TEXTBOOKS IN MATHEMATICS

DIFFERENTIAL EQUATIONS WITH APPLICATIONS AND HISTORICAL NOTES

Third Edition

George F. Simmons

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum \frac{1}{n^3} = ?$$

CRC Press
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For Hope and Nancy

my wife and daughter

who still make it all worthwhile

Preface to the Third Edition

I have taken advantage of this new edition of my book on differential equations to add two batches of new material of independent interest:

First, a fairly substantial appendix at the end of Chapter 1 on the famous bell curve. This curve is the graph of the normal distribution function, with many applications in the natural sciences, the social sciences, mathematics—in statistics and probability theory—and engineering. We shall be especially interested how the differential equation for this curve arises from very simple considerations and can be solved to obtain the equation of the curve itself.

And second, a brief section on the van der Pol nonlinear equation and its historical background in World War II that gave it significance in the development of the theory of radar. This consists, in part, of personal recollections of the eminent physicist Freeman Dyson.

Finally, I should add a few words on the meaning of the cover design, for this design amounts to a bit of self-indulgence.

The chapter on Fourier series is there mainly to provide machinery needed for the following chapter on partial differential equations. However, one of the minor offshoots of Fourier series is to find the exact sum of the infinite series formed from the reciprocals of the squares of the positive integers (the first formula on the cover). This sum was discovered by the great Swiss mathematician Euler in 1736, and since his time, several other methods for obtaining this sum, in addition to his own, have been discovered. This is one of the topics dealt with in Sections 34 and 35 and has been one of my own minor hobbies in mathematics for many years.

However, from 1736 to the present day, no one has ever been able to find the exact sum of the reciprocals of the cubes of the positive integers (the second formula on the cover). Some years ago, I was working with the zeroes of the Bessel functions. I thought for an exciting period of several days that I was on the trail of this unknown sum, but in the end it did not work out. Instead, the trail deviated in an unexpected direction and yielded yet another method for finding the sum in the first formula. These ideas will be found in Section 47.

Preface to the Second Edition

"As correct as a second edition"—so goes the idiom. I certainly hope so, and I also hope that anyone who detects an error will do me the kindness of letting me know, so that repairs can be made. As Confucius said, "A man who makes a mistake and doesn't correct it is making two mistakes."

I now understand why second editions of textbooks are always longer than first editions: as with governments and their budgets, there is always strong pressure from lobbyists to put things in, but rarely pressure to take things out.

The main changes in this new edition are as follows: the number of problems in the first part of the book has been more than doubled; there are two new chapters, on Fourier Series and on Partial Differential Equations; sections on higher order linear equations and operator methods have been added to Chapter 3; and further material on convolutions and engineering applications has been added to the chapter on Laplace Transforms.

Altogether, many different one-semester courses can be built on various parts of this book by using the schematic outline of the chapters given on page xix. There is even enough material here for a two-semester course, if the appendices are taken into account.

Finally, an entirely new chapter on Numerical Methods (Chapter 14) has been written especially for this edition by Major John S. Robertson of the United States Military Academy. Major Robertson's expertise in these matters is much greater than my own, and I am sure that many users of this new edition will appreciate his contribution, as I do.

McGraw-Hill and I would like to thank the following reviewers for their many helpful comments and suggestions: D. R. Arterburn, New Mexico Tech; Edward Beckenstein, St. John's University; Harold Carda, South Dakota School of Mines and Technology; Wenxiong Chen, University of Arizona; Jerald P. Dauer, University of Tennessee; Lester B. Fuller, Rochester Institute of Technology; Juan Gatica, University of Iowa; Richard H. Herman, The Pennsylvania State University; Roger H. Marty, Cleveland State University; Jean-Pierre Meyer, The Johns Hopkins University; Krzysztof Ostaszewski, University of Louisville; James L. Rovnyak, University of Virginia; Alan Sharples, New Mexico Tech; Bernard Shiffman, The Johns Hopkins University; and Calvin H. Wilcox, University of Utah.

George F. Simmons

Preface to the First Edition

To be worthy of serious attention, a new textbook on an old subject should embody a definite and reasonable point of view which is not represented by books already in print. Such a point of view inevitably reflects the experience, taste, and biases of the author, and should therefore be clearly stated at the beginning so that those who disagree can seek nourishment elsewhere. The structure and contents of this book express my personal opinions in a variety of ways, as follows.

The place of differential equations in mathematics. Analysis has been the dominant branch of mathematics for 300 years, and differential equations are the heart of analysis. This subject is the natural goal of elementary calculus and the most important part of mathematics for understanding the physical sciences. Also, in the deeper questions it generates, it is the source of most of the ideas and theories which constitute higher analysis. Power series, Fourier series, the gamma function and other special functions, integral equations, existence theorems, the need for rigorous justifications of many analytic processes—all these themes arise in our work in their most natural context. And at a later stage they provide the principal motivation behind complex analysis, the theory of Fourier series and more general orthogonal expansions, Lebesgue integration, metric spaces and Hilbert spaces, and a host of other beautiful topics in modern mathematics. I would argue, for example, that one of the main ideas of complex analysis is the liberation of power series from the confining environment of the real number system; and this motive is most clearly felt by those who have tried to use real power series to solve differential equations. In botany, it is obvious that no one can fully appreciate the blossoms of flowering plants without a reasonable understanding of the roots, stems, and leaves which nourish and support them. The same principle is true in mathematics, but is often neglected or forgotten.

Fads are as common in mathematics as in any other human activity, and it is always difficult to separate the enduring from the ephemeral in the achievements of one's own time. At present there is a strong current of abstraction flowing through our graduate schools of mathematics. This current has scoured away many of the individual features of the landscape and replaced them with the smooth, rounded boulders of general theories. When taken in moderation, these general theories are both useful and satisfying; but one unfortunate effect of their predominance is that if a student doesn't learn a little while he is an undergraduate about such colorful and worthwhile topics as the wave equation, Gauss's hypergeometric function, the gamma function, and the basic problems of the calculus of variations—among many others—then he is unlikely to do so later. The natural place for an informal acquaintance with such ideas is a leisurely introductory course on differential equations. Some of our current books on this subject remind me of a sightseeing bus whose driver is so obsessed with speeding along to meet a schedule that his passengers have little or no opportunity to enjoy the scenery. Let us be late occasionally, and take greater pleasure in the journey.

Applications. It is a truism that nothing is permanent except change; and the primary purpose of differential equations is to serve as a tool for the study of change in the physical world. A general book on the subject without a reasonable account of its scientific applications would therefore be as futile and pointless as a treatise on eggs that did not mention their reproductive purpose. This book is constructed so that each chapter except the last has at least one major "payoff"—and often several—in the form of a classic scientific problem which the methods of that chapter render accessible. These applications include

The brachistochrone problem

The Einstein formula $E = mc^2$

Newton's law of gravitation

The wave equation for the vibrating string

The harmonic oscillator in quantum mechanics

Potential theory

The wave equation for the vibrating membrane

The prey–predator equations

Nonlinear mechanics

Hamilton's principle

Abel's mechanical problem

I consider the mathematical treatment of these problems to be among the chief glories of Western civilization, and I hope the reader will agree.

The problem of mathematical rigor. On the heights of pure mathematics, any argument that purports to be a proof must be capable of withstanding the severest criticisms of skeptical experts. This is one of the rules of the game, and if you wish to play you must abide by the rules. But this is not the only game in town.

There are some parts of mathematics—perhaps number theory and abstract algebra—in which high standards of rigorous proof may be appropriate at all levels. But in elementary differential equations a narrow insistence on doctrinaire exactitude tends to squeeze the juice out of the subject, so that only the dry husk remains. My main purpose in this book is to help the student grasp the nature and significance of differential equations; and to this end, I much prefer being occasionally imprecise but understandable to being completely accurate but incomprehensible. I am not at all interested in building a logically impeccable mathematical structure, in which definitions, theorems, and rigorous proofs are welded together into a formidable barrier which the reader is challenged to penetrate.

In spite of these disclaimers, I do attempt a fairly rigorous discussion from time to time, notably in Chapter 13 and Appendices A in Chapters 5, 6 and 7, and B in Chapter 11. I am not saying that the rest of this book is nonrigorous, but only that it leans toward the activist school of mathematics, whose primary aim is to develop methods for solving scientific problems—in contrast to the contemplative school, which analyzes and organizes the ideas and tools generated by the activists.

Some will think that a mathematical argument either is a proof or is not a proof. In the context of elementary analysis I disagree, and believe instead that the proper role of a proof is to carry reasonable conviction to one's intended audience. It seems to me that mathematical rigor is like clothing: in its style it ought to suit the occasion, and it diminishes comfort and restricts freedom of movement if it is either too loose or too tight.

History and biography. There is an old Armenian saying, "He who lacks a sense of the past is condemned to live in the narrow darkness of his own generation." Mathematics without history is mathematics stripped of its greatness: for, like the other arts—and mathematics is one of the supreme arts of civilization—it derives its grandeur from the fact of being a human creation.

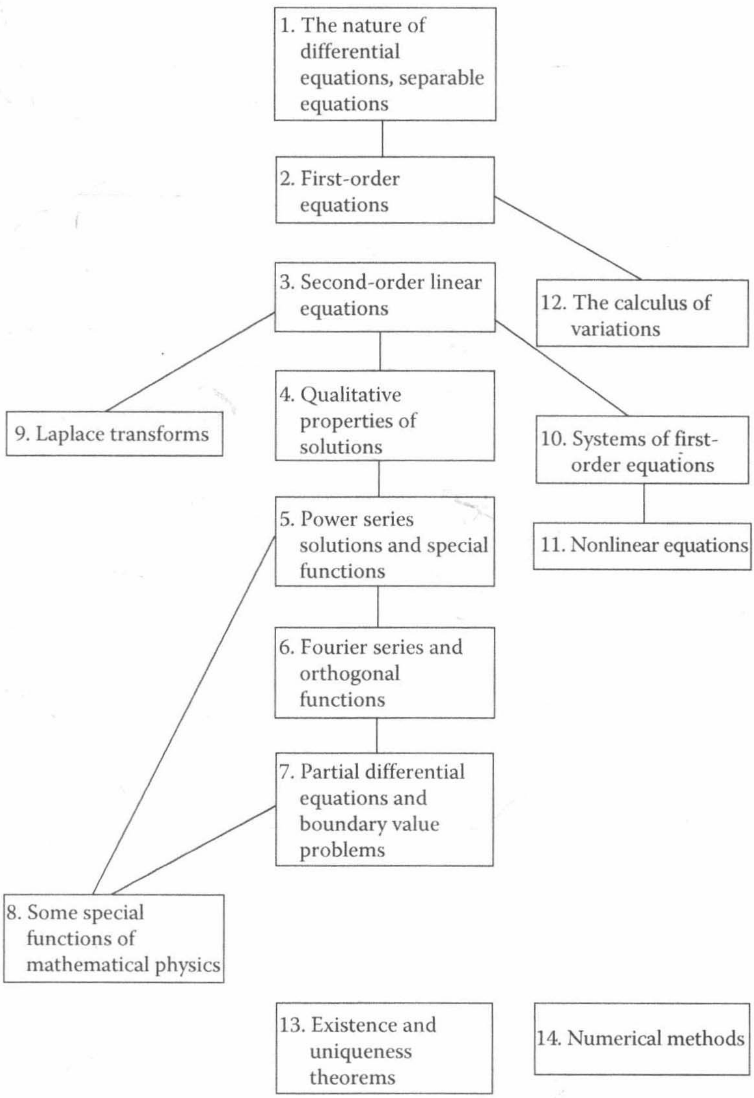
In an age increasingly dominated by mass culture and bureaucratic impersonality, I take great pleasure in knowing that the vital ideas of mathematics were not printed out by a computer or voted through by a committee, but instead were created by the solitary labor and individual genius of a few remarkable men. The many biographical notes in this book reflect my desire to convey something of the achievements and personal qualities of these astonishing human beings. Most of the longer notes are placed in the appendices, but each is linked directly to a specific contribution discussed in the text. These notes have as their subjects all but a few of the greatest mathematicians of the past three centuries: Fermat, Newton, the Bernoullis, Euler, Lagrange, Laplace, Fourier, Gauss, Abel, Poisson, Dirichlet, Hamilton, Liouville, Chebyshev, Hermite, Riemann, Minkowski, and Poincaré. As T. S. Eliot wrote in one of his essays, "Someone said: 'The dead writers are remote from us because we *know* so much more than they did.' Precisely, and they are that which we know."

History and biography are very complex, and I am painfully aware that scarcely anything in my notes is actually quite as simple as it may appear. I must also apologize for the many excessively brief allusions to mathematical ideas most student readers have not yet encountered. But with the aid of a good library, sufficiently interested students should be able to unravel most of them for themselves. At the very least, such efforts may help to impart a feeling for the immense diversity of classical mathematics—an aspect of the subject that is almost invisible in the average undergraduate curriculum.

George F. Simmons

Suggestions for the Instructor

The following diagram gives the logical dependence of the chapters and suggests a variety of ways this book can be used, depending on the purposes of the course, the tastes of the instructor, and the backgrounds and needs of the students.



The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living. Of course I do not here speak of that beauty that strikes the senses, the beauty of qualities and appearances; not that I undervalue such beauty, far from it, but it has nothing to do with science; I mean that profounder beauty which comes from the harmonious order of the parts, and which a pure intelligence can grasp.

—Henri Poincaré

As a mathematical discipline travels far from its empirical source, or still more, if it is a second or third generation only indirectly inspired by ideas coming from "reality," it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely l'art pour l'art. *This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration.*

—John von Neumann

Just as deduction should be supplemented by intuition, so the impulse to progressive generalization must be tempered and balanced by respect and love for colorful detail. The individual problem should not be degraded to the rank of special illustration of lofty general theories. In fact, general theories emerge from consideration of the specific, and they are meaningless if they do not serve to clarify and order the more particularized substance below. The interplay between generality and individuality, deduction and construction, logic and imagination—this is the profound essence of live mathematics. Any one or another of these aspects of mathematics can be at the center of a given achievement. In a far-reaching development all of them will be involved. Generally speaking, such a development will start from the "concrete" ground, then discard ballast by abstraction and rise to the lofty layers of thin air where navigation and observation are easy; after this flight comes the crucial test of landing and reaching specific goals in the newly surveyed low plains of individual "reality." In brief, the flight into abstract generality must start from and return to the concrete and specific.

—Richard Courant

About the Author

George Simmons has academic degrees from the California Institute of Technology, the University of Chicago, and Yale University. He taught at several colleges and universities before joining the faculty of Colorado College in 1962, where he is a Professor of Mathematics. He is also the author of *Introduction to Topology and Modern Analysis* (McGraw-Hill, 1963), *Precalculus Mathematics in a Nutshell* (Janson Publications, 1981), and *Calculus with Analytic Geometry* (McGraw-Hill, 1985).

When not working or talking or eating or drinking or cooking, Professor Simmons is likely to be traveling (Western and Southern Europe, Turkey, Israel, Egypt, Russia, China, Southeast Asia), trout fishing (Rocky Mountain states), playing pocket billiards, or reading (literature, history, biography and autobiography, science, and enough thrillers to achieve enjoyment without guilt).

Contents