain with state space $\mathbb{Z}^d$ and transition probabilities

$$p(y, z) := \mathbb{P}\{S_{n+1} = z \mid S_n = y\} = p(z - y).$$

We can also write

$$S_n = S_0 + X_1 + \cdots + X_n$$

where $X_1, X_2, \ldots$ are independent random variables, independent of S_0, with distribution p. (Most of the time, we will choose S_0 to have a trivial distribution.) We will use the phrase $\mathcal{P}$-walk or $\mathcal{P}_d$-walk for such a random walk. We will use the term *simple random walk* for the particular p with

$$p(\mathbf{e}_j) = p(-\mathbf{e}_j) = \frac{1}{2d}, \quad j = 1, \ldots, d.$$

We call p the *increment distribution* for the walk. Given that $p \in \mathcal{P}$, we write p_n for the n-step distribution

$$p_n(x, y) = \mathbb{P}\{S_n = y \mid S_0 = x\}$$

and $p_n(x) = p_n(0, x)$. Note that $p_n(\cdot)$ is the distribution of $X_1 + \cdots + X_n$ where $X_1, \ldots, X_n$ are independent with increment distribution p.

♣ In many ways the main focus of this book is simple random walk, and a first-time reader might find it useful to consider this example throughout. We have chosen to generalize this slightly, because it does not complicate the arguments much and allows the results to be extended to other examples. One particular example is simple random walk on other regular lattices such as the planar triangular lattice. In Section 1.3, we show that walks on other d-dimensional lattices are isomorphic to p-walks on $\mathbb{Z}^d$.

If $S_n = (S_n^1, \ldots, S_n^d)$ is a $\mathcal{P}$-walk with $S_0 = 0$, then $\mathbb{P}\{S_{2n} = 0\} > 0$ for every even integer n; this follows from the easy estimate $\mathbb{P}\{S_{2n} = 0\} \geq [\mathbb{P}\{S_2 = 0\}]^n \geq p(x)^{2n}$ for every $x \in \mathbb{Z}^d$. We will call the walk *bipartite* if $p_n(0, 0) = 0$ for every odd n, and we will call it *aperiodic* otherwise. In the latter case, $p_n(0, 0) > 0$ for all n sufficiently large (in fact, for all $n \geq k$ where k is the first odd integer with $p_k(0, 0) > 0$). Simple random walk is an example of a bipartite walk since $S_n^1 + \cdots + S_n^d$ is odd for odd n and even for even n. If p is bipartite, then we can partition $\mathbb{Z}^d = (\mathbb{Z}^d)_e \cup (\mathbb{Z}^d)_o$ where $(\mathbb{Z}^d)_e$ denotes the points that can be reached from the origin in an even number of steps and $(\mathbb{Z}^d)_o$ denotes the set of points that can be reached in an odd number of steps. In algebraic language, $(\mathbb{Z}^d)_e$ is an additive subgroup of $\mathbb{Z}^d$ of index 2 and $(\mathbb{Z}^d)_o$ is the nontrivial coset. Note that if $x \in (\mathbb{Z}^d)_o$, then $(\mathbb{Z}^d)_o = x + (\mathbb{Z}^d)_e$.

♣ It would suffice and would perhaps be more convenient to restrict our attention to aperiodic walks. Results about bipartite walks can easily be deduced from them. However, since our main example, simple random walk, is bipartite, we have chosen to allow such p.

If $p \in \mathcal{P}_d$ and $j_1, \ldots, j_d$ are nonnegative integers, the $(j_1, \ldots, j_d)$ moment is given by

$$\mathbb{E}[(X_1^1)^{j_1} \cdots (X_1^d)^{j_d}] = \sum_{x \in \mathbb{Z}^d} (x^1)^{j_1} \cdots (x^d)^{j_d} \, p(x).$$

We let Γ denote the *covariance matrix*

$$\Gamma = \left[\mathbb{E}[X_1^j X_1^k] \right]_{1 \le j,k \le d}.$$

The covariance matrix is symmetric and positive definite. Since the random walk is truly d-dimensional, it is easy to verify (see Proposition 1.1.1 (a)) that the matrix Γ is invertible. There exists a symmetric positive definite matrix Λ such that $\Gamma = \Lambda\,\Lambda^T$ (see Section A.3). There is a (not unique) orthonormal basis $u_1, \ldots, u_d$ of $\mathbb{R}^d$ such that we can write

$$\Gamma x = \sum_{j=1}^{d} \sigma_j^2\,(x \cdot u_j)\,u_j, \qquad \Lambda x = \sum_{j=1}^{d} \sigma_j\,(x \cdot u_j)\,u_j.$$

If X_1 has covariance matrix $\Gamma = \Lambda\,\Lambda^T$, then the random vector $\Lambda^{-1} X_1$ has covariance matrix I.

For future use, we define norms $\mathcal{J}^*, \mathcal{J}$ by

$$\mathcal{J}^*(x)^2 = |x \cdot \Gamma^{-1} x| = |\Lambda^{-1} x|^2 = \sum_{j=1}^{d} \sigma_j^{-2}\,(x \cdot u_j)^2, \qquad \mathcal{J}(x) = d^{-1/2}\,\mathcal{J}^*(x). \tag{1.1}$$

If $p \in \mathcal{P}_d$,

$$\mathbb{E}[\mathcal{J}(X_1)^2] = \frac{1}{d}\,\mathbb{E}[\mathcal{J}^*(X_1)^2] = \frac{1}{d}\,\mathbb{E}\left[|\Lambda^{-1} X_1|^2\right] = 1.$$

For simple random walk in $\mathbb{Z}^d$,

$$\Gamma = d^{-1}\,I, \qquad \mathcal{J}^*(x) = d^{1/2}\,|x|, \qquad \mathcal{J}(x) = |x|.$$

We will use $\mathcal{B}_n$ to denote the discrete ball of radius n,

$$\mathcal{B}_n = \{x \in \mathbb{Z}^d : |x| < n\},$$

and $\mathcal{C}_n$ to denote the discrete ball under the norm $\mathcal{J}$,

$$\mathcal{C}_n = \{x \in \mathbb{Z}^d : \mathcal{J}(x) < n\} = \{x \in \mathbb{Z}^d : \mathcal{J}^*(x) < d^{1/2}\,n\}.$$

We choose to use $\mathcal{J}$ in the definition of $\mathcal{C}_n$ so that for simple random walk, $\mathcal{C}_n = \mathcal{B}_n$. We will write $R = R_p = \max\{|x| : p(x) > 0\}$ and we will call R the