



Earthquake-Resistant Structures

Design and Analytical Aspects

Bruno Crump

Earthquake-Resistant Structures: Design and Analytical Aspects

Edited by **Bruno Crump**

NYRESEARCH
P R E S S

New York

Published by NY Research Press,
23 West, 55th Street, Suite 816,
New York, NY 10019, USA
www.nyresearchpress.com

Earthquake-Resistant Structures: Design and Analytical Aspects

Edited by Bruno Crump

© 2015 NY Research Press

International Standard Book Number: 978-1-63238-119-4 (Hardback)

This book contains information obtained from authentic and highly regarded sources. Copyright for all individual chapters remain with the respective authors as indicated. A wide variety of references are listed. Permission and sources are indicated; for detailed attributions, please refer to the permissions page. Reasonable efforts have been made to publish reliable data and information, but the authors, editors and publisher cannot assume any responsibility for the validity of all materials or the consequences of their use.

The publisher's policy is to use permanent paper from mills that operate a sustainable forestry policy. Furthermore, the publisher ensures that the text paper and cover boards used have met acceptable environmental accreditation standards.

Trademark Notice: Registered trademark of products or corporate names are used only for explanation and identification without intent to infringe.

Printed in China.

Preface

This book has been an outcome of determined endeavour from a group of educationists in the field. The primary objective was to involve a broad spectrum of professionals from diverse cultural background involved in the field for developing new researches. The book not only targets students but also scholars pursuing higher research for further enhancement of the theoretical and practical applications of the subject.

The book provides an overview of the latest developments and advances related to earthquake-resistant structures. It comprises of research works contributed by various experts and researchers in the field of earthquake engineering. The book discusses seismic-resistance design of masonry and reinforcement of concrete structures with safety measurements of strengthening and rehabilitation of existing structures against earthquake loads. It also covers topics dedicated to assessment and rehabilitation of jacket platforms, electromagnetic sensing mechanisms for health assessment of structures, post-earthquake examining of steel buildings in fire environment and response of underground pipes to blast loads. This book will be of help to graduate students, researchers and practicing structural engineers.

It was an honour to edit such a profound book and also a challenging task to compile and examine all the relevant data for accuracy and originality. I wish to acknowledge the efforts of the contributors for submitting such brilliant and diverse chapters in the field and for endlessly working for the completion of the book. Last, but not the least; I thank my family for being a constant source of support in all my research endeavours.

Editor

Contents

	Preface	VII
Chapter 1	Design Principles of Seismic Isolation George C. Lee and Zach Liang	1
Chapter 2	Seismic Design Forces and Risks Junichi Abe, Hiroyuki Sugimoto and Tadatomo Watanabe	37
Chapter 3	Seismic Bearing Capacity of Shallow Foundations Francesco Castelli and Ernesto Motta	57
Chapter 4	Advanced Base Isolation Systems for Light Weight Equipments Chong-Shien Tsai	79
Chapter 5	Seismic Damage Estimation in Buried Pipelines Due to Future Earthquakes – The Case of the Mexico City Water System Omar A. Pineda-Porras and Mario Ordaz	131
Chapter 6	Masonry and Earthquakes: Material Properties, Experimental Testing and Design Approaches Thomas Zimmermann and Alfred Strauss	151
Chapter 7	The Equivalent Non-Linear Single Degree of Freedom System of Asymmetric Multi-Storey Buildings in Seismic Static Pushover Analysis Triantafyllos K. Makarios	179
Chapter 8	Seismic Performance of Masonry Building Xiaosong Ren, Pang Li, Chuang Liu and Bin Zhou	205

Chapter 9	Seismic Response of Reinforced Concrete Columns Halil Sezen and Muhammad S. Lodhi	227
Chapter 10	Seismic Behavior and Retrofit of Infilled Frames Mohammad Reza Tabeshpour, Amir Azad and Ali Akbar Golafshani	251
Chapter 11	Seismic Vulnerability Analysis of RC Buildings in Western China Zhu Jian	279

Permissions

List of Contributors

Design Principles of Seismic Isolation

George C. Lee and Zach Liang

*Multidisciplinary Center for Earthquake Engineering Research,
University at Buffalo, State University of New York
USA*

1. Introduction

In earthquake resistance design of structures, two general concepts have been used. The first is to increase the capacity of the structures to resist the earthquake load effects (mostly horizontal forces) or to increase the dynamic stiffness such as the seismic energy dissipation ability by adding damping systems (both devices and/or structural fuses). The second concept includes seismic isolation systems to reduce the input load effects on structures. Obviously, both concepts can be integrated to achieve an optimal design of earthquake resilient structures. This chapter is focused on the principles of seismic isolation.

It should be pointed out that from the perspective of the structural response control community, earthquake protective systems are generally classified as passive, active and semi-active systems. The passive control area consists of many different categories such as energy dissipation systems, tuned-mass systems and vibration isolation systems. This chapter addresses only the passive, seismic isolation systems [Soong and Dargush, 1997; Takewaki, 2009; Liang et al, 2011]

Using seismic isolation devices/systems to control earthquake induced vibration of bridges and buildings is considered to be a relatively matured technology and such devices have been installed in many structures world-wide in recent decades. Design guidelines have been established and they are periodically improved as new information based on research and/or field observations become available during the past 20-30 years [ATC 1995; SEAONC 1986; FEMA 1997; IBC 2000; ECS 2000; AASHTO 2010, ASCE 2007, 2010].

Besides the United States, base isolation technologies are also used in Japan, Italy, New Zealand, China, as well as many other countries and regions. [Naiem and Kelly, 1999; Komodromos, 2000; Christopoulos, C. and Filiatrault 2006]

Affiliated with the increased use of seismic isolation systems, there is an increased demand of various isolation devices manufactured by different vendors. This growth of installing seismic isolation devices in earthquake engineering has been following the typical pattern experienced in structural engineering development, which begins from a “statics” platform by gradually modifying the design approach to include the seismic effects based on structural dynamics principles as they develop and new field observations on the responses of real-world structures. The process is typically slow because most studies and laboratory observations have been concentrated on the performances of the devices with scaled-down experiments. Results could not be readily scaled-up for design purposes. At the same time, there were very limited field data on the actual performances of seismically isolated

structures. In recent years, some limited successful stories were reported in the literature on the seismic performance of base-isolated bridges and buildings during real earthquakes, as well as reports of unsuccessful cases including the failure of isolation bearings and falling spans of bridges and magnification (rather than reduction) of vibration levels of buildings. These structural failures have not been systematically examined for their contributing factors. Some of them include: construction quality, improper choice of the type of isolation bearings, incomplete design principles and methods, unknown ground motion and soil characteristics, etc. In summary, current practice is mainly based on past research and observations on the performance of the isolation devices themselves, with minimum information on the dynamic performance of the structure-device as a system.

The working principle of seismic isolation may be explained in several ways. It is a general understanding that isolation devices/systems are used to reduce the seismic force introduced base shear. Designers often understand the working principles from the viewpoint of design spectrum in that, when the vibration period of a structure is longer than a certain level, continue to increase the period will reduce the magnitude of the spectral value and thus reduce the base shear accordingly. To qualitatively explain the working principle of seismic isolation in this manner is reasonable, but it is insufficient to use it in actual design. Refinements and additional design principles are necessary. Another commonly used explanation of seismic isolation is the “decoupling” of the superstructure vibration from the ground motions excitations to reduce the vibration of structures. This statement again requires quantitative elaborations from the viewpoint of isolation design. In general, an isolation device/system can be viewed as a low-pass mechanical filter of the structure being isolated, to filter out excitations with the undesirable high frequencies to reduce the level of acceleration. In order to establish the cut-off frequency the period of the isolation system must be carefully addressed, and this requires a basic design principle to guide the design.

In order to reduce the base shear, an isolation system must be allowed to deform. This relative displacement cannot be filtered like the absolute acceleration. In general working range, the longer the relative displacement associated with longer periods, the more reduction in base shear can be achieved, except for the fact that the latter will introduce certain negative effects. The most significant issue of large relative displacement is the large P-delta effect and for falling spans in bridges. In this regard, a design principle is needed to achieve the best compromise in seismic isolation design. In reality, the only approach to effectively reduce the relative displacement is to increase damping, which, in turn, will result in higher level of acceleration. This conflicting demand of controlling acceleration and acceptable displacement in essence defines the limiting range of the effectiveness of seismic isolation systems. Quantitatively this issue can be addressed, and this is an important design principle to be conceptually discussed in this chapter.

In Section 2 of this chapter, several important design issues (e.g. P-delta effect, vertical motions, etc.) will be discussed and seismic isolation design principles will be described. In Section 3, the quantitative basis of treating seismically-isolated structures will be briefly reviewed and simplified models will be established for the dynamic analysis and design of the structure-device system. Design methods will be briefly discussed in Section 4, and a newly developed seismic isolation device to address some of the issues facing today's practice is briefly introduced in Section 5. Finally, the key issues and parameters in seismic isolation is summarized and future research needs are briefly noted.

2. Some issues and principles of seismic isolation

In this section, the theories, design and practical considerations of seismic isolation are briefly discussed.

2.1 State-of-practice on seismic isolation

The principle of base isolation is typically conceptually explained by using figure 2.1.

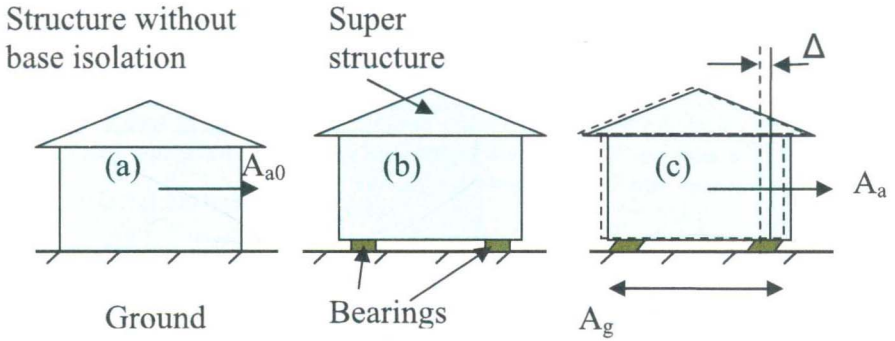


Fig. 2.1. Concept of base isolation

Figure 2.1 (a) and (b) show a structure without and with base isolation, respectively. It is seen that the major difference between (a) and (b) is that in (b), the structure is set on top of isolation several bearings. When the ground moves with acceleration A_g , the superstructure will move accordingly with a displacement Δ , and a reduced level of absolute acceleration, denoted by A_a , rather than the original level A_{a0} .

This principle has been the basis for many research and development efforts in design guidelines and devices for seismic isolation. These have been many important contributions during the past 25 years led by Kelly and his associates at Berkeley and Constantinou and his associates at Buffalo as well as many others. (See references listed at the end of this chapter.)

2.2 Basic concept

The major purpose of using the seismic isolation is to reduce the base-shear of the structure. Physically, large base shear is one of the main reasons of structural damages due to strong horizontal ground accelerations. Thus, to reduce the lateral acceleration is a basic principle.

From the viewpoint of design, many aseismic codes use the base shear as a control parameter. For example, if the base shear of a building is reduced, then the upper story lateral forces floor drifts are also reduced. In the case of a bridge, base shear reduction will minimize damage to the piers.

2.2.1 Base shear

Base shear V can be calculated through various approaches. The following are several examples, first

$$V = C_s W \tag{2.1}$$

where C_s is the seismic response factor and W is the total weight of a structure. Base isolation is intended for reducing C_s , second

$$V = \sum f_{Lj} \text{ (kN)} \quad (2.2)$$

where f_{Lj} is the later force of the j^{th} story of the structure. Base isolation is intended to reduce f_{Lj} simultaneously, so that the base shear will be reduced, in addition

$$V = K_b \Delta \text{ (kN)} \quad (2.3)$$

where K_b is the lateral stiffness of the bearing system; Δ is the nominal relative displacement of the bearing. The stiffness K_b of the bearing system will be much smaller than the structure without the bearing, so that the base shear is reduced.

2.2.2 Lateral acceleration

In equation (2.1), C_s is in fact the normalized lateral absolute acceleration A_a , which is in general zero-valued unless earthquake occurs.

$$C_s = A_a / g \quad (2.4)$$

Also note that

$$W = Mg \text{ (kN)} \quad (2.5)$$

where M is the total mass and $g = 9.8 \text{ m/s}^2$ is the gravity.

In equation (2.2), f_{Lj} is also caused by lateral absolute acceleration a_{aj} of the j^{th} story, that is

$$f_{Lj} = m_j a_{aj} \text{ (kN)} \quad (2.6)$$

where m_j is the mass of the j^{th} floor.

From (2.5) and (2.6), it is seen that, it is difficult to change or reduce the mass M or m_j in a design; however, if the acceleration can be reduced, the lateral forces will be reduced. Therefore, we will focus the discussion on the acceleration.

2.3 Issues of base isolation

Seismic isolation is considered as a relatively matured technology as evidenced by the many practical applications. These applications have been designed based on codes and provisions that have been established incrementally over time. In the following, seismic isolation principles are examined from a structural dynamics perspective with an objective to suggest additional future research needs.

2.3.1 Absolute acceleration vs. relative displacement

Seen in figure 2.1, to achieve the goal of acceleration reduction, in between the ground and the super structure, there will be installed in a group of bearings, which have much soft stiffness so that the period of the total system will be elongated.

Thus, to achieve acceleration reduction, a major sacrifice is the relative displacement between base and structure must be significantly large. Due to nonlinearities of isolation system, the dynamic displacement can be multiple-centered, which can further notably enlarge the displacement. In addition, permanent residual deformation of bearing may worsen the situation.

Generally speaking, the simplest model of a base isolation system can be expressed as

$$M \ddot{a}_a(t) + C \dot{v}(t) + K_b d(t) = 0 \quad (2.7)$$

where $C \dot{v}(t)$ is the viscous damping force and C is the damping coefficient; $v(t)$ is relative velocity.

In most civil engineering structures, the damping force is very small, that is

$$C \dot{v}(t) \approx 0 \quad (2.8)$$

Thus, (2.7) can be re-written as

$$M \ddot{A}_a \approx K_b \Delta \quad (2.9)$$

where A_a and Δ are amplitudes of the absolute acceleration $a_a(t)$ and relative displacement $d(t)$. Equation (2.9) describes the relationship between acceleration and displacement of a single-degree-of-freedom (SDOF) system, which can be used to generate the design spectra. Since the damping force is omitted, the generated acceleration is not exactly real, which is referred to as pseudo acceleration, denoted by A_s . Thus, (2.9) is rewritten as

$$M \ddot{A}_s = K_b \Delta \quad (2.10)$$

Furthermore, we have

$$A_s = \omega_b^2 \Delta \quad (2.11)$$

where ω_b is the angular natural frequency of the isolation system

$$\omega_b = \sqrt{K_b/M} \quad (\text{rad/s}) \quad (2.12)$$

Since the natural period T_b is

$$T_b = 2\pi / \omega_b \quad (\text{s}) \quad (2.13)$$

Equation (2.11) can be rewritten as

$$A_s = 4\pi^2 / T_b^2 \Delta \quad (2.14)$$

From (2.14), the acceleration A_s and displacement Δ are proportional, that is

$$A_s \propto \Delta \quad (2.15)$$

Since A_s and Δ are deterministic functions, (2.15) indicates that between A_s and Δ , only one parameter is needed, usually, the acceleration is considered.

Combine (2.12), (2.13) and (2.14), it is also seen that A_s is proportional to the stiffness K_b , that is

$$A_s \propto K_b \quad (2.16)$$

That is, the weaker the stiffness is chosen, the smaller value of A_s can be achieved, which is the basis for current design practice of seismic isolation.

From the above discussion, it seems that, as long as K_b is smaller than a certain level, the base isolation would be successful.

However, the above mentioned design principle may have several problems if the assumptions and limitations are not examined. First, from (2.10), it is seen that, only if Δ is fixed, (2.16) holds. On the other hand, if V , that is A_a , is fixed, one can have

$$\Delta \propto 1/K_b \quad (2.17)$$

That is, the weaker the stiffness is chosen, the larger value of Δ can result.

A more accurate model will unveil that, to realize the base isolation, only one parameter, say A_a or Δ , is not sufficient. This is because A_a and Δ are actually independent. In fact, both of them are needed. That is, whenever a claim of the displacement being considered in an isolation design, as long as they are not treated as two independent parameters, the design is questionable. Later, why they must be independent will be explained. Here, let us first use certain group of seismic ground motions as excitations applied on a SDOF system, the ground motion are suggested by Naiem and Kelly (1999) and normalized to have peak ground acceleration (PGA) to be 0.4 (g). The responses are mean plus one standard deviation values, plotted in figure 2.2.

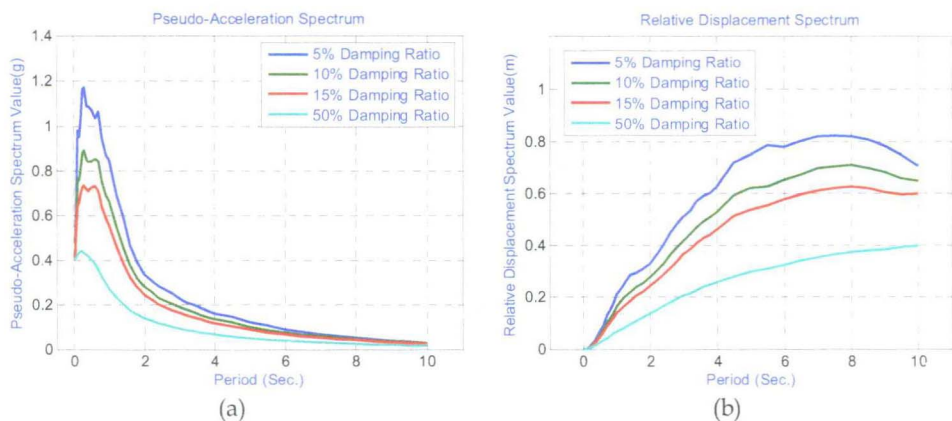


Fig. 2.2. Statistical seismic responses of SDOF systems

Figure 2.2 (a) shows the accelerations of SDOF systems as functions of period for selected damping ratios. When the period becomes larger, the accelerations do reduce, especially when $T_b > 2$ (s). The responses are all smaller than 0.4 (g) and namely, the acceleration is reduced.

Figure 2.2 (b) shows the displacements. It is seen that, when the periods increase, the displacements can become rather large. When $T_b > 2$ (s), the responses can be larger than 0.1 (m), especially if the damping ratio is small, say, 5%.

In figure 2.2 the parameter, damping ratio denoted by ξ is defined by

$$\xi = C / 2\sqrt{MK_b} \quad (2.18)$$

2.3.2 Displacement and center position

Another seismic isolation design issue is the self-centering capacity. Because the SDOF system used to generate the response is linear, and many commercially available bearings are nonlinear systems, the displacement time history can be multiple centered. Figure 2.3 shows examples of a bi-linear (nonlinear) system under Northridge earthquake excitations. The plot in Fig. 2.3 (a) is the displacement time history of the bi-linear system with load and unload stiffness ratio = 0.1 and damping ratio = 0.01. Here the damping ratio is calculated when the system is linear. The plot in Fig 2.3 (b) is the same system with damping ratio = 0.2. It is seen that, the biased deformations exist in both cases. This example illustrates that center shifting can enlarge the displacement significantly, even with heavy damping.

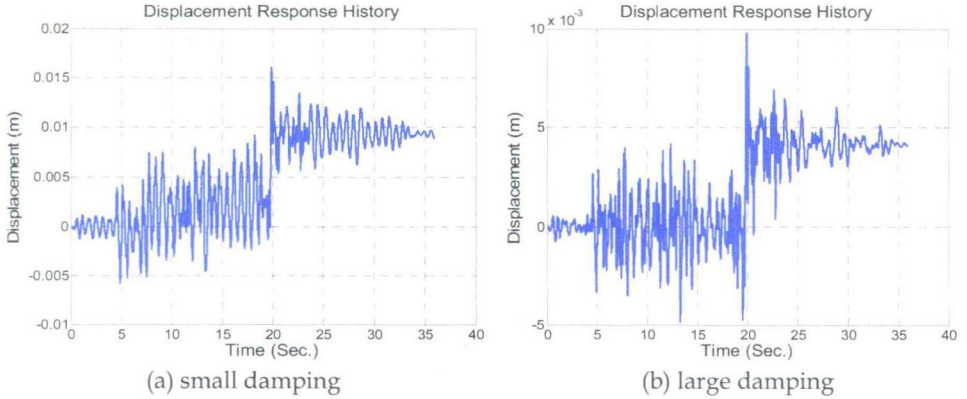


Fig. 2.3. Multiple centers of displacement responses

Briefly speaking, the current isolation design practice can be based on the spectral analysis, dealing with linear systems with the seismic response coefficient C_s given by

$$C_s = A_s / g = \frac{AS}{T_b B} \quad (2.19)$$

and the spectral displacement d_D given by

$$d_D = D = C_s T_b^2 / 4\pi^2 g = \frac{AST_b}{4B} \text{ (m)} \quad (2.20)$$

In the above equations, A is the input level of ground acceleration; S is the site factor; D , instead of Δ , is used to denote the dynamic amplitude and B is called the numerical damping coefficient. Approximately

$$B = 3\xi + 0.9 \quad (2.21)$$

Equations (2.19) and (2.20) will work for most (but not all) situations but cannot handle the position shifting which is a nonlinear response. Some of the nonlinear modeling issues are discussed in Section 3.

2.3.3 SDOF and MDOF models

Many analyses and design of seismic isolations are based on SDOF model, whereas realistic structures are mostly MDOF systems. The acceleration of a higher story of an MDOF system can be much more difficult to reduce.

Equation (2.15) is based on a SDOF system, when the superstructure can be treated as a lumped mass. Namely, the relative deformation among different stories of the structure is negligible. Realistically, such a case is rare; therefore, the acceleration a_{aj} at different stories can be different.

A conventional idea is to decouple a MDOF system into several vibration modes. Each mode is treated as a SDOF system. The total response of the MDOF can then obtained through certain method of modal combinations, such as SRSS method. That is, (2.19) can be rewritten as

$$C_{si} = \frac{AS}{T_{bi} B_i} \quad (2.22)$$

where the subscript i stands for the i^{th} mode. And the i^{th} spectral displacement d_{iD} , (2.20), is rewritten as:

$$d_{iD} = C_{si} T_{bi}^2 / 4\pi^2 g = \frac{AS T_{bi}}{4 B_i} \quad (\text{m}) \quad (2.23)$$

where

$$B_i = 3\xi_i + 0.9 \quad (2.24)$$

Furthermore, for multi-story structures, an additional parameter, mode shape, is needed to distribute the acceleration and displacement at different levels. By denoting the mode shape by \mathbf{P}_i , which is a vector with the j^{th} element representing the model displacement p_{ji} . The acceleration vector \mathbf{A}_{si} is given as

$$\mathbf{A}_{si} = \Gamma_i C_{si} \mathbf{P}_i g = \{a_{ji}\} \quad (\text{m/s}^2) \quad (2.25)$$

where a_{ji} is the acceleration of the i^{th} mode at the j^{th} story and Γ_i is the i^{th} modal participation factor.

The displacement vector is given by

$$\mathbf{d}_{si} = \Gamma_i d_{iD} \mathbf{P}_i = \{d_{ji}\} \quad (\text{m}) \quad (2.26)$$

where d_{ji} is the displacement of the i^{th} mode at the j^{th} story.

2.3.3.1 Acceleration of higher stories

One of the shortcomings of approximating a MDOF system by a SDOF model is the inability to estimate the responses at the higher levels of a MDOF structure. Typically, through SRSS, the acceleration of the j^{th} story of MDOF system can be calculated as

$$a_{sj} = \sqrt{\sum_i a_{ij}^2} \quad (\text{m/s}^2) \quad (2.27)$$

And the displacement of the j^{th} story is

$$d_j = \sqrt{\sum_i d_{ji}^2} \quad (\text{m}) \quad (2.28)$$

The reduction of MDOF accelerations is discussed in the literature. The general conclusion is that, (2.25) and (2.27) may not work well so that the acceleration of the higher story a_{sj} may not be calculated correctly. In fact, a_{sj} can be significantly larger.

The reasons of this inaccuracy mainly come from several factors. The first is the damping effect. The second is the error introduced by using only the first mode for design simplicity (the triangular shape function) as illustrated in figure 2.4

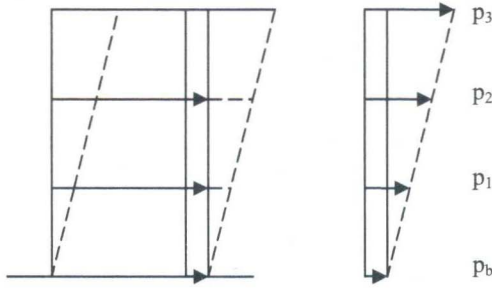


Fig. 2.4. Mode shape function of the 1st mode

In addition, seen from figure 2.4, even though the acceleration of the base, denoted by p_b is rather small by using isolators, the top story will have a rather large acceleration.

2.2.3.2 Cross effects

Another shortcoming of using SDOF models is the inability to estimate the cross effect. Typical MDOF structures have “cross effects” in their dynamic responses. Different from a single member of a structure, which has principal axes, a three-dimensional structure often does not have principal axes. This is conceptually shown in figure 2.5, which is a two story structure. Suppose the first story does have its own principal axes, marked as x_1 - y_1 , and the second story also has its own principal axes, marked as x_2 - y_2 . However, from the top view, if x_1 and x_2 are not pointing exactly the same direction, say, there exists an angle θ , then the entire structure will not have principal axes in general. In this case, the inputs from any two perpendicular directions will cause mutual responses. The resulted displacement will be further magnified. This the third reason of large displacement. At present, there are no available methods to quantify cross effects associated with seismic responses, although in general this effect in base isolation design may be small for regularly shaped structures.

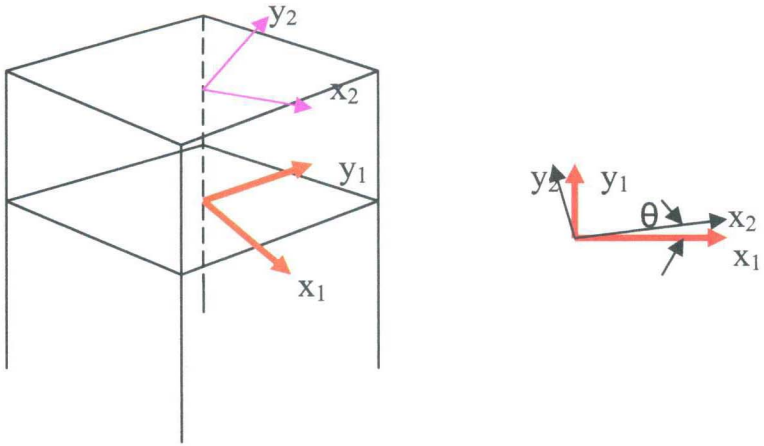
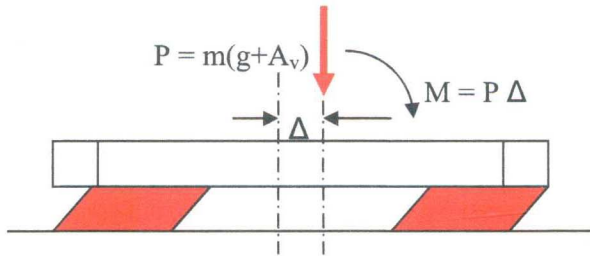


Fig. 2.5. Cross effect

2.3.4 Overturning moment

The product of large vertical load and large bearing displacement forms a large P- Δ effect, conceptually shown in figure 2.6.

Fig. 2.6. Isolation P- Δ effect

This large displacement will result in an overturning moment, given by

$$M = P\Delta = m(g + A_v)\Delta \quad (2.29)$$

where, the vertical load P is a product of the total weight and the vertical acceleration, the vertical acceleration is the sum of gravity g and earthquake induced acceleration A_v . For example, suppose the total mass is 1000 (ton), the displacement is 0.5 (m), and the additional vertical acceleration is 0.4 (g), the total overturning moment will be 6.86 (MN-m). This is a large magnitude, which requires special consideration in the design of foundations, structure base as well as bearings.

2.3.5 Horizontal and vertical vibrations

As mentioned above, the primary purpose of base isolation is to reduce the horizontal load and/or acceleration. By installing bearings, the lateral stiffness will be significantly reduced so the horizontal vibration can be suppressed. Moreover, by using bearings, the vertical

stiffness can also be reduced to a certain level and the vertical vibration can be magnified. Since the earthquake induced vertical load is often not significantly large, the vertical vibration is often ignored in design. However, due to the magnification of vertical acceleration as well as the above-mentioned large overturning moment, care must be taken to check the vertical load. In the worst scenario, there can be an uplift force acting on bearings with many of them not manufactured to take the uplift load (e.g. rubber bearings).

3. Dynamics of seismically isolated MDOF systems

In this section, base isolation is examined from a different perspective, as a second order mechanical filter of a dynamic system. The working principle of base isolation system is to increase the dynamic stiffness of acceleration without sacrifice too much “dynamic stiffness.” Dynamic stiffness is a function of effective period and damping.

3.1 Models of isolation systems

3.1.1 Linear SDOF model

The linear SDOF model is used here to provide a platform to explain the essence of isolation systems under sinusoidal excitation.

A more detailed base isolation model can be rewritten as

$$M a(t) + C v(t) + K_b d(t) = -M a_g(t) \quad (3.1)$$

where $a(t)$ and $a_g(t)$ are respectively the relative and ground accelerations. Note that

$$a_a(t) = a(t) + a_g(t) \quad (3.2)$$

$$a(t) = \dot{v}(t) = \ddot{d}(t) \quad (3.3)$$

where the overhead dot and double dot stand for the first and the second derivatives with respect to time t .

Let the ground displacement $d_g(t)$ be sinusoidal with driving frequency ω_f ,

$$d_g(t) = D \cos(\omega_f t) \quad (3.4)$$

The ground acceleration is

$$a_g(t) = A_g \cos(\omega_f t) = -D_g \omega_f^2 \cos(\omega_f t) \quad (3.5)$$

Here, D_g and A_g are respectively the amplitudes of the displacement and acceleration. The amplitude of steady state responses of the relative displacement D can be written as

$$D = \frac{D_g \frac{\omega_f^2}{\omega_b^2}}{\sqrt{\left(1 - \frac{\omega_f^2}{\omega_b^2}\right)^2 + \left(2\xi \frac{\omega_f}{\omega_b}\right)^2}} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} D_g = \beta_d D_g \quad (3.6)$$