DIFFERENTIAL EQUATIONS

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Differential Equations

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Preface

This book started in 1993, when the first author began to reorganize the teaching of ODEs at Rice University. It soon became apparent that a textbook was needed that brought to the students the expanded outlook that modern developments in the subject required, and the use of technology allowed. Over the ensuing years this book has evolved.

The mathematical subject matter of this book has not changed dramatically from that of many books published ten or even twenty years ago. The book strikes a balance between the traditional and the modern. It covers all of the traditional material and somewhat more. It does so in a way that makes it easily possible, but not necessary, to use modern technology, especially for the visualization of the ideas involved in ordinary differential equations. It offers flexibility of use that will allow instructors at a variety of institutions to use the book. In fact, this book could easily be used in a traditional differential equations course, provided the instructor carefully chooses the exercises assigned. However, there are changes in our students, in our world, and in our mathematics that require some changes in the ODE course, and the way we teach it.

Our students are now as likely to be majoring in the biological sciences or economics as in the physical sciences or engineering. These students are more interested in systems of equations than they are in second order equations. They are also more interested in applications to their own areas rather than to physics or engineering.

Our world is increasingly a technological world. In academia we are struggling with the problem of adapting to this new world. The easiest way to start a spirited discussion in a group of faculty is to raise the subject of the use of technology in our teaching. Regardless of one's position on this subject, it is widely agreed that the course where the use of technology makes the most sense, and where the impact of computer visualization is the most beneficial, is in the study of ODEs. The use of computer visualization pervades this book. The degree to which the student and the instructor are involved is up to the instructor.

The subject of ordinary differential equations has progressed, as has all of mathematics. To many it is now known by the new name, dynamical systems. Much of the progress, and many of the directions in which the research has gone, have been motivated by computer experiments. Much of the work is qualitative in nature. This is beautiful mathematics. Introducing some of these ideas to students at an early point is a move in the right direction. It gives them a better idea of what mathematics is about than the standard way of discussing one solution method after another. It should be emphasized that the introduction of qualitative methods is not, in itself, a move to less rigor.

The Use of Technology

The book covers the standard material with an appropriate level of rigor. However, it enables the instructor to be flexible in the use of modern technology. Available to all, without the use of any technology, is the large number of graphics in the book that display the ideas in ODEs. At the next level are a large number of exercises that require the student to compute and plot solutions. For these exercises, the student will have to have access to computer (or calculator) programs that will do this easily.

The tools needed for most of these exercises are two. The student will need a program that will plot the direction field for a single differential equation, and superimpose the solution with given initial conditions. In addition, the student will need a program that will plot the vector field for an autonomous planar system of equations, and superimpose the solution with given initial conditions. Such tools are available in MATLAB, Maple, and Mathematica. For many purposes it will be useful for the students to have computer (or calculator) tools for graphing functions of a single variable.

The book can also be used to teach a course in which the students learn numerical methods early and are required to use them regularly throughout the course. Students in such a course learn the valuable skill of solving equations and systems of equations numerically and interpreting the results using the subject matter of the course. The treatment of numerical methods is somewhat more substantial than in other books. However, just enough is covered so that readers get a sense of the complexity involved. Computational error is treated, but not so rigorously as to bog the reader down and interrupt the flow of the text. Students are encouraged to do some experimental analysis of computational error.

Modeling and Applications

It is becoming a common feature of mathematics books to include a large list of applications. Usually the students are presented with the mathematical model and they are required to apply it to a variety of cases. The derivation of the model is not done. There is some sense in this. After all, mathematics does not include all of the many application areas, and the derivation of the models is the subject of the application areas. Furthermore, the derivations are very time consuming.

However, mathematicians and mathematics are part of the modeling process. It should be a greater part of our teaching. This book takes a novel approach to the teaching of modeling. While a large number of applications are covered as examples, in some cases the applications are covered in more detail than is usual. There is a historical study of the models of motion, which demonstrates to students how models continue to evolve as knowledge increases. There is an in-depth study of several population models, including their derivation. Included are historical examples of how such models were applied both where they were appropriate and where they were not. This demonstrates to students that it is necessary to understand the assumptions that lie behind a model before using them, and that any model must be checked by experiments or observations before it is accepted.

In addition, models in personal finance are discussed. This is an area of potential interest to all students, but not one that is covered in any detail in college courses. Students majoring in almost all disciplines approach these problems on an even footing. As a result it is an area where students can be required to do some modeling on their own.

Linear Algebra and Systems

Most books at this level assume that students have an understanding of elementary matrix algebra, usually in two and three dimensions. In the experience of the authors this assumption is not valid. Accordingly, this book devotes a chapter to matrix algebra. The topics covered are carefully chosen to be those needed in the study of linear systems of ODEs. With this chapter behind them, the instructor can cover linear systems of ODEs in a more substantive way. On the other hand an instructor who is confident in the knowledge of the students can skip the matrix algebra chapter.

Projects

There are a number of projects discussed in the book. These involve students in an in-depth study of either mathematics or an application that uses ODEs. The projects provide students with the opportunity to bring together much of what they have learned, including analytical, computational, and interpretative skills. The level of difficulty of the projects varies. More projects will be made available to users of this book as they are developed.

Varied Approaches Possible

It should be noticed that the book has three authors from three very different schools. The ODE courses at these institutions are quite different. Indeed, there is no standard ODE course across the country. The authors set the understandable goal of writing a book that could be used in the ODE courses at each of their own institutions. Meeting this goal required some compromises, but the result is a book that is flexible enough to allow its use in a variety of courses at a variety of institutions.

On one hand, it is possible to use the book and teach a more or less standard course. The standard material is covered in the standard order, with or without the use of technology.

However, at Rice University, after the first three chapters the class moves to numerical methods, and then to matrix algebra. This is followed by linear systems. Once this material is covered, higher-order equations, including the second-order equations that are important in science and engineering, are covered as examples of systems. This approach allows the students to use linear algebra throughout the course, thereby gaining a working knowledge of the subject. Technology is used throughout to enhance the students' understanding of the mathematical ideas.

In another approach, used at College of the Redwoods, the chapter on numerical methods is done early, while discussing the qualitative analysis of single first-order equations. The students are taught the analytical, qualitative, and numerical approaches before moving on to Chapter 3. The chapter on matrix algebra is covered next. There follows an introduction of systems, both linear and nonlinear. Next, they return to second-order equations, including undetermined coefficients, the driven, damped oscillator, resonance, and so forth. The course ends with more on nonlinear systems. The ultimate goal is to get the students to use nullcline analysis and the Jacobian approximation to sketch a phase portrait without the use of software.

Mathematical Rigor

Mathematical ideas are not dodged. Proofs are given when the proof will add to the students' understanding of the material. Difficult proofs, or those that do not add to a student's understanding, are avoided. Suggestions of how to proceed, and examples that use these suggestions, are usually offered as motivation before one has to wade through the abstraction of a proof. The authors believe that proof is fundamental to mathematics, and that students at this level should be introduced gently to proof as an integral part of their training in mathematics. This is true for the future engineer or doctor as well as for the math major.

Supplements

Instructors who use this book will have available a number of resources. There are an Instructor's Solution Manual, containing the complete solutions to all of the exercises, and a Student's Solution Manual with the solutions to the odd-numbered exercises.

One way to meet the software needs of the student is to use the programs `dfield` and `pplane`, written by the first author for use with MATLAB . These programs are described in the book *Ordinary Differential Equations Using* MATLAB (ISBN 0-13-011381-6), written by two of the authors of this book. That book is available shrink-wrapped with this one at no extra cost (ISBN 0-13-059318-4). However, it should be emphasized that it is not necessary to use `dfield` and `pplane` with this book. There are many other possibilities.

The Website http://www.prenhall.com/polking is a resource that will ultimately become very valuable to both instructors and students. Interactive java versions of the direction field program `dfield` and the phase plane program `pplane` will be accessible from this site. It will also provide animations of the examples in the book, links to other web resources involving differential equations, and true-false quizzes on the subject matter. As additional projects are developed for use with the book, they will be accessible from the Website.

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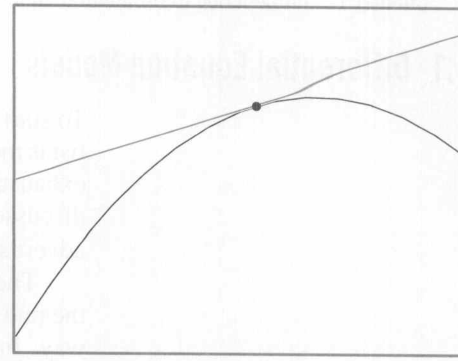

Introduction to Differential Equations

With the systematic study of differential equations, the calculus of functions of a single variable reaches a state of completion. Modeling by differential equations greatly expands the list of possible applications. The list continues to grow as we discover more differential equation models in old and in new areas of application. The use of differential equations makes available to us the full power of the calculus.

When explicit solutions to differential equations are available, they can be used to predict a variety of phenomena. Whether explicit solutions are available or not, we can usually compute useful and very accurate approximate numerical solutions. The use of modern computer technology makes possible the visualization of the results. Furthermore, we continue to discover ways to analyze solutions without knowing the solutions explicitly.

The subject of differential equations is solving problems and making predictions. In this book, we will exhibit many examples of this—in physics, chemistry, and biology, and also in such areas as personal finance and forensics. This is the process of mathematical modeling. If it were not true that differential equations were so useful, we would not be studying them, so we will spend a lot of time on the modeling process and with specific models. In the first section of this chapter we will present some examples of the use of differential equations.

The study of differential equations, and their application, uses the derivative and the integral, the concepts that make up the calculus. We will review these ideas starting in Sections 1.2 and 1.3.

1.1 Differential Equation Models

To start our study of differential equations, we will give a number of examples. This list is meant to be indicative of the many applications of the topic. It is far from being exhaustive. In each case, our discussion will be brief. Most of the examples will be discussed later in the book in greater detail. This section should be considered as advertising for what will be done in the rest of the book.

The theme that you will see in the examples is that in every case we compute the rate of change of a variable in two different ways. First there is the mathematical way. In mathematics, the rate at which a quantity changes is the derivative of that quantity. This is the same for each example. The second way of computing the rate of change comes from the application itself and is different from one application to another. When these two ways of expressing the rate of change are equated, we get a differential equation, the subject we will be studying.

Mechanics

Isaac Newton was responsible for a large number of discoveries in physics and mathematics, but perhaps the three most important are the following:

- The systematic development of the calculus. Newton's achievement was the realization and utilization of the fact that integration and differentiation are operations inverse to each other.
- The discovery of the laws of mechanics. Principal among these was Newton's second law, which says that force is equal to the rate of change of momentum with respect to time. Momentum is defined to be the product of mass and velocity, or mv. Thus the force is equal to the derivative of the momentum. If the mass is constant,

$$\frac{d}{dt}mv = m\frac{dv}{dt} = ma,$$

where a is the acceleration. Newton's second law says that the rate of change of momentum is equal to the force F. Expressing the equality of these two ways of looking at the rate of change, we get the equation

$$F = ma, \tag{1.1}$$

the standard expression for Newton's second law.
- The discovery of the universal law of gravitation. This law says that any body with mass M attracts any other body with mass m directly toward the mass M, with a magnitude proportional to the product of the two masses and inversely proportional to the square of the distance separating them. This means that there is a constant G, which is universal, such that the magnitude of the force is

$$\frac{GMm}{r^2}, \tag{1.2}$$

where r is the distance between the two bodies.

All of these discoveries were made in the period between 1665 and 1671. The discoveries were presented originally in Newton's *Philosophiae Naturalis Principia Mathematica*, better known as *Principia Mathematica*, published in 1687.

Newton's development of the calculus is what makes the theory and use of differential equations possible. His laws of mechanics create a template for a model for motion in almost complete generality. It is necessary in each case to figure out what forces are acting on a body. His law of gravitation does just that in one very important case.

The simplest example is the motion of a ball thrown into the air near the surface of the earth. If x measures the distance the ball is above the earth, then the velocity and acceleration of the ball are

$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

Since the ball is assumed to move only a short distance in comparison to the radius of the earth, the force given by (1.2) may be assumed to be constant. Notice that m, the mass of the ball, occurs in (1.2). We can write the force as $F = -mg$, where $g = GM/r^2$ and r is the radius of the earth. The constant g is called the earth's acceleration due to gravity. The minus sign reflects the fact that the displacement x is measured positively above the surface of the earth, and the force of gravity tends to decrease x. Newton's second law, (1.1), becomes

$$-mg = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}.$$

The masses cancel, and we get the differential equation

$$\frac{d^2x}{dt^2} = -g, \tag{1.3}$$

which is our mathematical model for the motion of the ball.

The equation in (1.3) is called a differential equation because it involves an unknown function $x(t)$ and at least one of its derivatives. In this case the highest derivative occurring is the second order, so this is called a differential equation of second order.

A more interesting example of the application of Newton's ideas has to do with planetary motion. For this case, we will assume that the sun with mass M is fixed and put the origin of our coordinate system at the center of the sun. We will denote by $\mathbf{x}(t)$ the vector that gives the location of a planet relative to the sun. The vector $\mathbf{x}(t)$ has three components. Its derivative is

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt},$$

which is the vector valued velocity of the planet. For this example, Newton's second law and his law of gravitation become

$$m\frac{d^2\mathbf{x}}{dt^2} = -\frac{GMm}{|\mathbf{x}|^2}\frac{\mathbf{x}}{|\mathbf{x}|}.$$

This system of three second-order differential equations is Newton's model of planetary motion. Newton solved these and verified that the three laws observed by Kepler follow from his model.

Population models

Consider a population $P(t)$ that is varying with time.[1] A mathematician will say that the rate at which the population is changing with respect to time is given by the derivative

$$\frac{dP}{dt}.$$

However, the population biologist will say that the rate of change is roughly proportional to the population. This means that there is a constant r, called the reproductive rate, such that the rate of change is equal to rP. Putting together the ideas of the mathematician and the biologist, we get the equation

$$\frac{dP}{dt} = rP. \tag{1.4}$$

This is an equation for the function $P(t)$. It involves both P and its derivative, so it is a differential equation. It is not difficult to show by direct substitution into (1.4) that the exponential function

$$P(t) = P_0 e^{rt}$$

is a solution. Thus, assuming that the reproductive rate r is positive, our population will grow exponentially.

If at this point you go back to the biologist he or she will undoubtedly say that the reproductive rate is not really a constant. While that assumption works for small populations, over the long term you have to take into account the fact that resources of food and space are limited. When you do, a better model for the the reproductive rate is the function $r(1 - P/K)$, and then the rate at which the population changes is better modeled by $r(1 - P/K)P$. Here both r and K are constants.

When we equate our two ideas about the rate at which the population changes, we get the equation

$$\frac{dP}{dt} = r(1 - P/K)P. \tag{1.5}$$

This differential equation for the function $P(t)$ is called the logistic equation. It is much harder to solve than (1.4), but it does a creditable job of predicting how single populations grow in isolated circumstances.

Pollution

Consider a lake that has a volume of $V = 100\ \text{km}^3$. It is fed by an input river, and there is another river which is fed by the lake at a rate that keeps the volume of the lake constant. The flow of the input river varies with the season, and assuming that $t = 0$ corresponds to January 1 of the first year of the study, the input rate is

$$r(t) = 50 + 20\cos(2\pi(t - 1/4)).$$

Notice that we are measuring time in years. Thus the maximum flow into the lake occurs when $t = 1/4$, or at the beginning of April.

[1] For the time being, the population can be anything—humans, paramecia, butterflies, etc. We will be more careful later.

In addition, there is a factory on the lake that introduces a pollutant into the lake at the rate of 2 km^3/year. Let $x(t)$ denote the total amount of pollution in the lake at time t. If we make the assumption that the pollutant is rapidly mixed throughout the lake, then we can show that $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = 2 - \big(52 + 20\cos(2\pi(t - 1/4))\big)\frac{x}{100}.$$

This equation can be solved and we can then answer questions about how dangerous the pollution problem really is. For example, if we know that a concentration of less than 2% is safe, will there there be a problem? The solution will tell us.

The assumption that the pollutant is rapidly mixed into the lake is not very realistic. We know that this does not happen, especially in this situation, where there is a flow of water through the lake. This assumption can be removed, but to do so, we need to allow the concentration of the pollutant to vary with position in the lake as well as with time. Thus the concentration is a function $c(t, x, y, z)$, where (x, y, z) represents a position in the three-dimensional lake. Instead of assuming perfect mixing, we will assume that the pollutant diffuses through water at a certain rate.

Once again we can construct a mathematical model. Again it will be a differential equation, but now it will involve partial derivatives with respect to the spatial coordinates x, y, and z, as well as the time t.

Personal finance

How much does a person need to save during his or her work life in order to be sure of a retirement without money worries? How much is it necessary to save each year in order to accumulate these assets? Suppose one's salary increases over time. What percent of one's salary should be saved to reach one's retirement goal?

All of these questions, and many more like them, can be modeled using differential equations. Then, assuming particular values for important parameters like return on investment and rate of increase of one's salary, answers can be found.

Other examples

We have given four examples. We could have given a hundred more. We could talk about electrical circuits, the behavior of musical instruments, the shortest paths on a complicated-looking surface, finding a family of curves that are orthogonal to a given family, discovering how two coexisting species interact, and many others.

All of these examples use ordinary differential equations. The applications of partial differential equations go much farther. We can include electricity and magnetism; quantum chromodynamics, which unifies electricity and magnetism with the weak and strong nuclear forces; the flow of heat; oscillations of many kinds, such as vibrating strings; the fair pricing of stock options; and many more.

The use of differential equations provides a way to reduce many areas of application to mathematical analysis. In this book, we will learn how to do the modeling and how to use the models after we make them.

EXERCISES

The phrase "y is proportional to x" implies that y is related to x via the equation $y = kx$, where k is a constant. In a similar manner, "y is proportional to the square of x" implies $y = kx^2$, "y is proportional to the product of x and z" implies $y = kxz$, and "y is inversely proportional to the cube of x" implies $y = k/x^3$. For example, when Newton proposed that the force of attraction of one body on another is proportional to the product of the masses and inversely proportional to the square of the distance between them, we can immediately write

$$F = \frac{GMm}{r^2},$$

where G is the constant of proportionality, usually known as the universal gravitational constant. In Exercises 1–10, use these ideas to model each application with a differential equation. All rates are assumed to be with respect to time.

1. The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.
2. The rate of growth of a population of field mice is inversely proportional to the square root of the population.
3. A certain area can sustain a maximum population of 100 ferrets. The rate of growth of a population of ferrets in this area is proportional to the product of the population and the difference between the actual population and the maximum sustainable population.
4. The rate of decay of a given radioactive substance is proportional to the amount of substance remaining.
5. The rate of decay of a certain substance is inversely proportional to the amount of substance remaining.
6. A potato that has been cooking for some time is removed from a heated oven. The room temperature of the kitchen is 65°F. The rate at which the potato cools is proportional to the difference between the room temperature and the temperature of the potato.
7. A thermometer is placed in a glass of ice water and allowed to cool for an extended period of time. The thermometer is removed from the ice water and placed in a room having temperature 77°F. The rate at which the thermometer warms is proportional to the difference in the room temperature and the temperature of the thermometer.
8. A particle moves along the x-axis, its position from the origin at time t given by $x(t)$. A single force acts on the particle that is proportional to but opposite the object's displacement. Use Newton's law to derive a differential equation for the object's motion.
9. Use Newton's law to develop the equation of motion for the particle in Exercise 8 if the force is proportional to but opposite the square of the particle's velocity.
10. Use Newton's law to develop the equation of motion for the particle in Exercise 8 if the force is inversely proportional to but opposite the square of the particle's displacement from the origin.