Introduction to Stochastic Calculus with Applications

随机分析应用导论

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Preface

This book aims at providing a concise presentation of Stochastic Calculus with some of its applications in Finance, Engineering and Science. The diversity of applications are reflected in the range of methods covered in the text.

During the past twenty years, there has been an increasing demand for tools and methods of Stochastic Calculus in various disciplines. One of the greatest demands appears to have come from the growing area of Mathematical Finance where Stochastic Calculus is used for pricing and hedging of financial derivatives, such as options. In Engineering, most popular applications of Stochastic Calculus are in filtering and control theory. In Physics, Stochastic Calculus is used to study the effects of random excitations on various physical phenomena. In Biology, Stochastic Calculus is used to model the effects of stochastic variability in reproduction and environment on populations.

From an applied perspective Stochastic Calculus can be loosely described as a field of Mathematics that is concerned with infinitesimal calculus on non-differentiable functions. The need for this calculus comes from the necessity to include unpredictable factors into modelling. This is where probability comes in, and the result is a calculus for random functions or stochastic processes. As such, most ideas and techniques in Stochastic Calculus come from the Stochastic Processes theory.

This is a mathematical text which presents a highly technical area, yet only a basic knowledge of calculus and probability is required for using the book. This text is aimed at taking the reader from a fairly low technical level to a sophisticated one with a reasonable gradient. This is achieved by making use of many solved examples and by giving simple heuristic ideas before precise results are stated and proved. I tried to make the presentation as simple as possible while keeping it mathematically correct. Oversimplification in this field tends to lead to mistakes. Simple proofs are usually presented in order to exercise the new tools and techniques. By avoiding some of the more technical proofs or their details (for which references to texts are given), the reader arrives at advanced results much sooner. These results are then used for applications.

This text presumes less initial knowledge than most texts on the subject (Métivier (1982), Dellacherie and Meyer (1982), Protter (1992), Liptser and Shiryayev (1989), Jacod and Shiryayev (1987), Karatzas and Shreve (1988), Stroock and Varadhan (1979), Revuz and Yor (1991), Rogers and Williams (1990)) however, it still presents a fairly complete and mathematically rigorous treatment of Stochastic Calculus for continuous and discrete processes.

Abstract theory becomes alive in applications. Completely unrelated applied problems have their solutions rooted in the same mathematical result, for example, the problem of pricing of an option in Finance and the problem of optimal filtering in Engineering, both rely on the martingale representation property of Brownian motion. The reader interested mainly in applications may start from applications and consult theoretical results as the need arises, for example, a reader interested in Finance, may start from the chapter on Finance (chapter 11).

A brief description of the contents follows. The first chapter presents some key results of Calculus in order to have some continuity of ideas and results readily available for later use. Basic concepts of Probability Theory are given in Chapter 2. Some results of this chapter may be skipped and referred to later. In Chapter 3, two main stochastic processes used in Stochastic Calculus are given: Brownian motion (for calculus of continuous processes) and Poisson process (for calculus of processes with jumps). Integration with respect to Brownian motion and closely related processes (Itô processes) is introduced in Chapter 4. It allows one to define a stochastic differential equation. Such equations arise in applications when a random noise is introduced into ordinary differential equations. Stochastic differential equations are treated in Chapter 5. Diffusion processes arise as solutions to stochastic differential equations, they are presented in Chapter 6. As the name suggests, diffusions describe a real physical phenomenon, and are met in many real life applications. Chapter 7 contains information about martingales, examples of which are provided by Itô processes and compensated Poisson processes, introduced in earlier chapters. Main tools of stochastic calculus include optional stopping, localization and martingale representations. These are abstract concepts, but they arise in applied problems where their use is demonstrated. Chapter 8 gives a brief account of calculus for most general processes called semimartingales. Basic results include Itô's formula and stochastic exponential. The reader have already met these concepts in Brownian motion calculus given in Chapter 4. Chapter 9 treats Pure Jump processes, where they are analyzed by using compensators. The change of measure is given in Chapter 10. This topic is important in mathematical Finance (for the problem of option pricing), and in inference for stochastic processes (for finding likelihoods). Chapters 11, 12 and 13 are devoted to applications. Chapter 11 gives an application to mathematical Finance, where the problem of pricing of financial derivatives (including exotic and interest rates derivatives) is presented and solved. Applications in Biology are given in Chapter 12. They include the branching diffusion model, Birth-Death processes with stabilizing rates, and diffusion models arising in Population Genetics. The filtering problem is presented as an application in Engineering. Random perturbations to two-dimensional differential equations are given as an application in Physics. These are covered in Chapter 13. Exercises are placed at the end of each chapter.

This text can be used for a variety of courses in stochastic calculus. The application to Finance is extensive enough to use it for a course in mathematical Finance and for self studies. This text is suitable for advanced undergraduate students, graduate students as well as research workers and practioners.

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Contents

Chapter 1

Preliminaries From Calculus

In this chapter some basic concepts of the infinitesimal calculus needed for further use are surveyed. The concept of variation of a function is central for stochastic calculus. Some more advanced results from the theory of functions are also given, so that the reader is not too surprised when similar results in stochastic calculus are encountered.

1.1 Continuous and Differentiable Functions

A function g is called continuous at the point $t = t_0$ if the increment of g over small intervals is small,

$$\Delta g = g(t) - g(t_0) \to 0 \text{ as } \Delta t = t - t_0 \to 0$$

If g is continuous at every point of its domain of definition, it is simply called continuous.

g is called differentiable at the point $t = t_0$ if at that point

$$\Delta g \sim C\Delta t \quad \text{or} \quad \lim_{\Delta t \to 0} \frac{\Delta g(t)}{\Delta t} = C,$$

this constant C is denoted by $g'(t_0)$. If g is differentiable at every point of its domain, it is called differentiable.

An important application of the derivative is a theorem on finite increments.

Theorem 1.1 (Mean Value Theorem) *If f is continuous on $[a,b]$ and has a derivative on (a,b), then there is c, $a<c<b$, such that*

$$f(b)-f(a)=f'(c)(b-a). \tag{1.1}$$

Clearly, differentiability implies continuity, but not the other way around, as continuity states that the increment Δg converges to zero together with $\Delta t \to 0$, whereas differentiability states that this convergence is at the same rate or faster.

Example 1.1: The function $g(t)=\sqrt{t}$ is not differentiable at 0, as at this point

$$\frac{\Delta g}{\Delta t}=\frac{\sqrt{\Delta t}}{\Delta t}=\frac{1}{\sqrt{\Delta t}}\to\infty$$

as $t \to 0$.

It is surprisingly difficult to construct an example of a continuous function which is not differentiable at *any* point.

Example 1.2: Example of a continuous but not differentiable at any point function.

$$f(x)=\sum_{n=0}^{\infty}\frac{\sin(3^n x)}{2^n}. \tag{1.2}$$

We don't give a proof of these properties, justification for continuity is given by the fact that if a sequence of continuous functions converges uniformly, then the limit is continuous; and a justification for non differentiability can be provided in some sense by differentiating term by term, which results in a divergent series.

To save repetition the following notations are used: *a continuous function f is said to be a C function; a differentiable function f with continuous derivative is said to be a C^1 function; a twice differentiable function f with continuous second derivative is said to be a C^2 function; etc.*

The same information is conveyed more formally by using classes of functions. C^m denotes the class of functions which are m times differentiable with continuous m-th derivative. Thus $f \in C^2$ means that the second derivative f'' exists and is continuous.

1.2 Right and Left-Continuous Functions

We can rephrase the definition of a continuous function: a function g is called continuous at the point $t=t_0$ if

$$\lim_{t\to t_0} g(t)=g(t_0). \tag{1.3}$$

In this definition it is not important how t approaches t_0, the limit is the same.

A function g is called right-continuous at t_0 if the values of the function approach $g(t_0)$ when t approaches t_0 from the right, that is, staying larger than t_0, $t > t_0$,

$$\lim_{t \downarrow t_0} g(t) = g(t_0). \tag{1.4}$$

A function g is called left-continuous at t_0 if the values of the function approach $g(t_0)$ when t approaches t_0 from the left, that is, staying smaller than t_0, $t < t_0$,

$$\lim_{t \uparrow t_0} g(t) = g(t_0). \tag{1.5}$$

If g is continuous, it is, clearly both right and left-continuous. As an exercise draw a graph of a right-continuous and a left-continuous functions.

The left-continuous version of g is $g(t-)$, which is the limit of values of $g(t)$ when $s \uparrow t$, that is, $s < t$ and $s \to t$,

$$g(t-) = \lim_{s \uparrow t} g(s). \tag{1.6}$$

From the definitions we have: g is left-continuous if $g(t) = g(t-)$.

The concept of $g(t+)$ is defined similarly,

$$g(t+) = \lim_{s \downarrow t} g(s). \tag{1.7}$$

If g is a right-continuous function then $g(t+) = g(t)$ for any t, so that $g_+ = g$.

Definition 1.2 *A point t is called a discontinuity of the first kind or a jump point if both limits $g(t+)$ and $g(t-)$ exist. The jump at t is defined as $\Delta g(t) = g(t+) - g(t-)$. Any other discontinuity is said to be of the second kind.*

An important result is that a function can have at most countably many jump discontinuities, see for example Hobson (1921), p.286.

Theorem 1.3 *A function defined on an interval $[a, b]$ can have no more than countably many jumps.*

PROOF: For an arbitrary $h > 0$ consider the set S_h of all the points at which the size of the jump of g, $|\Delta g(t)| = |g(t+) - g(t-)| \geq h$. If this set is not finite, then it has an accumulation point. It is easy to see that this point is not a discontinuity of the first kind. Therefore S_h does not contain its accumulation points (all points are isolated). Therefore this set is countable. The total number of jumps is the union over n of the jumps of size greater or equal $1/n$, which is countable.

□

Note that a function can have more than countably many discontinuities, but then they are not all jumps. Another useful observation is that a derivative cannot have jump discontinuities at all.

Theorem 1.4 *If f is differentiable with a finite derivative $f'(t)$ in an interval, then at all points $f'(t)$ is either continuous or has a discontinuity of the second kind.*

PROOF: If t is such that $f'(t+) = \lim_{s \downarrow t} f'(s)$ exists (finite or infinite), then (by the Mean Value Theorem) the same value is taken by the derivative from the right $\lim_{\Delta t \downarrow 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}$. Similarly for the left side of t. Hence $f'(t)$ is continuous at t. The result follows.

□

Functions considered in stochastic calculus

We make clear as to what kind of functions stochastic calculus deals with. Functions considered in stochastic calculus are functions without discontinuities of the second kind.

Thus the class of functions under consideration are functions that have both right and left limits at any point of the domain and have one-sided limits at the boundary. These functions are called *regular* functions. It is often agreed to identify functions if they have the same right and left limits at any point. In particular one can take the right-continuous version of the function or a left-continuous one.

The class $D = D[0,T]$ of right-continuous functions on $[0,T]$, with left limits has a special name *càdlàg* functions, (which is the abbreviation of "right continuous with left limits" in French). Sometimes these processes are called R.R.C. for regular right continuous. Notice that this class of processes includes C, the class of continuous functions.

Let $g \in D$ be a càdlàg function, then by definition, $g(t+) = g(t)$ and $g(t-)$ exist at any point. Therefore all the discontinuities of g are jumps. Introduce the function of jumps Δg, defined by $\Delta g(t) = g(t) - g(t-)$. The above result on the number of discontinuities of a càdlàg function is used in the following form in stochastic calculus.

Corollary 1.4.1 *Let g be a regular function which is left-continuous with right limits (càglàd), or right-continuous with left limits (càdlàg). Then g is continuous at all points but jumps. The set of jumps, $\{t : |\Delta g(t)| > 0\}$ is at most countable.*

Example 1.3: $g(t) = 0$ for $t < 0$, $g(0) = 1$, and $g(t) = 2$ for $t > 0$ is a regular function but not càglàd nor càdlàg.

Remark 1.1: In stochastic calculus $\Delta g(t)$ usually stands for the size of the jump at t. In standard calculus $\Delta g(t)$ usually stands for the increment of g over $[t, t+\Delta]$, $\Delta g(t) = g(t+\Delta) - g(t)$. The meaning of $\Delta g(t)$ will be clear from the context.

1.3 Variation of a Function

If g is a function of real variable, its variation over the interval $[a, b]$ is defined as

$$V_g([a,b]) = \sup \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)|, \tag{1.8}$$

where supremum is taken over partitions:

$$a = t_0^n < t_1^n < \ldots < t_n^n = b. \tag{1.9}$$

Clearly, (by the triangle inequality) the sums in (1.8) increase as new points are added to the partitions. Therefore variation of g is

$$V_g([a,b]) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)|, \tag{1.10}$$

where $\delta_n = \max_{1 \le i \le n}(t_i - t_{i-1})$. If $V_g([a,b])$ is finite then g is said to be a function of finite variation on $[a, b]$. If g is a function of $t \ge 0$, then the variation function of g as a function of t is defined by

$$V_g(t) = V_g([0,t]).$$

Clearly, $V_g(t)$ is a nondecreasing function of t.

Definition 1.5 *g is of finite variation if $V_g(t) < \infty$ for all t. g is of bounded variation if $\sup_t V_g(t) < \infty$, in other words, if for all t, $V_g(t) < C$, a constant independent of t.*

Example 1.4:

1. If $g(t)$ is increasing then for any i, $g(t_i) > g(t_{i-1})$ resulting in a telescoping sum, where all the terms excluding the first and the last cancel out, leaving

$$V_g(t) = g(t) - g(0).$$

2. If $g(t)$ is decreasing then, similarly,

$$V_g(t) = g(0) - g(t).$$

Example 1.5: If $g(t)$ is differentiable with continuous derivative $g'(t)$, $g(t) = \int_0^t g'(s)ds$, and $\int_0^t |g'(s)|ds < \infty$, then

$$V_g(t) = \int_0^t |g'(s)|ds.$$

This can be seen by using the definition and the mean value theorem. $\int_{t_{i-1}}^{t_i} g'(s)ds = g'(\xi_i)(t_i - t_{i-1})$, for a $\xi_i \in (t_{i-1}, t_i)$. Thus $|\int_{t_{i-1}}^{t_i} g'(s)ds| = |g'(\xi_i)|(t_i - t_{i-1})$, and

$$\begin{aligned} V_g(t) = \lim \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| &= \lim \sum_{i=1}^{n} |\int_{t_{i-1}}^{t_i} g'(s)ds| \\ = \sup \sum_{i=1}^{n} |g'(\xi_i)|(t_i - t_{i-1}) &= \int_0^t |g'(s)|ds. \end{aligned}$$

The last equality is due to the last sum being a Riemann sum for the final integral.

Alternatively, the result can be seen from the decomposition of the derivative into the positive and negative parts,

$$g(t) = \int_0^t g'(s)ds = \int_0^t [g'(s)]^+ ds - \int_0^t [g'(s)]^- ds.$$

Notice that $[g'(s)]^-$ is zero when $[g'(s)]^+$ is positive, and the other way around. Using this it is not hard to see that the total variation of g is given by the sum of the variation of the above integrals. But these integrals are monotone functions with the value zero at zero. Hence

$$\begin{aligned} V_g(t) &= \int_0^t [g'(s)]^+ ds + \int_0^t [g'(s)]^- ds \\ &= \int_0^t ([g'(s)]^+ + [g'(s)]^-)ds = \int_0^t |g'(s)|ds. \end{aligned}$$

Example 1.6: (Variation of a pure jump function).
If g is a regular right-continuous (càdlàg) function or regular left-continuous (càglàd), and changes only by jumps,

$$g(t) = \sum_{0 \le s \le t} \Delta g(s),$$

then it is easy to see from the definition that

$$V_g(t) = \sum_{0 \le s \le t} |\Delta g(s)|.$$

The following theorem gives necessary and sufficient conditions for a function to have finite variation.

Theorem 1.6 (Jordan Decomposition) *Any function $g(t) : [0, \infty) \to \mathbf{R}$ of finite variation can be expressed as the difference of two increasing functions*

$$g(t) = a(t) - b(t).$$

If g is right-continuous then it can be expressed as the difference of two right-continuous increasing functions.

One such decomposition is given by

$$a(t) = V_g(t) \quad b(t) = V_g(t) - g(t). \tag{1.11}$$

The representation of a function of finite variation as difference of two right-continuous increasing functions is not unique. Another decomposition is

$$g(t) = \frac{1}{2}(V_g(t) + g(t)) - \frac{1}{2}(V_g(t) - g(t))$$

The sum, the difference and the product of functions of finite variation are also functions of finite variation. This is also true for the ratio of two functions of finite variation provided the modulus of the denominator is larger than a positive constant.

The following result follows by Theorem 1.3, and its proof is easy.

Theorem 1.7 *A finite variation function can have no more than countably many discontinuities. Moreover, all discontinuities are jumps.*

PROOF: A monotone function has left and right limits at any point, therefore any discontinuity is a jump. It can have only finitely many jumps of a fixed size on $[a, b]$, the number of jumps of size greater or equal to $\frac{1}{n}$ is no more than $(g(b) - g(a))n$. This implies that the total number of jumps is at most countable. Since a function of finite variation is a difference of two monotone functions, the result follows.

□

A sufficient condition for a continuous function to be of finite variation is

Theorem 1.8 *If g is continuous, g' exists and $\int |g'(t)|dt < \infty$ then g is of finite variation.*

A partial converse also holds. See, for example, Saks (1964), Freedman (1983) p.209, for the following results.

Theorem 1.9 (Lebesgue) *A finite variation function g on $[a, b]$ is almost everywhere differentiable on $[a, b]$.*

Continuous and Discrete Parts of a Function

Let $g(t)$, $t \geq 0$, be a right-continuous increasing function. Then it can have at most countably many jumps, moreover the sum of the jumps is finite over finite time intervals. Define the discontinuous part g^d of g by

$$g^d(t) = \sum_{s \leq t} (g(s) - g(s-)) = \sum_{0 < s \leq t} \Delta g(s), \tag{1.12}$$