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# Quantum Kinematics and Dynamics

JULIAN SCHWINGER

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# QUANTUM KINEMATICS AND DYNAMICS

JULIAN SCHWINGER

*Harvard University*



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# Foreword

Early in 1955 I began to write an article on the Quantum Theory of Fields. The introduction contained this description of its plan. "In part A of this article a general scheme of quantum kinematics and dynamics is developed within the nonrelativistic framework appropriate to systems with a finite number of dynamical varibles. Apart from specific physical consequences of the relativistic invariance requirement, the extension to fields in part B introduces relatively little that is novel, which permits the major mathematical features of the theory of fields to be discussed in the context of more elementary physical systems."

A preliminary and incomplete version of part A was used as the basis of lectures delivered in July, 1955 at the Les Houches Summer School of Theoretical Physics. Work on part A ceased later that year and part B was never begun. Several years after, I used some of the material in a series of notes published in the Proceedings of the National Academy of Sciences. And there the matter rested until, quite recently, Robert Kohler (State University College at Buffalo) reminded me of the continuing utility of the Les Houches notes and suggested their publication. He also volunteered to assist in this process. Here is the result. The main text is the original and still incomplete 1955 manuscript, modified only by the addition of subheadings. To it is appended excerpts from the Proc. Nat. Acad. of Sciences articles that supplement the text, together with two papers that illustrate and further develop its methods.

Belmont, Massachusetts
1969

Julian Schwinger

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# CHAPTER ONE

# THE ALGEBRA OF MEASUREMENT


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The classical theory of measurement is built upon the conception of an interaction between the system of interest and the measuring apparatus that can be made arbitrarily small, or at least precisely compensated, so that one can speak meaningfully of an idealized measurement that disturbs no property of the system. But it is characteristic of atomic phenomena that the interaction between system and instrument is not arbitarily small. Nor can the disturbance produced by the interaction be compensated precisely since to some extent it is uncontrollable and unpredictable. Accordingly, a measurement on one property can produce unavoidable changes in the value previously assigned to another property, and it is without meaning to speak of a microscopic system possessing precise values for all its attributes. This contradicts the classical representation of all physical quantities by numbers. The laws of atomic physics must be expressed, therefore, in a nonclassical mathematical language that constitutes a symbolic expression of the properties of microscopic measurement.

## 1.1 MEASUREMENT SYMBOLS

We shall develop the outlines of this mathematical structure by discussing simplified physical systems which are such that any physical quantity $A$ assumes only a finite number of distinct values, $a',..a^n$. In the most elementary type of measurement, an ensemble of independent similar systems is sorted by the apparatus into subensembles, distinguished by definite values of the physical quantity being measured. Let $M(a')$ symbolize the selective measurement that accepts systems possessing the value $a'$ of property $A$ and rejects all others. We define the addition of such symbols to signify less specific selective measurements that produce a subensemble associated with any of the values in the summation, none of these being distinguished by the measurement.

The multiplication of the measurement symbols represents the successive performance of measurements (read from right to left). It follows from the physical meaning of these operations that addition is commutative and associative, while multiplication is associative. With 1 and 0 symbolizing the measurements that, respectively, accept and reject all systems, the properties of the elementary selective measurements are expressed by

$$M(a')M(a') = M(a') \tag{1.1}$$

$$M(a')M(a'') = 0 \quad , \quad a' \neq a'' \tag{1.2}$$

$$\sum_{a'} M(a') = 1 \; . \tag{1.3}$$

Indeed, the measurement symbolized by $M(a')$ accepts every system produced by $M(a')$ and rejects every system produced by $M(a'')$, $a'' \neq a'$, while a selective measurement that does not distinguish any of the possible values of $a'$ is the measurement that accepts all systems.

According to the significance of the measurements denoted as 1 and 0, these symbols have the algebraic properties

$$\begin{aligned} 1\,1 &= 1 \quad , \quad 0\,0 = 0 \\ 1\,0 &= 0\,1 = 0 \\ 1 + 0 &= 1 \; , \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} 1M(a') &= M(a')1 = M(a') \; , \\ 0M(a') &= M(a')0 = 0 \\ M(a') + 0 &= M(a') \; , \end{aligned} \tag{1.5}$$

which justifies the notation. The various properties of 0, $M(a')$ and 1 are consistent, provided multiplication is distributive. Thus,

$$\sum_{a''} M(a')M(a'') = M(a') = M(a')1$$
$$= M(a')\sum_{a''} M(a'') . \qquad (1.6)$$

The introduction of the numbers 1 and 0 as multipliers, with evident definitions, permits the multiplication laws of measurement symbols to be combined in the single statement

$$M(a')M(a'') = \delta(a',a'')M(a') , \qquad (1.7)$$

where

$$\delta(a',a'') = \begin{cases} 1 , & a' = a'' \\ 0 , & a' \neq a'' . \end{cases} \qquad (1.8)$$

## 1.2 COMPATIBLE PROPERTIES. DEFINITION OF STATE

Two physical quantities $A_1$ and $A_2$ are said to be compatible when the measurement of one does not destroy the knowledge gained by prior measurement of the other. The selective measurements $M(a_1')$ and $M(a_2')$ , performed in either order, produce an ensemble of systems for which one can simultaneously assign the values $a_1'$ to $A_1$ and $a_2'$ to $A_2$ . The symbol of this compound measurement is

$$M(a_1'a_2') = M(a_1')M(a_2') = M(a_2')M(a_1') \quad . \qquad (1.9)$$

By a complete set of compatible physical quantities, $A_1, \ldots A_k$ , we mean that every pair of these quantities is compatible and that no other quantities exist, apart from functions of the set $A$ , that are compatible with every member of this set. The measurement symbol

$$M(a') = \prod_{i=1}^{k} M(a_i') \qquad (1.10)$$

then describes a complete measurement, which is such that the systems chosen possess definite values for the maximum number of attributes; any attempt to determine the value of still another independent physical quantity will produce uncontrollable changes in one or more of the previously assigned values. Thus the optimum state of knowledge concerning a given system is realized by subjecting it to a complete selective measurement. The systems admitted by the complete measurement $M(a')$ are said to be in the state $a'$ . The symbolic properties of complete measurements are also given by (1.1), (1.2) and (1.3).

## 1.3 MEASUREMENTS THAT CHANGE THE STATE

A more general type of measurement incorporates a disturbance that produces a change of state. The symbol $M(a', a'')$ indicates a selective measurement in which systems are accepted only in the state $a''$ and emerge in the state $a'$. The measurement process $M(a')$ is the special case for which no change of state occurs,

$$M(a') = M(a', a') \quad . \tag{1.11}$$

The properties of successive measurements of the type $M(a', a'')$ are symbolized by

$$M(a', a'')M(a''', a^{iv}) = \delta(a'', a''')\, M(a', a^{iv}) \quad , \tag{1.12}$$

for, if $a'' \neq a'''$, the second stage of the compound apparatus accepts none of the systems that emerge from the first stage, while if $a'' = a'''$, all such systems enter the second stage and the compound measurement serves to select systems in the state $a^{iv}$ and produce them in the state $a'$. Note that if the two stages are reversed, we have