Finite Element Method and its Applications

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Preface

During the past decades, giant needs for ever more sophisticated mathematical models and increasingly complex and extensive computer simulations have arisen. In this fashion, two indissoluble activities, mathematical modeling and computer simulation, have gained a major status in all aspects of science, technology, and industry. Since the partial differential equations are very important in the theory and practice, mathematicians, physicians and scientists and engineer pay greatly attention to its numerical computations. In this background, Finite element method, as numerical computational methods for solving partial differential equations is established to occur in last century. Within only a few decades, this technique has evolved from one with initial applications in structural engineering to a widely utilized and richly varied computational approach for many scientific and technological areas.

Finite Element; perhaps no other family of approximation methods has had a greater impact on the theory and practice of numerical methods during the twentieth century. Finite element methods have now been used in virtually every conceivable area of engineering that can make use of models of nature characterized by partial differential equations. There are dozens of textbooks, monographs, handbooks, memoirs, and journals devoted to its further study; numerous conferences, symposia, and workshops on various aspects of finite element methodology are held regularly throughout the world. There exist easily over one hundred reference on finite elements today, and this number is growing exponentially with further revelations of the power and versatility of the method. Today, finite element methodology is making significant inroads into fields in which many thought were outside its realm, for example computational fluid dynamics. In time, finite element methods may assume a position in this area of comparable or greater importance than classical finite difference schemes which have long dominated the subject.

As well known, the nature phenomenon can be described by instantaneous and local or global and process methodology. Two difference methodologies describe the same nature phenomenon. A main task of computational mathematics is making transformation from infinite dimensional space into an finite dimensional space, transform a continuum problem into a finite dimensional system of a discrete structure. Finite element is that a field function of infinite dimensional continuum is instead by a finite dimensional system consists of a piecewise polynomials. This means that the base functions of finite element subspace are a system of functions with finite support. Variational formulation shows the relationship any different two local points, an energy of each other acting from two different field functions which can be described by bilinear form $B(\varphi_i, \varphi_j)$ where φ_i, φ_j are based functions associated with different point.

The first part(chapter 1-3) of this book provides computational aspect of the method, second part(chapter 4,5), the mathematical background to the finite element methods and the mathematical fundamental to the method are explored, third part (chapter 6) provides nonstandard finite element method, fourth part (chapter 7-9) is concerning applications to elastic mechanics, fluid mechanics, electro-magnetic field and some engineering problems. The book should not only provide mathematical aspect, computational construct of the method and engineering application, but also it should provide a useful starting point for further research.

Contents

Chapter 1

The Structure of Finite Element Method

The finite element method is a numerical computational method for differential equations and partial differential equations. In order to solve the general field problem by using finite element method, it must pass through the following processes:

1) Find the variational formulation associated with original field problem.

2) Establish finite element subspace. For example, select the element type and associated phase functions.

3) Establish element stiffness matrix, element column and assemble global stiffness matrix-full column.

4) Treatment of the boundary conditions and solving of the system of finite element equations.

5) Come back to the real world.

In this book, the first four processes will be systematic formulations in the first chapter till third chapter.

1.1 Galerkin Variational Principle and Ritz Variational Principle

As an example, we consider the linear elliptic boundary value problem of two dimension,

$$\begin{cases} -\left[\dfrac{\partial}{\partial x}\left(p(x,y)\dfrac{\partial u}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(p(x,y)\dfrac{\partial u}{\partial y}\right)\right]=f(x,y), \\ u|_{\Gamma_1}=0, \\ \left[p(x,y)\dfrac{\partial u}{\partial n}+\sigma(x,y)u\right]_{\Gamma_2}=g(x,y), \end{cases} \tag{1.1.1}$$

where, Ω is a connected domain in R^2, $\partial\Omega=\Gamma_1\cup\Gamma_2$ is a piecewise smooth boundary. Let $\boldsymbol{n}$ denote the unit outward normal vector to $\partial\Omega$ defined almost everywhere on $\partial\Omega$. $p(x,y)\in C^1(\Omega)$, $p(x,y)\geqslant p_0>0$, $\sigma(x,y)\in C^0(\Omega)$ and $\sigma(x,y)\geqslant 0$.

Throughout this chapter we make notation: $C^0(\Omega)=$ the set of all continuous function in an open subset in R^n. $C^k(\Omega)=$ the set of functions $v\in C^0(\Omega)$, whose derivatives of order $\leqslant k$, exist and are continuous;

$$D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n},\quad D_j=\frac{\partial}{\partial x^j},\quad D_j^0=\text{identity},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Assume that $u(x, y) \in C^2(\Omega)$ satisfies (1.1.1) in Ω and on $\partial\Omega$, the function $u(x, y)$ is called *classical solution* of problem (1.1.1).

Next, we consider weak solution of (1.1.1). Define the norm

$$\|u\|_{1,\Omega}^2 = \iint_\Omega (u_x^2 + u_y^2 + u^2) dxdy. \tag{1.1.2}$$

Sobolev space $H^1(\Omega)$ is a closure of $C^\infty(\Omega)$, under the norm (1.1.2) with the inner product

$$(u, v)_1 = \iint_\Omega (u_x v_x + u_y v_y + uv) dxdy, \tag{1.1.3}$$

$H^1(\Omega)$ is a Hilbert space which is called *one order Sobolev space*. Let

$$C_0^\infty(\Omega) = \{v : v \text{ is an infinite differentiable function and support of } v \subset \Omega\},$$

$$H_0^1(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ under the norm (1.1.2)},$$

it is equivalent to

$$H_0^1(\Omega) = \{v : v \in H^1(\Omega),\ v|_{\partial\Omega} = 0\}.$$

In addition, let

$$C_\#^\infty(\Omega) = \{v : v \in C^\infty(\Omega),\ v|_{\Gamma_1} = 0\},$$

$$V(\Omega) = \text{closure of } C_\#^\infty(\Omega) \text{ under the norm (1.1.2)},$$

which is equivalent to

$$V = \{v : v \in H^1(\Omega),\ v|_{\Gamma_1} = 0\}.$$

It is clear that V is a Hilbert space with inner product (1.1.3). Furthermore,

$$H_0^1(\Omega) \subset V \subset H^1(\Omega).$$

Let us introduce bilinear functional

$$B(u, v) = \iint_\Omega (p u_x v_x + p u_y v_y) dxdy + \int_{\Gamma_2} \sigma uv ds, \quad \forall u, v \in H^1(\Omega). \tag{1.1.4}$$

In (1.1.4), fixed u, then $B(u, v)$ is a linear functional of v, while v is fixed, it is a linear functional of u. In other words, suppose $\alpha_1, \alpha_2, \beta_1, \beta_2$ are arbitrary constants, then

$$\begin{aligned} B(\alpha_1 u_1 + \alpha_2 u_2, \beta_1 v_1 + \beta_2 v_2) =& \alpha_1\beta_1 B(u_1, v_1) + \alpha_1\beta_2 B(u_1, v_2) \\ &+ \alpha_2\beta_1 B(u_2, v_1) + \alpha_2\beta_2 B(u_2, v_2), \quad \forall u_1, u_2, v_1, v_2 \in H^1(\Omega). \end{aligned}$$

It is clear that (1.1.4) satisfies