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The Maximal Subgroups of the Low-Dimensional Finite Classical Groups

John N. Bray, Derek F. Holt and
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Foreword

In this book the authors determine the maximal subgroups of all the finite classical groups of dimension 12 or less. This work fills a long-standing gap in the literature. Behind this gap there is a story which I am pleased to have the opportunity to tell.

The completion of the classification of finite simple groups was first announced in the early 1980s. It was clear then (and before) that for many applications of the classification one would need detailed knowledge of the maximal subgroups of the simple groups and of their automorphism groups. Around that time, I gave a Part III course at Cambridge about the classification and its impact. Full of enthusiasm, I set a fearsome exam – I remember giving it to John Conway to check, and him saying that he couldn't do any of the questions, but he thought it was probably OK. The second highest mark was 18%, scored by a rather strong student. The top mark was 97%, scored by Peter Kleidman, a young American.

Soon afterwards, Kleidman started as my first research student. Michael Aschbacher had just published his fundamental theorem on maximal subgroups of the finite classical groups. The time seemed right to attempt to use this to determine all the maximal subgroups of the classical groups of low dimensions (up to 20, say, I thought optimistically). This was Kleidman's initial project. As it turned out, in his thesis he solved many other maximal subgroup problems, and this project occupied just one chapter. Nevertheless it was rather an interesting chapter, consisting of tables of all the maximal subgroups of finite simple classical groups of dimension up to 12. No proofs were given, just an outline of the strategy and a few examples of how the calculations were performed.

After he had graduated, Kleidman and I wrote a book on the subgroups of the finite classical groups, which was an analysis of the structure, conjugacy and maximality of the subgroups arising in Aschbacher's theorem now known as geometric subgroups. For the maximality questions we assumed that the dimen-

Preface

The aim of this book is to classify the maximal subgroups of the almost simple finite classical groups of dimension at most 12. We also include tables describing the maximal subgroups of the almost simple finite exceptional groups that have faithful representations of degree at most 12.

A group G is *simple* if it has order greater than 1 and has no normal subgroups other than the trivial subgroup and G itself, and is *almost simple* if $S \leqslant G \leqslant \mathrm{Aut}\, S$ for some non-abelian simple group S. A group is *perfect* if it is equal to its derived group. A group G is called *quasisimple* if G is perfect and G modulo its centre is a non-abelian simple group.

The study and classification of the (maximal) subgroups of the finite simple groups and their variations has a long history, and the completion of the classification of the finite simple groups provided further motivation.

The term 'classical group' is used frequently in the literature, but it is rarely, if ever, defined precisely. We shall not attempt a formal definition here, and we shall avoid using it in a precise sense. Our general intention is to use it very inclusively. We shall certainly regard all of the named groups (like $\mathrm{GL}_n(q)$, $\mathrm{O}_n^{\epsilon}(q)$, $\mathrm{PCSp}_n(q)$, etc.) in Table 1.2 as classical groups, but we also include among the classical groups arbitrary subgroups between the Ω-groups and the A-groups in the table, and also quotients of the groups in the first of each of the paired rows in the table by arbitrary subgroups of the scalars. Furthermore, we include all almost simple extensions of the simple classical groups.

Maximal subgroups of classical groups. In [1], Aschbacher proved a fundamental theorem that describes the subgroups of almost all of the finite almost simple classical groups (the only exceptions are certain extensions of $\mathrm{S}_4(2^e)$ and $\mathrm{O}_8^+(q)$). This theorem divides these subgroups into nine classes. The first eight of these consist roughly of groups that preserve some kind of geometric structure; for example the first class consists (roughly) of the reducible groups, which fix a proper non-zero subspace of the vector space on which the group sion was more than 12, and it was the intention that Kleidman would extend and write up his thesis work on the low dimensions as a separate book. Indeed, this project was accepted as a volume to appear in the Longman Research Notes series, and has been referred to as such in many articles. Unfortunately, he did not write this book, and left mathematics at the age of about thirty to pursue other interests such as working on Wall Street and producing Hollywood movies.

The non-appearance of Kleidman's book left a yawning gap in the literature for over twenty years. We are fortunate indeed that a number of years ago the authors of this volume took it upon themselves to fill this gap. They have done this in marvellously complete fashion, presenting the material with great clarity and attention to detail. Full proofs and comprehensive background material are given, making the book easily accessible to graduate students. It should also be said that their results go quite a way beyond Kleidman's thesis, in that they handle almost simple classical groups rather than just simple ones, which is important for applications.

It is marvellous to have this volume on the bookshelf where previously there was such an evident space, and I congratulate the authors on their achievement.

Martin Liebeck
Imperial College London

acts naturally. Subgroups of classical groups that lie in the first eight classes are of *geometric type*. The ninth class, denoted by $\mathscr{C}_9$ or $\mathscr{S}$, consists (roughly) of those subgroups that are not of geometric type and which, modulo the subgroup of scalar matrices, are almost simple. An alternative proof of Aschbacher's theorem, as a corollary to a version of the theorem for algebraic groups, can be found in [82]. We present a detailed version of Aschbacher's theorem, based on the treatment in [66], in Section 2.2. An interesting version of Aschbacher's theorem is presented in [91], which emphasises the links between the subgroup structure of the finite classical groups and of the algebraic groups of Lie type.

In [66], Kleidman and Liebeck provide an impressively detailed enumeration of the maximal subgroups of geometric type of the almost simple finite classical groups of dimension greater than 12. In this book, we shall extend the work of Kleidman and Liebeck to handle dimensions at most 12, and also classify the maximal subgroups of these groups that are in Class $\mathscr{S}$.

With the exception of Kleidman's work [62] on $\Omega_8^+(q)$, and classifications of Kleidman [62, 64, 63] and Cooperstein [14] of maximal subgroups of some of the exceptional groups of Lie type, which we simply cite and reproduce in our tables, our approach in this book has been to use these previous classifications for checking purposes only: our proofs make no use of the references below, although we have compared our results with them and, where there are differences, verified that our tables are correct.

The most complete previous work on low-dimensional classical groups is undoubtedly Peter Kleidman's PhD thesis [61], where he presents a classification, without proof, of the maximal subgroups of the simple classical groups in dimensions up to 12. This is a remarkable achievement. Kleidman intended to publish a subsequent book, with the same goal as ours: the classification of maximal subgroups of the almost simple classical groups in dimension up to 12. Unfortunately, this has not been published, and the present work is an attempt to carry out Kleidman's plan.

We base the following historical summary on the surveys by King [60], and Kleidman and Liebeck [65]. The complete description of the subgroup structure of the groups $\mathrm{L}_2(q)$ is usually attributed to Dickson [22], but this topic was also investigated by E. H. Moore [94] and Wiman [115]. The subgroups of $\mathrm{L}_3(q)$ were described by Mitchell for q odd [92] and then by Hartley for q even [40]. In both cases the subgroups that lie in $\mathrm{U}_3(\sqrt{q})$ were identified. A more modern treatment of subgroups of $\mathrm{L}_3(q)$ is provided by Bloom in [3]. The maximal subgroups of $\mathrm{L}_4(q)$ for even q were listed independently by Mwene [95] and Zalesskii [116]. A partial classification for odd q can be found in [96] and, independently, for $p > 5$ in [117]. Further results on this case can be found in [59, Section 5]. Mitchell classified the maximal subgroups of $\mathrm{S}_4(q)$ for odd q in [93]. Flesner [26, 27] partially classified the maximal subgroups of $\mathrm{S}_4(2^e)$.

The maximal subgroups of $L_5(q)$ were determined by Di Martino and Wagner for q odd [23], and independently by Wagner [110] and Zalesskii [116] for q even. Kondrat'ev classified the quasisimple absolutely irreducible subgroups of $GL_6(q)$ [71]. There is a brief survey of many of these results in [72].

For higher dimensions, many people have concentrated on the case $q = 2$. In 1984, Darafsheh classified the maximal subgroups of $GL_6(2)$ [21], building on Harada and Yamaki's paper [39], which classified the insolvable irreducible subgroups of $GL_n(2)$ with $n \leqslant 6$. The subgroups of $GL_n(2)$ for $n \leqslant 10$ were studied extensively by Kondrat'ev: see [67, 68, 69, 70].

In the work of King and others (see [60] and the references therein) a different approach is taken: rather than concentrating on a family of almost simple groups, such as those with socle $L_6(q)$, one concentrates on a family of potentially maximal subgroups, such as those of type $\Gamma L_{n/2}(q^2)$ in $SL_n(q)$, and tries to determine maximality. A great deal is known in this direction: see the results cited in [60], together with [15, 16, 17, 99], amongst others. Again, we have not used these works in our proofs, but have compared our results with them: we mention them in the relevant sections of Chapters 2, 3 and 6 of this book.

An approach which is slightly orthogonal to our present purposes, but which has been used as a tool in the proof of many deep theorems, is to classify subgroups of classical groups containing elements of specified orders. We shall not use these results, so will not provide an extensive list of papers, but the interested reader could start by looking at [35, 36] and the references therein.

Maximal subgroups of non-classical simple groups. See [83, 114] for an excellent survey and introduction, respectively, on the whole of this field. The ATLAS [12] is also an essential reference in this area.

For the alternating groups, the O'Nan–Scott Theorem (see, for example, Chapter 4 of [8]) provides a subgroup classification similar to, but much simpler than, the Aschbacher classification of matrix groups over finite fields, and results of Liebeck, Praeger and Saxl [78] enable us to determine maximality. As in the case of finite matrix groups, we have a final class of almost simple primitive permutation groups which need to be listed individually by degree, and these lists are currently complete up to degree 4095 [18, 98].

The maximal subgroups of the almost simple exceptional Lie type groups have not yet been fully classified, although a great deal is known about them. A discussion of the overall strategy for their classification, and a brief summary of the main theorems in this area, appear in [91, Chapter 29]; whilst further references to the literature can be found in [66, Table 1.3.B]. In particular, the maximal subgroups of all of the simple, and most of the almost simple groups of this type with representations of degree up to 12 have been classified: see [106, Section 15] for the Suzuki groups ${}^2B_2(q)$, [76, 64] for the Ree groups

${}^2G_2(q)$, [2, 14, 64] for $G_2(q)$ and [63] for ${}^3D_4(q)$. In this book we extend these classifications to the remaining almost simple groups with representations of degree at most 12.

The sporadic groups can be handled on a case-by-case basis. Most of the necessary information is available, including references to the literature, in [12] or, more usefully for computational purposes, in [111]. At the time of writing, all of the maximal subgroups of the almost simple groups with socle a sporadic group are known except for almost simple maximal subgroups of the Monster whose socle is one of a small list of groups. See [111] for a description of the state of play: note that it has recently been shown by R.A. Wilson that $L_2(41)$ is a maximal subgroup of the Monster, correcting an earlier error in the literature.

Computational applications. We were partly motivated to carry out this classification by its applications to computational group theory. In [9], results of Kovács, Aschbacher and Scott dating from the mid 1980s are used to reduce the computation of the maximal subgroups of a general finite group G to the case when G is almost simple. Polynomial-time algorithms for constructing the geometric-type maximal subgroups of the classical groups (in all dimensions) are presented in [45] for the linear, unitary and symplectic groups, and in [46] for the orthogonal groups, and they have been implemented in MAGMA [5].

The Class $\mathscr{S}$ subgroups arising from representations of almost simple groups in their defining characteristic are generally moderately straightforward to construct using standard functionality for computing with modules over groups. Most of the quasisimple Class $\mathscr{S}$ subgroups that are not in defining characteristic can be constructed by restricting a representation of a quasisimple group in characteristic 0 to the required finite field. The associated almost simple groups can be constructed over the finite field from a knowledge of the relevant group automorphisms and the computation of module isomorphisms.

Various databases of characteristic 0 representations are available either directly on the web, or via computer algebra systems such as GAP [29] and MAGMA. A facility of this type [111] has been under construction and continuous development for several years now. More recently, Steel [103] has constructed almost all of the characteristic 0 representations in [42] of quasisimple groups in dimensions up to 250. We used data from these databases to carry out some of the calculations needed to complete the classification.

Although the bulk of the arguments used in our classification theorems are theoretical, a substantial number of them make use of computer calculations. These calculations require only small amounts of computer time (generally at most a few seconds) and could be easily carried out using existing functionality and databases in either GAP or MAGMA. The MAGMA commands for each of these individual calculations are given in the files on the webpage `http://www.cambridge.org/9780521138604`, to enable the user to verify them easily. Whenever we use such a calculation in a proof in this book, we refer to it as a "computer calculation" in the text, and direct the user to the individual online file that contains the commands to carry it out. The files have names like `4l34d8calc`, which contains calculations with the 8-dimensional characteristic 0 representation of the group $4^{\cdot}L_3(4)$. The matrices defining the images of these representations are stored in data files, which are also on the website, and are accessed by the commands that carry out the calculations.

We shall assume that the reader has a general knowledge of group theory and of group representation theory as might be acquired from advanced undergraduate courses on these topics. Some knowledge of the general theory of classical groups over finite fields and of their associated bilinear, sesquilinear and quadratic forms would also be helpful, because we shall only briefly summarise what we need for this book. Good sources are the books by Rob Wilson [114] (which also includes a great deal of information about maximal subgroups of simple groups), Don Taylor [108] or Chapter 2 of [66]. We do not require any familiarity with algebraic groups, but the interested reader should consult [91] for an introduction which is especially well-suited to our current purposes.

Finally, in any classification project of this scale, it is inevitable that some mistakes will have slipped into our tables. At the time of publication, we know of no such errors, but an errata list has been created at

`http://www.cambridge.org/9780521138604`,

and we shall keep this up to date. We would be extremely grateful to be informed of any errata.

Acknowledgements. Writing this book has been a long-running project, and a great many people have helped us at various stages of it. We would like to thank Martin Liebeck for kindly agreeing to write the foreword to this volume. Many thanks go to Gunter Malle for his heroic proof-reading of this entire volume, mostly over the Christmas holidays preceding its publication; the book has been vastly improved by his efforts. Special thanks go also to Rob Wilson for proof-reading large portions of the book, and his thoughtful and extremely helpful comments. In addition we would like to thank Jeff Burdges, Peter Cameron, John Cannon, Ian Gent, Steve Linton, Frank Lübeck, Chris Parker and Anna Schroeder. Needless to say, any and all remaining mistakes and infelicities in this volume are entirely the responsibility of the authors! We gratefully acknowlege the financial support that we have received from EPSRC grants GR/S30580/01 and EP/C523229/1, the Mathematisches Forschungsinstitut Oberwolfach, and the seminar budgets of our respective universities. Finally, Derek thanks Cathy for systems support, and Colva thanks Struan for his encouragement, tolerance and sense of humour during the many years it has taken to write this book.

Contents

1
Introduction

1.1 Background

Given a group G, we write $\mathrm{Soc}\, G$ for the *socle* of G: the subgroup of G generated by its minimal normal subgroups. A group G is *almost simple* if $S \leqslant G \leqslant \mathrm{Aut}\, S$ for some non-abelian simple group S. Note that $S = \mathrm{Soc}\, G$. A group G is *perfect* if $G = G'$. A group G is *quasisimple* if G is perfect and $G/\mathrm{Z}(G)$ is a non-abelian simple group.

Aschbacher [1] proves a classification theorem, which subdivides the subgroups of the finite classical groups into nine classes. The first eight of these consist roughly of groups that preserve some kind of geometric structure; for example the first class, $\mathscr{C}_1$, consists (roughly) of the reducible groups, which fix a proper non-zero subspace of the vector space on which the group acts naturally. Subgroups of classical groups that lie in the first eight classes are of *geometric type*. The ninth class, denoted by $\mathscr{C}_9$ or $\mathscr{S}$, consists (roughly) of those absolutely irreducible subgroups that are not of geometric type and which, modulo the central subgroup of scalar matrices, are almost simple.

In [66], Kleidman and Liebeck provide an impressively detailed enumeration of the maximal subgroups of geometric type of the finite classical groups of dimension greater than 12. More precisely, they classify the conjugacy classes of maximal subgroups $\bar{H}$ of those almost simple groups $\overline{G}$ for which $\overline{\Omega} := \mathrm{Soc}\,\overline{G} = \Omega/\mathrm{Z}(\Omega)$ for some classical quasisimple group Ω, with $\bar{H} \cap \overline{\Omega} = K/\mathrm{Z}(\Omega)$ for a subgroup K of Ω of geometric type.

In this book, we determine the maximal subgroups of all such almost simple groups $\overline{G}$ with dimension at most 12. For the subgroups of geometric type, Kleidman and Liebeck proved that their lists contain all such maximal subgroups even in dimensions at most 12. But their determination of when these subgroups are actually maximal applies only to dimensions greater than 12. It turns out that they are nearly all maximal, with just a few exceptions in small dimensions: all of the exceptions are in dimension at most 8.

We do not, however, restrict ourselves to the subgroups of geometric type, and include those subgroups in Aschbacher Class $\mathscr{S}$ in our classification. It is a feature of the groups in this class that they are not, as far as we know, susceptible to a uniform description across all dimensions, but can only be listed for each individual dimension and type of classical group. Fortunately, lists are available of all irreducible representations of degree up to 250 of all finite quasisimple groups G. These have been compiled by Lübeck [84] for representations of G in defining characteristic (when G is a group of Lie type), and by Hiß and Malle [42] for all other representations. These lists provide us with a complete set of candidates for the quasisimple normal subgroups S of maximal subgroups in Class $\mathscr{S}$ of the finite classical groups of dimension up to 250.

We are, however, left with two major problems. Firstly, in order to find the *almost* simple maximal subgroups of the almost simple classical groups $\overline{G} = G/\mathrm{Z}(\Omega)$, we need to determine which of the automorphisms of the simple groups $S/\mathrm{Z}(S)$ in the lists of candidates can be adjoined within $\overline{G}$. Secondly, we need to determine which of the candidates that we construct are actually maximal subgroups of the almost simple groups. Indeed, our approach to the project as a whole follows the same general pattern as [66]: first we find the candidates for the maximal subgroups within each of the nine Aschbacher classes, then we determine which are maximal within their own class, and finally we decide maximality itself by identifying cases in which maximal groups in one class are properly contained in a subgroup in another class.

The $\mathrm{O}_8^+(q)$ case is handled in detail in [62], so we shall not repeat that work here: we will simply reproduce the table of maximal subgroups from [62], but in the format we are using for the remainder of our tables.

Structure of this book. In the remainder of this chapter we present basic results on the structure and representations of simple groups; this material will be required both for the study of geometric type groups and of groups in Class $\mathscr{S}$. Topics covered include: novelty maximal subgroups; finite fields; sesquilinear and quadratic forms, including the specification of our standard forms; introduction to the classical groups, including the specification of our standard outer automorphisms; some relevant representation theory; tensor products; exceptional properties of various small classical groups; permutation and matrix representations of the classical groups; properties of the natural matrix representations of the classical groups; Zsigmondy primes; quadratic reciprocity.

In Chapter 2 we first state our main theorem, Theorem 2.1.1. Then in Section 2.2 we introduce the *types* of geometric subgroups: these are families of subgroups with the property that if H is a geometric maximal subgroup of a quasisisimple classical group, then H is a member of one of these families. For each geometric Aschbacher class, we define the corresponding types, give