Introduction to Differentiable Manifolds

Second Edition

Universitext

Serge Lang

Springer

Serge Lang

Introduction to Differentiable Manifolds

Second Edition

With 12 Illustrations

Springer

Serge Lang
Department of Mathematics
Yale University
New Haven, CT 06520
USA

Mathematics Subject Classification (2000): 58Axx, 34M45, 57Nxx, 57Rxx

Library of Congress Cataloging-in-Publication Data
Lang, Serge, 1927–
Introduction to differentiable manifolds / Serge Lang. — 2nd ed.
p. cm. — (Universitext)
Includes bibliographical references and index.
ISBN 0-387-95477-5 (acid-free paper)
1. Differential topology. 2. Differentiable manifolds. I. Title.
QA649.L3 2002
516.3′6—dc21 2002020940

The first edition of this book was published by Addison-Wesley, Reading, MA, 1972.

ISBN 0-387-95477-5 Printed on acid-free paper.

Printed in the United States of America.

9 8 7 6 5 4 3 2 1 SPIN 10874516

www.springer-ny.com

Springer-Verlag New York Berlin Heidelberg
A member of BertelsmannSpringer Science+Business Media GmbH

Universitext

Springer
New York
Berlin
Heidelberg
Hong Kong
London
Milan
Paris
Tokyo

BOOKS OF RELATED INTEREST BY SERGE LANG

Introduction to Linear Algebra, Second Edition
1986, ISBN 0-387-96205-0

Linear Algebra, Third Edition
1987, ISBN 0-387-96412-6

Undergraduate Algebra, Second Edition
1990, ISBN 0-387-97279-X

Undergraduate Analysis, Second Edition
1997, ISBN 0-387-94841-4

Complex Analysis, Third Edition
1993, ISBN 0-387-97886-0

Math Talks for Undergraduates
1999, ISBN 0-387-98749-5

OTHER BOOKS BY LANG PUBLISHED BY SPRINGER-VERLAG

Math! Encounters with High School Students • The Beauty of Doing Mathematics • Geometry: A High School Course • Basic Mathematics • Short Calculus • A First Course in Calculus • Calculus of Several Variables • Algebra • Real and Functional Analysis • Introduction to Differentiable Manifolds • Fundamentals of Differential Geometry • Algebraic Number Theory • Cyclotomic Fields I and II • Introduction to Diophantine Approximations • $SL_2(\mathbf{R})$ • Spherical Inversion on $SLn(\mathbf{R})$ (with Jay Jorgenson) • Elliptic Functions • Elliptic Curves: Diophantine Analysis • Introduction to Arakelov Theory • Riemann-Roch Algebra (with William Fulton) • Abelian Varieties • Introduction to Algebraic and Abelian Functions • Complex Multiplication • Introduction to Modular Forms • Modular Units (with Daniel Kubert) • Introduction to Complex Hyperbolic Spaces • Number Theory III • Survey on Diophantine Geometry • Collected Papers I–V, including the following: Introduction to Transcendental Numbers in volume I, Frobenius Distributions in GL2-Extensions (with Hale Trotter) in volume II, Topics in Cohomology of Groups in volume IV, Basic Analysis of Regularized Series and Products (with Jay Jorgenson in volume V, and Explicit Formulas for Regularized Products and Series (with Jay Jorgenson) in volume V • THE FILE • CHALLENGES

Foreword

This book is an outgrowth of my *Introduction to Differentiable Manifolds* (1962) and *Differential Manifolds* (1972). Both I and my publishers felt it worth while to keep available a brief introduction to differential manifolds.

The book gives an introduction to the basic concepts which are used in differential topology, differential geometry, and differential equations. In differential topology, one studies for instance homotopy classes of maps and the possibility of finding suitable differentiable maps in them (immersions, embeddings, isomorphisms, etc.). One may also use differentiable structures on topological manifolds to determine the topological structure of the manifold (for example, à la Smale [Sm 67]). In differential geometry, one puts an additional structure on the differentiable manifold (a vector field, a spray, a 2-form, a Riemannian metric, ad lib.) and studies properties connected especially with these objects. Formally, one may say that one studies properties invariant under the group of differentiable automorphisms which preserve the additional structure. In differential equations, one studies vector fields and their integral curves, singular points, stable and unstable manifolds, etc. A certain number of concepts are essential for all three, and are so basic and elementary that it is worthwhile to collect them together so that more advanced expositions can be given without having to start from the very beginnings. The concepts are concerned with the general basic theory of differential manifolds. My *Fundamentals of Differential Geometry* (1999) can then be viewed as a continuation of the present book.

Charts and local coordinates. A chart on a manifold is classically a representation of an open set of the manifold in some euclidean space. Using a chart does not necessarily imply using coordinates. Charts will be used systematically.

I don't propose, of course, to do away with local coordinates. They are useful for computations, and are also especially useful when integrating differential forms, because the $dx_1 \wedge \cdots \wedge dx_n$. corresponds to the $dx_1 \cdots dx_n$ of Lebesgue measure, in oriented charts. Thus we often give the local coordinate formulation for such applications. Much of the literature is still covered by local coordinates, and I therefore hope that the neophyte will thus be helped in getting acquainted with the literature. I also hope to convince the expert that nothing is lost, and much is gained, by expressing one's geometric thoughts without hiding them under an irrelevant formalism.

Since this book is intended as a text to follow advanced calculus, say at the first year graduate level or advanced undergraduate level, manifolds are assumed finite dimensional. Since my book *Fundamentals of Differential Geometry* now exists, and covers the infinite dimensional case as well, readers at a more advanced level can verify for themselves that there is no essential additional cost in this larger context. I am, however, following here my own admonition in the introduction of that book, to assume from the start that all manifolds are finite dimensional. Both presentations need to be available, for mathematical and pedagogical reasons.

New Haven 2002 — Serge Lang

Acknowledgments

I have greatly profited from several sources in writing this book. These sources are from the 1960s.

First, I originally profited from Dieudonné's *Foundations of Modern Analysis*, which started to emphasize the Banach point of view.

Second, I originally profited from Bourbaki's *Fascicule de résultats* [Bou 69] for the foundations of differentiable manifolds. This provides a good guide as to what should be included. I have not followed it entirely, as I have omitted some topics and added others, but on the whole, I found it quite useful. I have put the emphasis on the differentiable point of view, as distinguished from the analytic. However, to offset this a little, I included two analytic applications of Stokes' formula, the Cauchy theorem in several variables, and the residue theorem.

Third, Milnor's notes [Mi 58], [Mi 59], [Mi 61] proved invaluable. They were of course directed toward differential topology, but of necessity had to cover ad hoc the foundations of differentiable manifolds (or, at least, part of them). In particular, I have used his treatment of the operations on vector bundles (Chapter III, §4) and his elegant exposition of the uniqueness of tubular neighborhoods (Chapter IV, §6, and Chapter VII, §4).

Fourth, I am very much indebted to Palais for collaborating on Chapter IV, and giving me his exposition of sprays (Chapter IV, §3). As he showed me, these can be used to construct tubular neighborhoods. Palais also showed me how one can recover sprays and geodesics on a Riemannian manifold by making direct use of the canonical 2-form and the metric (Chapter VII, §7). This is a considerable improvement on past expositions.

Contents

Foreword v

Acknowledgments vii

CHAPTER I

Differential Calculus 1

§1. Categories 2
§2. Finite Dimensional Vector Spaces 4
§3. Derivatives and Composition of Maps 6
§4. Integration and Taylor's Formula 9
§5. The Inverse Mapping Theorem 12

CHAPTER II

Manifolds 20

§1. Atlases, Charts, Morphisms 20
§2. Submanifolds, Immersions, Submersions 23
§3. Partitions of Unity 31
§4. Manifolds with Boundary 34

CHAPTER III

Vector Bundles 37

§1. Definition, Pull Backs 37
§2. The Tangent Bundle 45
§3. Exact Sequences of Bundles 46
§4. Operations on Vector Bundles 52
§5. Splitting of Vector Bundles 57

CHAPTER IV
Vector Fields and Differential Equations ... 60
§1. Existence Theorem for Differential Equations ... 61
§2. Vector Fields, Curves, and Flows ... 77
§3. Sprays ... 85
§4. The Flow of a Spray and the Exponential Map ... 94
§5. Existence of Tubular Neighborhoods ... 98
§6. Uniqueness of Tubular Neighborhoods ... 101

CHAPTER V
Operations on Vector Fields and Differential Forms ... 105
§1. Vector Fields, Differential Operators, Brackets ... 105
§2. Lie Derivative ... 111
§3. Exterior Derivative ... 113
§4. The Poincaré Lemma ... 126
§5. Contractions and Lie Derivative ... 127
§6. Vector Fields and 1-Forms Under Self Duality ... 132
§7. The Canonical 2-Form ... 137
§8. Darboux's Theorem ... 139

CHAPTER VI
The Theorem of Frobenius ... 143
§1. Statement of the Theorem ... 143
§2. Differential Equations Depending on a Parameter ... 148
§3. Proof of the Theorem ... 149
§4. The Global Formulation ... 150
§5. Lie Groups and Subgroups ... 153

CHAPTER VII
Metrics ... 158
§1. Definition and Functoriality ... 158
§2. The Metric Group ... 162
§3. Reduction to the Metric Group ... 165
§4. Metric Tubular Neighborhoods ... 168
§5. The Morse Lemma ... 170
§6. The Riemannian Distance ... 173
§7. The Canonical Spray ... 176

CHAPTER VIII
Integration of Differential Forms ... 180
§1. Sets of Measure 0 ... 180
§2. Change of Variables Formula ... 184
§3. Orientation ... 193
§4. The Measure Associated with a Differential Form ... 195

CHAPTER IX

Stokes' Theorem 200

§1. Stokes' Theorem for a Rectangular Simplex 200
§2. Stokes' Theorem on a Manifold 203
§3. Stokes' Theorem with Singularities 207

CHAPTER X

Applications of Stokes' Theorem 214

§1. The Maximal de Rham Cohomology 214
§2. Volume forms and the Divergence 221
§3. The Divergence Theorem 230
§4. Cauchy's Theorem 234
§5. The Residue Theorem 237

Bibliography 243

Index 247

CHAPTER I

Differential Calculus

We shall recall briefly the notion of derivative and some of its useful properties. My books on analysis [La83/97], [La 93] give a self-contained and complete treatment. We summarize basic facts of the differential calculus. *The reader can actually skip this chapter* and start immediately with Chapter II if the reader is accustomed to thinking about the derivative of a map as a linear transformation. (In the finite dimensional case, when bases have been selected, the entries in the matrix of this transformation are the partial derivatives of the map.) We have repeated the proofs for the more important theorems, for the ease of the reader.

It is convenient to use throughout the language of categories. The notion of category and morphism (whose definitions we recall in §1) is designed to abstract what is common to certain collections of objects and maps between them. For instance, euclidean vector spaces and linear maps, open subsets of euclidean spaces and differentiable maps, differentiable manifolds and differentiable maps, vector bundles and vector bundle maps, topological spaces and continuous maps, sets and just plain maps. In an arbitrary category, maps are called morphisms, and in fact the category of differentiable manifolds is of such importance in this book that from Chapter II on, we use the word morphism synonymously with differentiable map (or p-times differentiable map, to be precise). All other morphisms in other categories will be qualified by a prefix to indicate the category to which they belong.

I, §1. CATEGORIES

A **category** is a collection of objects $\{X, Y, \ldots\}$ such that for two objects X, Y we have a set $\mathrm{Mor}(X, Y)$ and for three objects X, Y, Z a mapping (composition law)

$$\mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$$

satisfying the following axioms:

CAT 1. *Two sets* $\mathrm{Mor}(X, Y)$ *and* $\mathrm{Mor}(X', Y')$ *are disjoint unless* $X = X'$ *and* $Y = Y'$, *in which case they are equal.*

CAT 2. *Each* $\mathrm{Mor}(X, X)$ *has an element* id_X *which acts as a left and right identity under the composition law.*

CAT 3. *The composition law is associative.*

The elements of $\mathrm{Mor}(X, Y)$ are called **morphisms**, and we write frequently $f\colon X \to Y$ for such a morphism. The composition of two morphisms f, g is written fg or $f \circ g$.

Elements of $\mathrm{Mor}(X, X)$ are called **endomorphisms** of X, and we write

$$\mathrm{Mor}(X, X) = \mathrm{End}(X).$$

For a more extensive description of basic facts about categories, see my *Algebra* [La 02], Chapter I, §1. Here we just remind the reader of the basic terminology which we use. The main categories for us will be:

Vector spaces, whose morphisms are linear maps.

Open sets in a finite dimensional vector space over **R**, whose morphisms are differentiable maps (of given degree of differentiability, $C^0, C^1, \ldots, C^\infty$).

Manifolds, with morphisms corresponding to the morphisms just mentioned. See Chapter II, §1.

In any category, a morphism $f\colon X \to Y$ is said to be an **isomorphism** if it has an inverse in the category, that is, there exists a morphism $g\colon Y \to X$ such that fg and gf are the identities (of Y and X respectively). An isomorphism in the category of topological spaces (whose morphisms are continuous maps) has been called a **homeomorphism**. We stick to the functorial language, and call it a **topological isomorphism**. In general, we describe the category to which a morphism belongs by a suitable prefix. In the category of sets, a set-isomorphism is also called a **bijection**. **Warning:** A map $f\colon X \to Y$ may be an isomorphism in one category but not in another. For example, the map $x \mapsto x^3$ from $\mathbf{R} \to \mathbf{R}$ is a C^0-isomorphism, but not a C^1 isomorphism (the inverse is continuous, but not differentiable at the origin). In the category of vector spaces, it is true that a bijective morphism is an isomorphism, but the example we just gave shows that the conclusion does not necessarily hold in other categories.

An **automorphism** is an isomorphism of an object with itself. The set of automorphisms of an object X in a category form a group, denoted by $\mathrm{Aut}(X)$.

If $f\colon X \to Y$ is a morphism, then a **section** of f is defined to be a morphism $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

A **functor** $\lambda\colon \mathfrak{A} \to \mathfrak{A}'$ from a category $\mathfrak{A}$ into a category $\mathfrak{A}'$ is a map which associates with each object X in $\mathfrak{A}$ an object $\lambda(X)$ in $\mathfrak{A}'$, and with each morphism $f\colon X \to Y$ a morphism $\lambda(f)\colon \lambda(X) \to \lambda(Y)$ in $\mathfrak{A}'$ such that, whenever f and g are morphisms in $\mathfrak{A}$ which can be composed, then $\lambda(fg) = \lambda(f)\lambda(g)$ and $\lambda(\mathrm{id}_X) = \mathrm{id}_{\lambda(X)}$ for all X. This is in fact a covariant functor, and a contravariant functor is defined by reversing the arrows (so that we have $\lambda(f)\colon \lambda(Y) \to \lambda(X)$ and $\lambda(fg) = \lambda(g)\lambda(f)$).

In a similar way, one defines functors of many variables, which may be covariant in some variables and contravariant in others. We shall meet such functors when we discuss multilinear maps, differential forms, etc.

The functors of the same variance from one category $\mathfrak{A}$ to another $\mathfrak{A}'$ form themselves the objects of a category $\mathrm{Fun}(\mathfrak{A}, \mathfrak{A}')$. Its morphisms will sometimes be called **natural transformations** instead of functor morphisms. They are defined as follows. If λ, μ are two functors from $\mathfrak{A}$ to $\mathfrak{A}'$ (say covariant), then a natural transformation $t\colon \lambda \to \mu$ consists of a collection of morphisms

$$t_X\colon \lambda(X) \to \mu(X)$$

as X ranges over $\mathfrak{A}$, which makes the following diagram commutative for any morphism $f\colon X \to Y$ in $\mathfrak{A}$:

$$\begin{array}{ccc} \lambda(X) & \xrightarrow{t_X} & \mu(X) \\ {\scriptstyle \lambda(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mu(f)} \\ \lambda(Y) & \xrightarrow[t_Y]{} & \mu(Y) \end{array}$$

Vector spaces form a category, the morphisms being the linear maps. Note that $(E, F) \mapsto L(E, F)$ is a functor in two variables, contravariant in the first variable and covariant in the second. If many categories are being considered simultaneously, then an isomorphism in the category of vector spaces and linear map is called a **linear** isomorphism. We write $\mathrm{Lis}(E, F)$ and $\mathrm{Laut}(E)$ for the vector spaces of linear isomorphisms of E onto F, and the linear automorphisms of E respectively.

The vector space of r-multilinear maps

$$\psi\colon E \times \cdots \times E \to F$$

of E into F will be denoted by $L^r(E, F)$. Those which are symmetric (resp. alternating) will be denoted by $L^r_s(E, F)$ or $L^r_{\text{sym}}(E, F)$ (resp. $L^r_a(E, F)$). **Symmetric** means that the map is invariant under a permutation of its variables. **Alternating** means that under a permutation, the map changes by the sign of the permutation.

We find it convenient to denote by $L(\mathbf{E})$, $L^r(\mathbf{E})$, $L^r_s(\mathbf{E})$, and $L^r_a(\mathbf{E})$ the linear maps of $\mathbf{E}$ into $\mathbf{R}$ (resp. the r-multilinear, symmetric, alternating maps of $\mathbf{E}$ into $\mathbf{R}$). Following classical terminology, it is also convenient to call such maps into $\mathbf{R}$ **forms** (of the corresponding type). If $\mathbf{E}_1, \ldots, \mathbf{E}_r$ and $\mathbf{F}$ are vector spaces, then we denote by $L(\mathbf{E}_1, \ldots, \mathbf{E}_r; \mathbf{F})$ the multilinear maps of the product $\mathbf{E}_1 \times \cdots \times \mathbf{E}_r$ into $\mathbf{F}$. We let:

$$\operatorname{End}(\mathbf{E}) = L(\mathbf{E}, \mathbf{E}),$$

$$\operatorname{Laut}(\mathbf{E}) = \text{elements of End}(\mathbf{E}) \text{ which are invertible in End}(\mathbf{E}).$$

Thus for our finite dimensional vector space E, an element of $\operatorname{End}(\mathbf{E})$ is in $\operatorname{Laut}(\mathbf{E})$ if and only if its determinant is $\neq 0$.

Suppose E, F are given norms. They determine a natural norm on $L(E, F)$, namely for $A \in L(E, F)$, the **operator norm** $|A|$ is the greatest lower bound of all numbers K such that

$$|Ax| \leqq K|x|$$

for all $x \in \mathbf{E}$.

I, §2. FINITE DIMENSIONAL VECTOR SPACES

Unless otherwise specified, **vector spaces** *will be finite dimensional over the real numbers.* Such vector spaces are linearly isomorphic to euclidean space $\mathbf{R}^n$ for some n. They have norms. If a basis $\{e_1, \ldots, e_n\}$ is selected, then there are two natural norms: the euclidean norm, such that for a vector v with coordinates $(x_1, \ldots, x_n)$ with respect to the basis, we have

$$|v|^2_{\text{euc}} = x_1^2 + \cdots + x_n^2.$$

The other natural norm is the sup norm, written $|v|_\infty$, such that

$$|v|_\infty = \max_i |x_i|.$$

It is an elementary lemma that all norms on a finite dimensional vector space $\mathbf{E}$ are equivalent. In other words, if $|\ |_1$ and $|\ |_2$ are norms on $\mathbf{E}$, then there exist constants $C_1, C_2 > 0$ such that for all $v \in \mathbf{E}$ we have

$$C_1|v|_1 \leqq |v|_2 \leqq C_2|v|_1.$$

A vector space with a norm is called a **normed vector space**. They form a category whose morphisms are the norm preserving linear maps, which are then necessarily injective.

By a **euclidean space** we mean a vector space with a positive definite scalar product. A morphism in the euclidean category is a linear map which preserves the scalar product. Such a map is necessarily injective. An isomorphism in this category is called a **metric** or **euclidean isomorphism**. An orthonormal basis of a euclidean vector space gives rise to a metric isomorphism with $\mathbf{R}^n$, mapping the unit vectors in the basis on the usual unit vectors of $\mathbf{R}^n$.

Let $\mathbf{E}$, $\mathbf{F}$ be vector spaces (so finite dimensional over $\mathbf{R}$ by convention). The set of linear maps from $\mathbf{E}$ into $\mathbf{F}$ is a vector space isomorphic to the space of $m \times n$ matrices if $\dim \mathbf{E} = m$ and $\dim \mathbf{F} = n$.

Note that $(\mathbf{E}, \mathbf{F}) \mapsto L(\mathbf{E}, \mathbf{F})$ is a functor, contravariant in $\mathbf{E}$ and covariant in $\mathbf{F}$. Similarly, we have the vector space of multilinear maps

$$L(\mathbf{E}_1, \ldots, \mathbf{E}_r, \mathbf{F})$$

of a product $\mathbf{E}_1 \times \cdots \times \mathbf{E}_r$ into $\mathbf{F}$. Suppose norms are given on all $\mathbf{E}_i$ and $\mathbf{F}$. Then a natural norm can be defined on $L(\mathbf{E}_1, \ldots, \mathbf{E}_r, \mathbf{F})$, namely the norm of a multilinear map

$$A\colon \mathbf{E}_1 \times \cdots \times \mathbf{E}_r \to \mathbf{F}$$

is defined to be the greatest lower bound of all numbers K such that

$$|A(x_1, \ldots, x_r)| \leqq K|x_1| \cdots |x_r|.$$

We have:

Proposition 2.1. *The canonical map*

$$L(\mathbf{E}_1, L(\mathbf{E}_2, \ldots, L(\mathbf{E}_r, \mathbf{F}))) \to L^r(\mathbf{E}_1, \ldots, \mathbf{E}_r, \mathbf{F})$$

from the repeated linear maps to the multilinear maps is a linear isomorphism which is norm preserving.

For purely differential properties, which norms are chosen are irrelevant since all norms are equivalent. The relevance will arise when we deal with metric structures, called Riemannian, in Chapter VII.

We note that a linear map and a multilinear map are necessarily continuous, having assumed the vector spaces to be finite dimensional.

I, §3. DERIVATIVES AND COMPOSITION OF MAPS

For the calculus in vector spaces, see my *Undergraduate Analysis* [La 83/97]. We recall some of the statements here.

A real valued function of a real variable, defined on some neighborhood of 0 is said to be $o(t)$ if

$$\lim_{t \to 0} o(t)/t = 0.$$

Let **E**, **F** be two vector spaces (assumed finite dimensional), and φ a mapping of a neighborhood of 0 in **E** into **F**. We say that φ is **tangent to** 0 if, given a neighborhood W of 0 in **F**, there exists a neighborhood V of 0 in **E** such that

$$\varphi(tV) \subset o(t)W$$

for some function $o(t)$. If both **E**, **F** are normed, then this amounts to the usual condition

$$|\varphi(x)| \leqq |x|\psi(x)$$

with $\lim \psi(x) = 0$ as $|x| \to 0$.

Let **E**, **F** be two vector spaces and U open in **E**. Let $f\colon U \to \mathbf{F}$ be a continuous map. We shall say that f is **differentiable** at a point $x_0 \in U$ if there exists a linear map λ of **E** into **F** such that, if we let

$$f(x_0 + y) = f(x_0) + \lambda y + \varphi(y)$$

for small y, then φ is tangent to 0. It then follows trivially that λ is uniquely determined, and we say that it is the **derivative** of f at x_0. We denote the derivative by $Df(x_0)$ or $f'(x_0)$. It is an element of $L(\mathbf{E}, \mathbf{F})$. If f is differentiable at every point of U, then f' is a map

$$f'\colon U \to L(\mathbf{E}, \mathbf{F}).$$

It is easy to verify the chain rule.

Proposition 3.1. *If $f\colon U \to V$ is differentiable at x_0, if $g\colon V \to W$ is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'\big(f(x_0)\big) \circ f'(x_0).$$

Proof. We leave it as a simple (and classical) exercise.

The rest of this section is devoted to the statements of the differential calculus.

Let U be open in **E** and let $f\colon U \to \mathbf{F}$ be differentiable at each point of U. If f' is continuous, then we say that f is **of class** C^1. We define maps