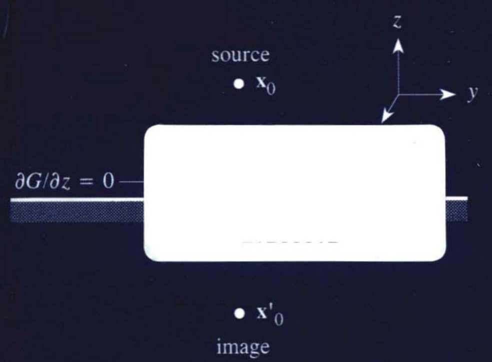

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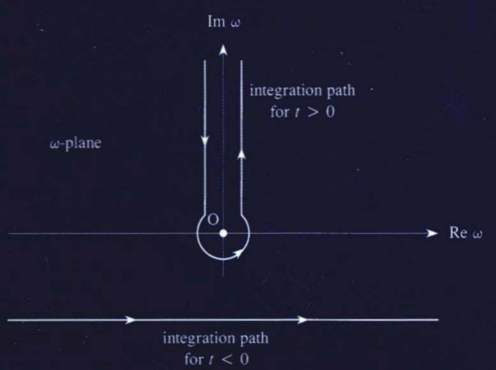

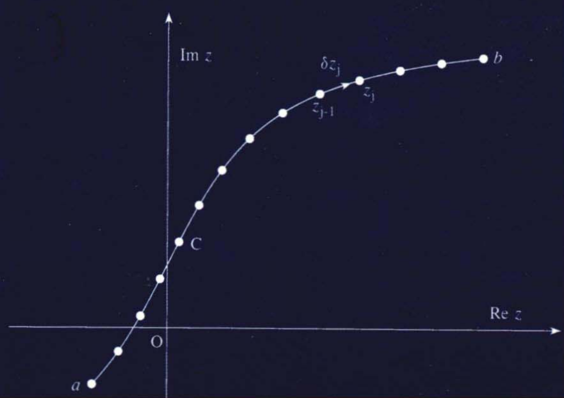

MATHEMATICAL METHODS for MECHANICAL SCIENCES

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MATHEMATICAL METHODS for MECHANICAL SCIENCES

PREFACE

A mathematical model of a physical system provides the engineer with the insight and intuitive understanding generally required to make efficient system design changes or other modifications. A simple formula is often worth a thousand numerical simulations, and can reveal connections between different control parameters that might otherwise take hours or weeks to deduce from a computational analysis. This book is intended to supply the undergraduate engineer with the basic mathematical tools for developing and understanding such models. A firm grasp of the topics covered will also enable the working engineer (educated to bachelor's degree level) to understand, write and otherwise make sensible use of technical reports and papers.

The book was orginally written for students taking the Boston University senior level, one-semester course in engineering mathematics for mechanical and aerospace engineers. This course marks the final exposure of these students to formal mathematical training prior to graduation, and includes material taken principally from Chapters 1–4. The intention is to consolidate earlier courses in ordinary differential equations, vector calculus, Fourier series and transforms, and linear algebra, and to introduce more advanced topics, including complex variable theory, partial differential equations and elementary generalised functions leading to Green's functions. The book has also formed the basis of a review course for first-year engineering graduate students. It is not possible to cover in a one-semester class all subjects with which an 'educated' engineer might reasonably be expected to be familiar; additional topics are included in the text, mainly for reference, on conformal transformations, special functions and variational methods. However, an overriding objective has been compactness of presentation, and to avoid the currently fashionable trend of attempting to achieve encyclopaedic coverage with a text that typically runs to a thousand or more pages.

M. S. Howe

GREEK ALPHABET

alpha	α, A	nu	ν, N
beta	β, B	xi	ξ, Ξ
gamma	γ, Γ	omicron	o, O
delta	δ, Δ	pi	π, Π
epsilon	ϵ, E	rho	ρ, P
zeta	ζ, Z	sigma	σ, Σ
eta	η, H	tau	τ, T
theta	θ, Θ	upsilon	υ, Υ
iota	ι, I	phi	ϕ, Φ
kappa	κ, K	chi	χ, X
lambda	λ, Λ	psi	ψ, Ψ
mu	μ, M	omega	ω, Ω

MATHEMATICAL CONSTANTS

Euler's	$\gamma = 0.5772\ 15665$
Exponential	$\mathrm{e} = 2.7182\ 81828$
	$\pi = 3.1415\ 92654$

CONTENTS

1

LINEAR ORDINARY DIFFERENTIAL EQUATIONS

1.1 First-Order Equations

General form:

$$\frac{dy}{dx} + p(x)y = r(x), \quad \text{or} \quad y' + p(x)y = r(x), \quad \text{where } y' = \frac{dy}{dx}.$$

Homogeneous equation $y' + p(x)y = 0$.

Solve by separating the variables:

$$\int \frac{dy}{y} = -\int p(x)dx + C_1, \quad C_1 = \text{constant}$$

$$\therefore \quad \ln y = -\int p(x)dx + C_1$$

$\therefore$ The general solution is $y = C\mathrm{e}^{-\int p(x)dx}$,

$$C = \mathrm{e}^{C_1} = \text{arbitrary constant}$$

This solution may also be derived by means of an integrating factor, as described below for the inhomogeneous equation.

Example Find the general solution of $y' + x^2 y = 0$.

$$\int \frac{dy}{y} = -\int x^2 dx + C_1,$$

$$\therefore \quad \ln y = -\frac{1}{3}x^3 + C_1$$

$$\therefore \quad y = C\mathrm{e}^{-\frac{x^3}{3}}.$$

If $y = 2$ when $x = 0$, then $C = 2$ and $y = 2\mathrm{e}^{-\frac{x^3}{3}}$.

Inhomogeneous equation $y' + p(x)y = r(x)$.

This is solved by multiplying by the integrating factor $f(x) \equiv e^{\int p(x)dx}$:

$$fy' + fpy \equiv \frac{d}{dx}\left(y(x)e^{\int p(x)dx}\right) = r(x)e^{\int p(x)dx}$$

$$\therefore \quad y(x)e^{\int p(x)dx} = \int r(x)e^{\int p(x)dx}\, dx + C$$

$$\therefore \quad y = e^{-\int p(x)dx}\int r(x)e^{\int p(x)dx}\, dx + Ce^{-\int p(x)dx}$$

$$= \text{particular integral}$$

$$+ \text{ solution of the homogeneous equation}$$

Example Find the general solution of $y' + x^2y = x^2$.

$$\text{Integrating factor } = e^{\int x^2 dx} = e^{\frac{x^3}{3}}$$

$$\therefore \quad \frac{d}{dx}\left(y(x)e^{\frac{x^3}{3}}\right) = x^2e^{\frac{x^3}{3}}$$

$$\therefore \quad y(x)e^{\frac{x^3}{3}} = \int x^2e^{\frac{x^3}{3}}\, dx + C$$

$$\therefore \quad y = 1 + Ce^{-\frac{x^3}{3}}$$

If $y = 2$ when $x = 0$, then $C = 1$ and $y = 1 + e^{-\frac{x^3}{3}}$.

Problems 1A

Find the general solution of:

1. $y' - 4y = 2x - 4x^2$. $[y = x^2 + Ce^{4x}]$
2. $xy' + 2y = 4e^{x^2}$. $[y = (C + 2e^{x^2})/x^2]$
3. $y' + 2y\tan x = \sin^2 x$. $[y = C\cos^2 x + \cos^2 x(\tan x - x)]$
4. $y' + y\cot x = \sin 2x$. $[y = \frac{2}{3}\sin^2 x + C\text{cosec}x]$
5. $\sin xy' - y\cos x = \sin 2x$. $[y = 2\sin x\ln(\sin x) + C\sin x]$
6. $x\ln xy' + y = 2\ln x$. $[y = \ln x + C/\ln x]$
7. $y' + \frac{2y}{x} = e^x$. $[y = C/x^2 + (1 - 2/x + 2/x^2)e^x]$
8. $(x-1)y' + 3y = x^2$. $[(x-1)^3y = C + x^5/5 - x^4/2 + x^3/3]$
9. $(x+1)y' + (2x-1)y = e^{-2x}$. $[e^{2x}y = C(x+1)^3 - \frac{1}{3}]$

10. $y' + \frac{y}{x} = \frac{1}{2}\sin\left(\frac{x}{2}\right)$. $\left[y = -\cos\frac{x}{2} + \frac{2}{x}\sin\frac{x}{2} + \frac{C}{x}\right]$
11. $(1-x^2)y' + x(y-a) = 0$. $[y = a + C(1-x^2)^{\frac{1}{2}}]$
12. $y' - (1+\cot x)y = 0$. $[y = Ce^x \sin x]$
13. $(1+x^2)y' + xy = 3x + 3x^3$. $[y = 1 + x^2 + C(1+x^2)^{-\frac{1}{2}}]$
14. $\sin x \cos x y' + y = \cot x$. $[y = (C + \ln\tan x)/\tan x]$

Solve:

15. $y' + 2xy = 4x,\ y(0) = 3$. $[y = 2 + e^{-x^2}]$
16. $y' \coth 2x = 2y - 2,\ y(0) = 0$. $[y = 1 - \cosh 2x]$
17. $y' + ky = e^{-kx},\ y(0) = 1$. $[y = (1+x)e^{-kx}]$
18. $y' = a(y-g),\ y(0) = b$. $[y = g + (b-g)e^{ax}]$
19. $yy' = 2a,\ y(0) = 0$. $[y^2 = 4ax]$
20. $yy' + x = 0,\ y(0) = a$. $[x^2 + y^2 = a^2]$
21. $yy' + \frac{b^2 x}{a^2} = 0,\ y(0) = b$. $\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right]$
22. $(x+1)y' = y - 3,\ y(0) = 8$. $[y = 5x + 8]$
23. $2xy' + y = 0,\ y(1) = 1$. $[xy^2 = 1]$
24. $(1+x^2)y' = \sqrt{y},\ y(0) = 0$. $[y = \frac{1}{4}(\tan^{-1} x)^2]$
25. $\frac{di}{dt} + 3i = \sin 2t$, $i = 0$ when $t = 0$. $[i = \{\sin(2t - \alpha) + e^{-3t}\sin\alpha\}/\sqrt{13}$, where $\tan\alpha = \frac{2}{3}]$
26. Water runs out through a hole in the base of a circular cylindrical tank at speed $\sqrt{2gh}$ ft/s, where $g = 32\,\text{ft/s}^2$ and h is the water depth. If the tank is 2 ft in height, 1 ft in diameter and is full at time $t = 0$, calculate the time at which half the water has run out when the effective area of the hole is $0.25\,\text{in}^2$. [47 s]
27. The current i in a circuit satisfies $L\,di/dt + Ri = E$, where L, R, E are constants. Show that when t is large the current is approximately equal to E/R.

    If, instead, $E = E_o \cos\omega t$, where E_o, ω are constants, show that when t is large

$$i \approx \frac{E_o \cos(\omega t - \epsilon)}{\sqrt{R^2 + \omega^2 L^2}}, \quad \text{where } \tan\epsilon = \frac{\omega L}{R}.$$

1.2 Second-Order Equations with Constant Coefficients

Homogeneous equation $y'' + ay' + by = 0$, a, $b =$ constants.

Inhomogeneous equation $y'' + ay' + by = r(x)$.

General solution:

$$y = Ay_1(x) + By_2(x) + y_p(x), \quad A,\ B = \text{constant}$$

where $y_1,\ y_2$ are any two linearly independent solutions of the homogeneous equation, called *basis functions* or *complementary functions*, and y_p is a *particular integral* that yields $r(x)$ when substituted into the equation.

Solution of the homogeneous equation Because $d(\mathrm{e}^{\lambda x})/dx = \lambda \mathrm{e}^{\lambda x}$, $y = \mathrm{e}^{\lambda x}$ will be a solution of the homogeneous equation if λ is a solution of the *characteristic equation*

$$\lambda^2 + a\lambda + b = 0, \quad \text{i.e. for } \lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \lambda_1,\ \lambda_2. \qquad (1.2.1)$$

Case 1 $\lambda_1 \neq \lambda_2$:

$y_1 = \mathrm{e}^{\lambda_1 x}$ and $y_2 = \mathrm{e}^{\lambda_2 x}$ are linearly independent and the general solution is therefore

$$y = A\mathrm{e}^{\lambda_1 x} + B\mathrm{e}^{\lambda_2 x}. \qquad (1.2.2)$$

The values of the constants $A,\ B$ are fixed by the *boundary conditions*.

Example Solve $y'' + 2y' - 8y = 0, \quad y(0) = 1,\ y'(0) = 0.$

$$\text{Characteristic equation}: \lambda^2 + 2\lambda - 8 = 0$$

$$\therefore \quad \lambda = -4,\ 2$$

$$\therefore \quad y(x) = A\mathrm{e}^{-4x} + B\mathrm{e}^{2x}.$$

$$\text{At } x = 0: \ y = 1, \text{ and } y' = 0$$

$$\therefore \quad A + B = 1$$

$$\text{and} \quad -4A + 2B = 0.$$

$$\therefore \quad y = \frac{\mathrm{e}^{-4x} + 2\mathrm{e}^{2x}}{3}.$$

Case 2 $\lambda_1 = \lambda_2 \equiv \lambda$:

The two solutions in (1.2.2) are not independent. The differential equation can now be written in the factored form

$$y'' + ay' + b \equiv \left(\frac{d}{dx} - \lambda\right)\left(\frac{d}{dx} - \lambda\right) y = 0.$$

If $z = \dfrac{dy}{dx} - \lambda y$, then $z' - \lambda z = 0$, i.e. $z = B\mathrm{e}^{\lambda x}$, $B = \text{constant}$,

$$\therefore \quad y' - \lambda y = B\mathrm{e}^{\lambda x}.$$