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Klaus Janich

Topology

Klaus Jänich

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Translated by Silvio Levy

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Klaus Jänich
Fakultät für Mathematik
Universität Regensburg
8400 Regensburg
Federal Republic of Germany

Silvio Levy (*Translator*)
Mathematics Department
Princeton University
Princeton, NJ 08544
U.S.A.

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Preface

This volume covers approximately the amount of point-set topology that a student who does not intend to specialize in the field should nevertheless know. This is not a whole lot, and in condensed form would occupy perhaps only a small booklet. Our aim, however, was not economy of words, but a lively presentation of the ideas involved, an appeal to intuition in both the immediate and the higher meanings.

I wish to thank all those who have helped me with useful remarks about the German edition or the original manuscript, in particular, J. Bingener, Guy Hirsch and B. Sagraloff. I thank Theodor Bröcker for donating his "Last Chapter on Set-Theory" to my book; and finally my thanks are due to Silvio Levy, the translator. Usually, a foreign author is not very competent to judge the merits of a translation of his work, but he may at least be allowed to say: I like it.

Regensburg, May 1983 KLAUS JÄNICH

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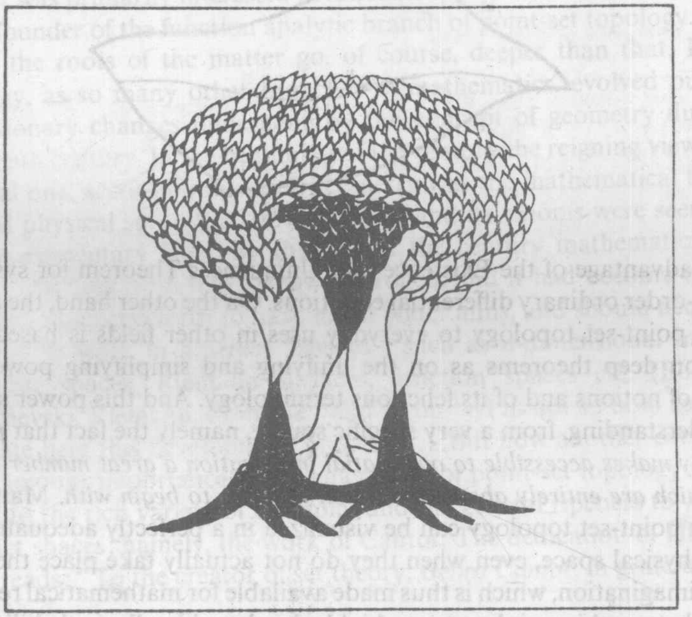
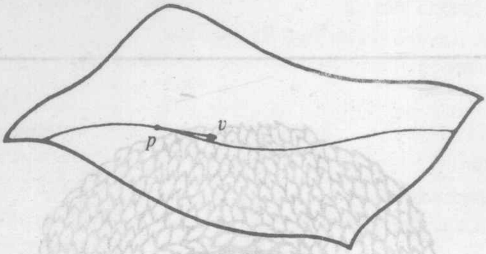
Introduction



§1. What Is Point-Set Topology About?

It is sometimes said that a characteristic of modern science is its high—and ever increasing—level of specialization; every one of us has heard the phrase "only a handful of specialists . . .". Now a general statement about so complex a phenomenon as "modern science" always has the chance of containing a certain amount of truth, but in the case of the above cliché about specialization the amount is fairly small. One might rather point to the great and ever increasing *interweaving* of formerly separated disciplines as a mark of modern science. What must be known today by, say, both a number theorist and a differential geometer, is much more, even relatively speaking, than it was fifty or a hundred years ago. This interweaving is a result of the fact that scientific development again and again brings to light hidden analogies whose further application represents such a great intellectual advance that the theory based on them very soon permeates all fields involved, connecting them together. Point-set topology is just such an analogy-based theory, comprising all that can be said in general about concepts related, though sometimes very loosely, to "closeness", "vicinity" and "convergence".

Theorems of one theory can be instruments in another. When, for instance, a differential geometer makes use of the fact that for each point and direction there is exactly one geodesic (which he does just about every day), he is



taking advantage of the Existence and Uniqueness Theorem for systems of second-order ordinary differential equations. On the other hand, the application of point-set topology to everyday uses in other fields is based not so much on deep theorems as on the unifying and simplifying power of its system of notions and of its felicitous terminology. And this power stems, in my understanding, from a very specific source, namely the fact that *point-set topology makes accessible to our spatial imagination a great number of problems which are entirely abstract and non-intuitive to begin with.* Many situations in point-set topology can be visualized in a perfectly adequate way in usual physical space, even when they do not actually take place there. Our spatial imagination, which is thus made available for mathematical reasoning about abstract things, is however a highly developed intellectual ability which is independent from abstraction and logical thinking; and this strengthening of our other mathematical talents is indeed the fundamental reason for the effectiveness and simplicity of topological methods.

§2. Origin and Beginnings

The emergence of fundamental mathematical concepts is almost always a long and intricate process. To be sure, one can point at a given moment and say: Here this concept, as understood today, is first defined in a clear-cut and plain way, from here on it "exists"—but by that time the concept had always passed through numerous preliminary stages, it was already known in important special cases, variants of it had been considered and discarded, etc., and it is often difficult, and sometimes impossible, to determine which mathematician supplied the decisive contribution and should be considered the originator of the concept in question.

In this sense one might say that the system of concepts of point-set topology "exists" since the appearance of Felix Hausdorff's book *Grundzüge der*

Mengenlehre (Leipzig, 1914). In its seventh chapter, "Point sets in general spaces", are defined the most important fundamental concepts of point-set topology. Maurice Fréchet, in his work "Sur quelques points du calcul fonctionnel" (*Rend. Circ. Mat. Palermo* **22**), had already come close to this mark, introducing the concept of metric spaces and attempting to grasp that of topological spaces as well (by axiomatizing the notion of convergence). Fréchet was primarily interested in function spaces and can perhaps be seen as the founder of the function analytic branch of point-set topology.

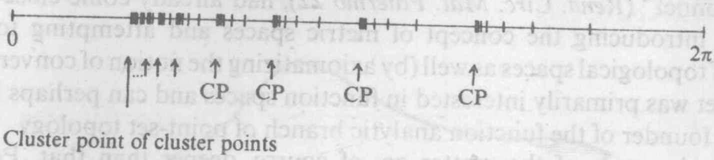
But the roots of the matter go, of course, deeper than that. Point-set topology, as so many other branches of mathematics, evolved out of the revolutionary changes undergone by the concept of geometry during the nineteenth century. In the beginning of the century the reigning view was the classical one, according to which geometry was the mathematical theory of the real physical space that surrounds us, and its axioms were seen as self-evident elementary facts. By the end of the century mathematicians had freed themselves from this narrow approach, and it had become clear that geometry was henceforth to have much wider aims, and should accordingly be made to work in abstract "spaces", such as n-dimensional manifolds, projective spaces, Riemann surfaces, function spaces etc. (Bolyai and Lobachevski, Riemann, Poincaré "and so on"—I'm not so bold as to try to delineate here this development process ...). But now another contribution of paramount importance to the emergence of point-set topology was to be added to the rich variety of examples and the general ripeness to work with abstract spaces: namely, the work of Cantor. The dedication of Hausdorff's book reads: "To the creator of set theory, *Georg Cantor*, in grateful admiration."

"A topological space is a pair consisting of a set and a set of subsets, such that ..."—it is indeed clear that the concept could never have been grasped in such generality were it not for the introduction of abstract sets in mathematics, a development which we owe to Cantor. But long before establishing his transfinite set theory Cantor had contributed to the genesis of point-set topology in an entirely diverse way, about which I would like to add something.

Cantor had shown in 1870 that two Fourier series that converge pointwise to the same limit function have the same coefficients. In 1871 he improved this theorem by proving that the coefficients have to be the same also when convergence and equality of the limits hold for all points outside a finite exception set $A \subset [0, 2\pi]$. In a work of 1872 he now dealt with the problem of determining for which *infinite* exception sets uniqueness would still hold.

An infinite subset of $[0, 2\pi]$ must of course have at least one cluster point:

This is a very "innocent" example of an infinite subset of $[0, 2\pi]$. A somewhat "wilder" set would be one whose cluster points themselves cluster around some point:



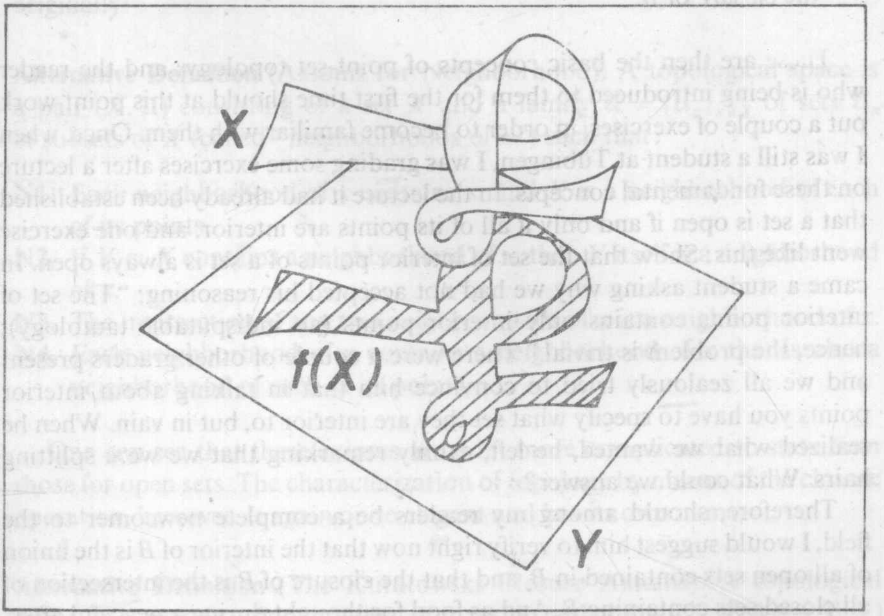
Cantor now showed that if the sequence of subsets of $[0, 2\pi]$ defined inductively by $A^0 := A$ and $A^{n+1} := \{x \in [0, 2\pi] \mid x \text{ is a cluster point of } A^n\}$ breaks up after finitely many terms, that is if eventually we have $A^k = \varnothing$, then uniqueness *does* hold with A as the exception set. In particular a function that vanishes outside such a set (but not identically in the interval) cannot be represented by a Fourier series. This result helps to understand the strange convergence behavior of Fourier series, and the motivation for Cantor's investigation stems from classical analysis and ultimately from physics. But because of it Cantor was led to the discovery of a new type of subset $A \subset \mathbb{R}$, which must have been felt to be quite exotic, especially when the sequence $A, A^1, A^2, \ldots$ takes a *long* time to break off. Now the subsets of $\mathbb{R}$ move to the fore as objects to be studied in themselves, and, what is more, studied from what we would recognize today as being a topological viewpoint. Cantor continued along this path when later, while investigating general point sets in $\mathbb{R}$ and $\mathbb{R}^n$, he introduced the point-set topological approach, upon which Hausdorff could now base himself.

*

I do not want to give the impression that Cantor, Fréchet and Hausdorff were the only mathematicians to take part in the development and clarification of the fundamental ideas of point-set topology; but a more detailed treatment of the subject would be out of the scope of this book. I just wanted to outline, with a couple of sketchy but vivid lines, the starting point of the theory we are about to study.

CHAPTER I

Fundamental Concepts



§1. The Concept of a Topological Space

Definition. A *topological space* is a pair $(X, \mathcal{O})$ consisting of a set X and a set $\mathcal{O}$ of subsets of X (called "open sets"), such that the following axioms hold:

Axiom 1. Any union of open sets is open.

Axiom 2. The intersection of any two open sets is open.

Axiom 3. $\varnothing$ and X are open.

One also says that $\mathcal{O}$ is the *topology* of the topological space $(X, \mathcal{O})$. In general one drops the topology from the notation and speaks simply of a topological space X, as we'll do from now on.

Definition. Let X be a topological space.

(1) $A \subset X$ is called *closed* when $X \setminus A$ is open.

(2) $U \subset X$ is called a *neighborhood* of $x \in X$ if there is an open set V with $x \in V \subset U$.

(3) Let $B \subset X$ be any subset. A point $x \in X$ is called an *interior*, *exterior* or *boundary* (or *frontier*) *point of B*, respectively, according to whether B, $X \setminus B$ or neither is a neighborhood of x.
(4) The set $\mathring{B}$ of the interior points of B is called the *interior* of B.
(5) The set $\bar{B}$ of the points of X which are not exterior points of B is called the *closure* of B.

These are then the basic concepts of point-set topology; and the reader who is being introduced to them for the first time should at this point work out a couple of exercises, in order to become familiar with them. Once, when I was still a student at Tübingen, I was grading some exercises after a lecture on these fundamental concepts. In the lecture it had already been established that a set is open if and only if all of its points are interior, and one exercise went like this: Show that the set of interior points of a set is always open. In came a student asking why we had not accepted his reasoning: "The set of interior points contains only interior points (an indisputable tautology); hence, the problem is trivial." There were a couple of other graders present and we all zealously tried to convince him that in talking about interior points you have to specify what set they are interior to, but in vain. When he realized what we wanted, he left, calmly remarking that we were splitting hairs. What could we answer?

Therefore, should among my readers be a complete newcomer to the field, I would suggest him to verify right now that the interior of B is the union of all open sets contained in B, and that the closure of B is the intersection of all closed sets containing B. And as food for thought during a peaceful afternoon let me add the following remarks.

Each of the three concepts defined above using open sets, namely, "closed sets", "neighborhoods" and "closure", can in its turn be used to characterize openness. In fact, a set $B \subset X$ is open if and only if $X \setminus B$ is closed, if and only if B is a neighborhood of each of its points, and if and only if $X \setminus B$ is equal to its closure. Thus the system of axioms defining a topological space must be expressible in terms of each one of these concepts, for instance:

Alternative Definition for Topological Spaces (Axioms for Closed Sets). A topological space is a pair $(X, \mathscr{A})$ consisting of a set X and a set $\mathscr{A}$ of subsets of X (called "closed sets"), such that the following axioms hold:

A1. Any intersection of closed sets is closed.
A2. The union of any two closed sets is closed.
A3. X and ϕ are closed.

This new definition is equivalent to the old in that $(X, \mathscr{O})$ is a topological space in the sense of the old definition if and only if $(X, \mathscr{A})$ is one in the sense of the new, where $\mathscr{A} = \{X \setminus V \mid V \in \mathscr{O}\}$. Had we given the second definition first, closedness would have become the primary concept, openness following by defining $X \setminus V$ to be open if and only if $V \subset X$ is closed. But the definition of concepts (2)–(5) would have been left untouched and given rise to the same system of concepts that we obtained in the beginning. It has become customary to start with open sets, but the idea of neighborhood is more intuitive, and indeed it was in terms of it that Hausdorff defined these notions originally:

Alternative Definition (Axioms for Neighborhood). A topological space is a pair $(X, \mathfrak{U})$ consisting of a set X and a family $\mathfrak{U} = \{\mathfrak{U}_x\}_{x \in X}$ of sets $\mathfrak{U}_x$ of subsets of X (called "neighborhoods of x") such that:

N1. Each neighborhood of x contains x, and X is a neighborhood of each of its points.
N2. If $V \subset X$ contains a neighborhood of x, then V itself is a neighborhood of x.
N3. The intersection of any two neighborhoods of x is a neighborhood of x.
N4. Each neighborhood of x contains a neighborhood of x that is also a neighborhood of each of its points.

One can see that these axioms are a bit more complicated to state than those for open sets. The characterization of topology by means of the closure operation, however, is again quite elegant and has its own name:

Alternative Definition (The Kuratowski Closure Axioms). A topological space is a pair $(X, {}^{-})$ consisting of a set X and a map ${}^{-}: \mathfrak{P}(X) \to \mathfrak{P}(X)$ from the set of all subsets of X into itself such that:

C1. $\overline{\varnothing} = \varnothing$.
C2. $A \subset \overline{A}$ for all $A \subset X$.
C3. $\overline{\overline{A}} = \overline{A}$ for all $A \subset X$.
C4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all $A, B \in X$.

Formulating what exactly the equivalence of all these definitions means and then proving it is, as we said, left as an exercise to the new reader. We will stick to our first definition.

§2. Metric Spaces

As we know, a subset of $\mathbb{R}^n$ is called open in the usual topology when every point in it is the center of some ball also contained in the set. This definition can be extended in a natural way if instead of $\mathbb{R}^n$ we consider a set X for which the notion of distance is defined; in particular every such space gives rise to a topological space. Let's recall the following

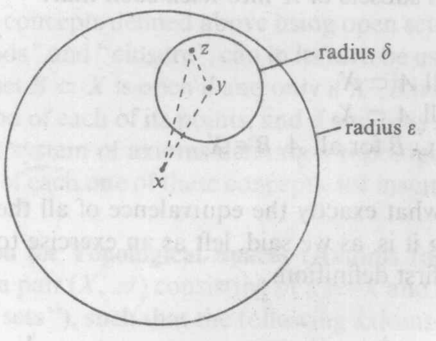
Definition (Metric Space). A metric space is a pair (X, d) consisting of a set X and a real function $d: X \times X \to \mathbb{R}$ (called the "metric"), such that:

M1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
M2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
M3. (Triangle Inequality). $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition (Topology of a Metric Space). Let (X, d) be a metric space. A subset $V \subset X$ is called open if for every $x \in V$ there is an $\varepsilon > 0$ such that the "ε-ball" $K_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$ centered at x is still contained in V. The set $\mathcal{O}(d)$ of all open sets of X is called the topology of the metric space (X, d).

Then $(X, \mathcal{O}(d))$ is really a topological space: and here again our hypothetic novice has an opportunity to practice. But at this point even the more experienced reader could well lean back on his chair, stare at the void and think for a few seconds about what role is played here by the triangle inequality.

So? Well, absolutely none. But as soon as we want to start doing something with these topological spaces $(X, \mathcal{O}(d))$, the inequality will become very useful. It allows us, for example, to draw the conclusion, familiar from $\mathbb{R}^n$, that around each point y such that $d(x, y) < \varepsilon$ there is a small δ-ball entirely contained in the ε-ball around x:

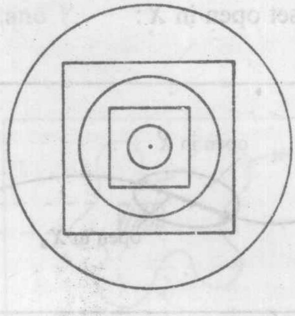


and consequently that the "open ball" $\{y \mid d(x, y) < \varepsilon\}$ is really open, whence in particular a subset $U \subset X$ is a neighborhood of x if and only if it contains a ball centered at x.

Metrics which are very different can in certain circumstances induce the same topology. If d and d' are metrics on X, and if every ball around x in the d metric contains a ball around x in the d' metric, we immediately have that every d-open set is d'-open, that is $\mathcal{O}(d) \subset \mathcal{O}(d')$. If furthermore the converse also holds, then the two topologies are the same: $\mathcal{O}(d) = \mathcal{O}(d')$. An example is the case $X = \mathbb{R}^2$ and

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$d'(x, y) := \max\{|x_1 - y_1|, |x_2 - y_2|\}:$$



And here there is a simple but instructive trick that should be noted right from the start, a veritable talisman against false assumptions about the relationship between metric and topology: If (X, d) is a metric space, then so is (X, d'), where d' is given by $d'(x, y) := d(x, y)/(1 + d(x, y))$; moreover, as can be readily verified, $\mathcal{O}(d) = \mathcal{O}(d')$! But since all distances in d' are less than 1, it follows in particular that if a metric happens to be bounded this property can by no means be traced back to its topology.

Definition (Metrizable Spaces). A topological space $(X, \mathcal{O})$ is called *metrizable* if there is a metric d on X such that $\mathcal{O}(d) = \mathcal{O}$.

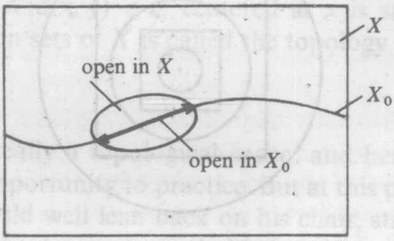
How can one determine whether or not a given topological space is metrizable? This question is answered by the "metrization theorems" of point-set topology. Are all but a few topological spaces metrizable, or is metrizability, on the contrary, a rare special case? The answer is neither, but rather the first than the second: there are a great many metrizable spaces. We will not deal with the metrization theorems in this book, but with the material in Chapters I, VI and VIII the reader will be quite well equipped for the further pursuit of this question.

§3. Subspaces, Disjoint Unions and Products

It often happens that new topological spaces are constructed out of old ones, and the three simplest and most important such constructions will be discussed now.

Definition (Subspace). If $(X, \mathcal{O})$ is a topological space and $X_0 \subset X$ a subset, the topology $\mathcal{O}|X_0 := \{U \cap X_0 | U \in \mathcal{O}\}$ on X_0 is called the *induced* or *subspace topology*, and the topological space $(X_0, \mathcal{O}|X_0)$ is called a *subspace* of $(X, \mathcal{O})$.

Instead of "open with respect to the topology of X_0" one says in short "open in X_0", and a subset $B \subset X_0$ is then open in X_0 if and only if it is the intersection of X_0 with a set open in X:



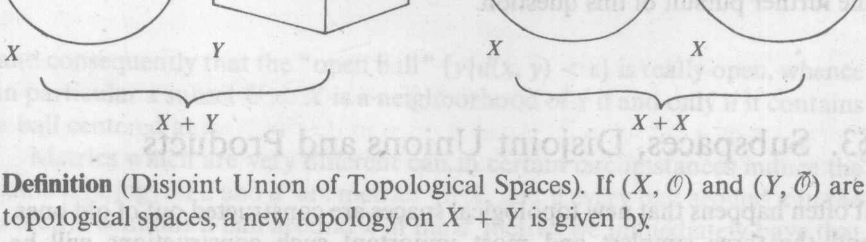
Thus such sets are not to be confused with sets "open and in X_0", since they *do not* have to be open—open, that is, in the topology of X.

Definition (Disjoint Union of Sets). If X and Y are sets, their *disjoint union* or *sum* is defined by means of some formal trick like for instance

$$X + Y := X \times \{0\} \cup Y \times \{1\}$$

—but we immediately start treating X and Y as subsets of $X + Y$, in the obvious way.

Intuitively this operation is nothing more than the disjoint juxtaposition of a copy of X and one of Y, and we obviously cannot write this as $X \cup Y$, since X and Y do not have to be disjoint to begin with, as for example when $X = Y$ and $X \cup X = X$ consists of only one copy of X.



Definition (Disjoint Union of Topological Spaces). If $(X, \mathcal{O})$ and $(Y, \tilde{\mathcal{O}})$ are topological spaces, a new topology on $X + Y$ is given by

$$\{U + V | U \in \mathcal{O}, V \in \tilde{\mathcal{O}}\}$$