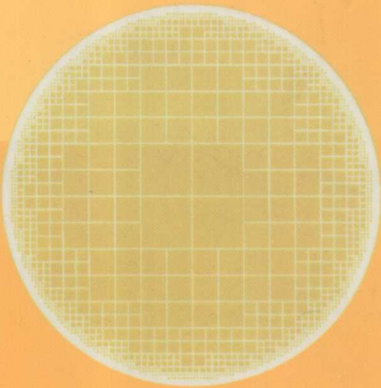
Graduate Texts in Mathematics

Loukas Grafakos

Classical Fourier Analysis

Second Edition



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Preface

The great response to the publication of the book *Classical and Modern Fourier Analysis* has been very gratifying. I am delighted that Springer has offered to publish the second edition of this book in two volumes: *Classical Fourier Analysis, 2nd Edition,* and *Modern Fourier Analysis, 2nd Edition.*

These volumes are mainly addressed to graduate students who wish to study Fourier analysis. This first volume is intended to serve as a text for a one-semester course in the subject. The prerequisite for understanding the material herein is satisfactory completion of courses in measure theory, Lebesgue integration, and complex variables.

The details included in the proofs make the exposition longer. Although it will behoove many readers to skim through the more technical aspects of the presentation and concentrate on the flow of ideas, the fact that details are present will be comforting to some. The exercises at the end of each section enrich the material of the corresponding section and provide an opportunity to develop additional intuition and deeper comprehension. The historical notes of each chapter are intended to provide an account of past research but also to suggest directions for further investigation. The appendix includes miscellaneous auxiliary material needed throughout the text.

A web site for the book is maintained at

http://math.missouri.edu/~loukas/FourierAnalysis.html

I am solely responsible for any misprints, mistakes, and historical omissions in this book. Please contact me directly (loukas@math.missouri.edu) if you have corrections, comments, suggestions for improvements, or questions.

Columbia, Missouri,
April 2008

Loukas Grafakos

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Contents

Chapter 1
L^p Spaces and Interpolation

Many quantitative properties of functions are expressed in terms of their integrability to a power. For this reason it is desirable to acquire a good understanding of spaces of functions whose modulus to a power p is integrable. These are called Lebesgue spaces and are denoted by L^p. Although an in-depth study of Lebesgue spaces falls outside the scope of this book, it seems appropriate to devote a chapter to reviewing some of their fundamental properties.

The emphasis of this review is basic interpolation between Lebesgue spaces. Many problems in Fourier analysis concern boundedness of operators on Lebesgue spaces, and interpolation provides a framework that often simplifies this study. For instance, in order to show that a linear operator maps L^p to itself for all $1 < p < \infty$, it is sufficient to show that it maps the (smaller) Lorentz space $L^{p,1}$ into the (larger) Lorentz space $L^{p,\infty}$ for the same range of p's. Moreover, some further reductions can be made in terms of the Lorentz space $L^{p,1}$. This and other considerations indicate that interpolation is a powerful tool in the study of boundedness of operators.

Although we are mainly concerned with L^p subspaces of Euclidean spaces, we discuss in this chapter L^p spaces of arbitrary measure spaces, since they represent a useful general setting. Many results in the text require working with general measures instead of Lebesgue measure.

1.1 L^p and Weak L^p

Let X be a measure space and let μ be a positive, not necessarily finite, measure on X. For $0 < p < \infty$, $L^p(X,\mu)$ denotes the set of all complex-valued μ-measurable functions on X whose modulus to the pth power is integrable. $L^\infty(X,\mu)$ is the set of all complex-valued μ-measurable functions f on X such that for some $B > 0$, the set $\{x : |f(x)| > B\}$ has μ-measure zero. Two functions in $L^p(X,\mu)$ are considered equal if they are equal μ-almost everywhere. The notation $L^p(\mathbf{R}^n)$ is reserved for the space $L^p(\mathbf{R}^n, |\cdot|)$, where $|\cdot|$ denotes n-dimensional Lebesgue measure. Lebesgue measure on $\mathbf{R}^n$ is also denoted by dx. Within context and in the absence of ambi-

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guity, $L^p(X,\mu)$ is simply written as L^p. The space $L^p(\mathbf{Z})$ equipped with counting measure is denoted by $\ell^p(\mathbf{Z})$ or simply ℓ^p.

For $0<p<\infty$, we define the L^p quasinorm of a function f by

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \tag{1.1.1}$$

and for $p=\infty$ by

$$\|f\|_{L^\infty(X,\mu)} = \text{ess.sup}\,|f| = \inf\big\{B>0 : \mu(\{x : |f(x)|>B\})=0\big\}. \tag{1.1.2}$$

It is well known that Minkowski's (or the triangle) inequality

$$\|f+g\|_{L^p(X,\mu)} \le \|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)} \tag{1.1.3}$$

holds for all f, g in $L^p = L^p(X,\mu)$, whenever $1\le p\le\infty$. Since in addition $\|f\|_{L^p(X,\mu)}=0$ implies that $f=0$ (μ-a.e.), the L^p spaces are normed linear spaces for $1\le p\le\infty$. For $0<p<1$, inequality (1.1.3) is reversed when $f,g\ge 0$. However, the following substitute of (1.1.3) holds:

$$\|f+g\|_{L^p(X,\mu)} \le 2^{(1-p)/p}\big(\|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)}\big), \tag{1.1.4}$$

and thus $L^p(X,\mu)$ is a quasinormed linear space. See also Exercise 1.1.5. For all $0<p\le\infty$, it can be shown that every Cauchy sequence in $L^p(X,\mu)$ is convergent, and hence the spaces $L^p(X,\mu)$ are complete. For the case $0<p<1$ we refer to Exercise 1.1.8. Therefore, the L^p spaces are Banach spaces for $1\le p\le\infty$ and quasi-Banach spaces for $0<p<1$. For any $p\in(0,\infty)\setminus\{1\}$ we use the notation $p'=\frac{p}{p-1}$. Moreover, we set $1'=\infty$ and $\infty'=1$, so that $p''=p$ for all $p\in(0,\infty]$. Hölder's inequality says that for all $p\in[1,\infty]$ and all measurable functions f,g on (X,μ) we have

$$\|fg\|_{L^1} \le \|f\|_{L^p}\|g\|_{L^{p'}}.$$

It is a well-known fact that the dual $(L^p)^*$ of L^p is isometric to $L^{p'}$ for all $1\le p<\infty$. Furthermore, the L^p norm of a function can be obtained via duality when $1\le p\le\infty$ as follows:

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}}=1}\left|\int_X fg\,d\mu\right|.$$

For the endpoint cases $p=1$, $p=\infty$, see Exercise 1.4.12(a), (b).

1.1.1 The Distribution Function

Definition 1.1.1. For f a measurable function on X, the *distribution function* of f is the function d_f defined on $[0,\infty)$ as follows:

$$d_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\}). \tag{1.1.5}$$

The distribution function d_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbf{R}^n$ and each of its translates have the same distribution function. It follows from Definition 1.1.1 that d_f is a decreasing function of α (not necessarily strictly).

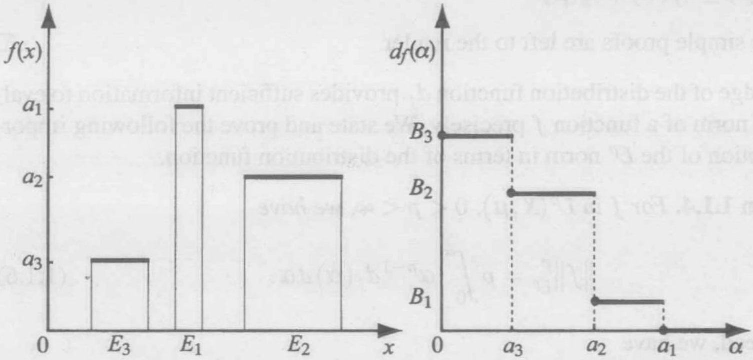

Fig. 1.1 The graph of a simple function $f = \sum_{k=1}^{3} a_k \chi_{E_k}$ and its distribution function $d_f(\alpha)$. Here $B_j = \sum_{k=1}^{j} \mu(E_k)$.

Example 1.1.2. Recall that simple functions are finite linear combinations of characteristic functions of sets of finite measure. For pedagogical reasons we compute the distribution function d_f of a nonnegative simple function

$$f(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x),$$

where the sets E_j are pairwise disjoint and $a_1 > \cdots > a_N > 0$. If $\alpha \geq a_1$, then clearly $d_f(\alpha) = 0$. However, if $a_2 \leq \alpha < a_1$ then $|f(x)| > \alpha$ precisely when $x \in E_1$, and in general, if $a_{j+1} \leq \alpha < a_j$, then $|f(x)| > \alpha$ precisely when $x \in E_1 \cup \cdots \cup E_j$. Setting

$$B_j = \sum_{k=1}^{j} \mu(E_k),$$

we have

$$d_f(\alpha) = \sum_{j=0}^{N} B_j \chi_{[a_{j+1}, a_j)}(\alpha),$$

where $a_0 = \infty$ and $B_0 = a_{N+1} = 0$. Figure 1.1 illustrates this example when $N = 3$.

We now state a few simple facts about the distribution function d_f.

Proposition 1.1.3. *Let f and g be measurable functions on (X,μ). Then for all $\alpha,\beta > 0$ we have*

(1) $|g| \le |f|$ μ-a.e. implies that $d_g \le d_f$;

(2) $d_{cf}(\alpha) = d_f(\alpha/|c|)$, for all $c \in \mathbf{C}\setminus\{0\}$;

(3) $d_{f+g}(\alpha+\beta) \le d_f(\alpha) + d_g(\beta)$;

(4) $d_{fg}(\alpha\beta) \le d_f(\alpha) + d_g(\beta)$.

Proof. The simple proofs are left to the reader. □

Knowledge of the distribution function d_f provides sufficient information to evaluate the L^p norm of a function f precisely. We state and prove the following important description of the L^p norm in terms of the distribution function.

Proposition 1.1.4. *For f in $L^p(X,\mu)$, $0 < p < \infty$, we have*

$$\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha\,. \tag{1.1.6}$$

Proof. Indeed, we have

$$\begin{aligned} p\int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha &= p\int_0^\infty \alpha^{p-1}\int_X \chi_{\{x:\,|f(x)|>\alpha\}}\, d\mu(x)\, d\alpha \\ &= \int_X \int_0^{|f(x)|} p\alpha^{p-1}\, d\alpha\, d\mu(x) \\ &= \int_X |f(x)|^p\, d\mu(x) \\ &= \|f\|_{L^p}^p\,, \end{aligned}$$

where we used Fubini's theorem in the second equality. This proves (1.1.6). □

Notice that the same argument yields the more general fact that for any increasing continuously differentiable function φ on $[0,\infty)$ with $\varphi(0) = 0$ we have

$$\int_X \varphi(|f|)\, d\mu = \int_0^\infty \varphi'(\alpha) d_f(\alpha)\, d\alpha\,. \tag{1.1.7}$$

Definition 1.1.5. For $0 < p < \infty$, the space *weak* $L^p(X,\mu)$ is defined as the set of all μ-measurable functions f such that

$$\begin{aligned} \|f\|_{L^{p,\infty}} &= \inf\Big\{C > 0 : d_f(\alpha) \le \frac{C^p}{\alpha^p} \quad \text{for all} \quad \alpha > 0\Big\} & (1.1.8) \\ &= \sup\big\{\gamma d_f(\gamma)^{1/p} : \gamma > 0\big\} & (1.1.9) \end{aligned}$$

is finite. The space *weak-$L^\infty(X,\mu)$* is by definition $L^\infty(X,\mu)$.

The reader should check that (1.1.9) and (1.1.8) are in fact equal. The weak L^p spaces are denoted by $L^{p,\infty}(X,\mu)$. Two functions in $L^{p,\infty}(X,\mu)$ are considered equal if they are equal μ-a.e. The notation $L^{p,\infty}(\mathbf{R}^n)$ is reserved for $L^{p,\infty}(\mathbf{R}^n, |\cdot|)$. Using Proposition 1.1.3 (2), we can easily show that

$$\|kf\|_{L^{p,\infty}} = |k| \|f\|_{L^{p,\infty}}, \tag{1.1.10}$$

for any complex nonzero constant k. The analogue of (1.1.3) is

$$\|f+g\|_{L^{p,\infty}} \le c_p \big(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}}\big), \tag{1.1.11}$$

where $c_p = \max(2, 2^{1/p})$, a fact that follows from Proposition 1.1.3 (3), taking both α and β equal to $\alpha/2$. We also have that

$$\|f\|_{L^{p,\infty}(X,\mu)} = 0 \Rightarrow f = 0 \qquad \mu\text{-a.e.} \tag{1.1.12}$$

In view of (1.1.10), (1.1.11), and (1.1.12), $L^{p,\infty}$ is a quasinormed linear space for $0 < p < \infty$.

The weak L^p spaces are larger than the usual L^p spaces. We have the following:

Proposition 1.1.6. *For any $0 < p < \infty$ and any f in $L^p(X,\mu)$ we have $\|f\|_{L^{p,\infty}} \le \|f\|_{L^p}$; hence $L^p(X,\mu) \subseteq L^{p,\infty}(X,\mu)$.*

Proof. This is just a trivial consequence of Chebyshev's inequality:

$$\alpha^p d_f(\alpha) \le \int_{\{x:\,|f(x)|>\alpha\}} |f(x)|^p \, d\mu(x). \tag{1.1.13}$$

The integral in (1.1.13) is at most $\|f\|_{L^p}^p$ and using (1.1.9) we obtain that $\|f\|_{L^{p,\infty}} \le \|f\|_{L^p}$. □

The inclusion $L^p \subseteq L^{p,\infty}$ is strict. For example, on $\mathbf{R}^n$ with the usual Lebesgue measure, let $h(x) = |x|^{-\frac{n}{p}}$. Obviously, h is not in $L^p(\mathbf{R}^n)$ but h is in $L^{p,\infty}(\mathbf{R}^n)$ with $\|h\|_{L^{p,\infty}(\mathbf{R}^n)} = v_n$, where v_n is the measure of the unit ball of $\mathbf{R}^n$.

It is not immediate from their definition that the weak L^p spaces are complete with respect to the quasinorm $\|\cdot\|_{L^{p,\infty}}$. The completeness of these spaces is proved in Theorem 1.4.11, but it is also a consequence of Theorem 1.1.13, proved in this section.

1.1.2 Convergence in Measure

Next we discuss some convergence notions. The following notion is important in probability theory.

Definition 1.1.7. Let f, f_n, $n = 1, 2, \dots$, be measurable functions on the measure space (X,μ). The sequence f_n is said to *converge in measure* to f if for all $\varepsilon > 0$ there exists an $n_0 \in \mathbf{Z}^+$ such that