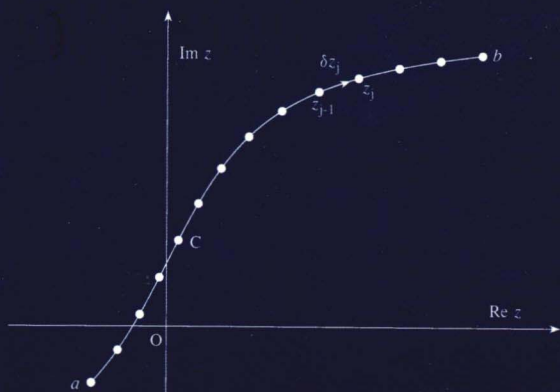
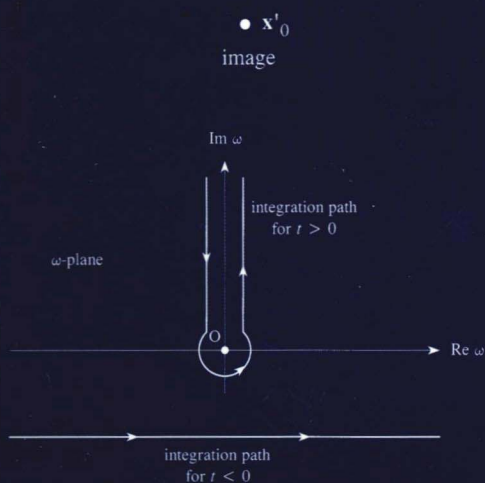


Michael Howe

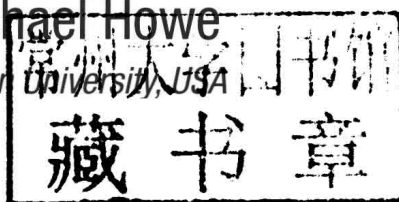


MATHEMATICAL METHODS for MECHANICAL SCIENCES

Imperial College Press

MATHEMATICAL METHODS for MECHANICAL SCIENCES

Michael Howe
Boston University, USA



Published by

Imperial College Press
57 Shelton Street
Covent Garden
London WC2H 9HE

Distributed by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Howe, Michael (Acoustical engineer)

Mathematical methods for mechanical sciences / Michael Howe (Boston University, USA).
pages cm

Includes bibliographical references.

ISBN 978-1-78326-664-7 (alk. paper)

1. Engineering mathematics--Textbooks. 2. Engineering--Mathematical models. 3. Engineering--Study and teaching (Higher). 4. Differential equations--Textbooks. I. Title.

TA347.D45H69 2015

515--dc23

2015019890

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Copyright © 2016 by Imperial College Press

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

In-house Editors: Thomas Stottor/Chandrima Maitra

Typeset by Stallion Press
Email: enquiries@stallionpress.com

Printed in Singapore by Mainland Press Pte Ltd.

**MATHEMATICAL
METHODS for
MECHANICAL
SCIENCES**

PREFACE

A mathematical model of a physical system provides the engineer with the insight and intuitive understanding generally required to make efficient system design changes or other modifications. A simple formula is often worth a thousand numerical simulations, and can reveal connections between different control parameters that might otherwise take hours or weeks to deduce from a computational analysis. This book is intended to supply the undergraduate engineer with the basic mathematical tools for developing and understanding such models. A firm grasp of the topics covered will also enable the working engineer (educated to bachelor's degree level) to understand, write and otherwise make sensible use of technical reports and papers.

The book was originally written for students taking the Boston University senior level, one-semester course in engineering mathematics for mechanical and aerospace engineers. This course marks the final exposure of these students to formal mathematical training prior to graduation, and includes material taken principally from Chapters 1–4. The intention is to consolidate earlier courses in ordinary differential equations, vector calculus, Fourier series and transforms, and linear algebra, and to introduce more advanced topics, including complex variable theory, partial differential equations and elementary generalised functions leading to Green's functions. The book has also formed the basis of a review course for first-year engineering graduate students. It is not possible to cover in a one-semester class all subjects with which

an ‘educated’ engineer might reasonably be expected to be familiar; additional topics are included in the text, mainly for reference, on conformal transformations, special functions and variational methods. However, an overriding objective has been compactness of presentation, and to avoid the currently fashionable trend of attempting to achieve encyclopaedic coverage with a text that typically runs to a thousand or more pages.

M. S. Howe

GREEK ALPHABET

alpha	α, A	nu	ν, N
beta	β, B	xi	ξ, Ξ
gamma	γ, Γ	omicron	o, O
delta	δ, Δ	pi	π, Π
epsilon	ϵ, E	rho	ρ, P
zeta	ζ, Z	sigma	σ, Σ
eta	η, H	tau	τ, T
theta	θ, Θ	upsilon	v, Υ
iota	ι, I	phi	ϕ, Φ
kappa	κ, K	chi	χ, X
lambda	λ, Λ	psi	ψ, Ψ
mu	μ, M	omega	ω, Ω

MATHEMATICAL CONSTANTS

Euler's	$\gamma = 0.5772\ 15665$
Exponential	$e = 2.7182\ 81828$
	$\pi = 3.1415\ 92654$

CONTENTS

PREFACE	ix
1. LINEAR ORDINARY DIFFERENTIAL EQUATIONS	1
1.1 First-Order Equations	1
1.2 Second-Order Equations with Constant Coefficients	3
1.3 Euler's Homogeneous Equation	6
1.4 Method of Reduction of Order	7
1.5 Particular Integrals of Second-Order Equations	9
1.6 Method of Variation of Parameters	13
1.7 Method of Frobenius	15
1.8 Bessel Functions of Integer Order	25
1.9 The Sturm–Liouville Equation	27
1.10 Fourier Series	32
1.11 Generalised Functions and Green's Function	38
2. VECTOR CALCULUS	53
2.1 Elementary Operations with Vectors	53
2.2 Scalar and Vector Fields	56
2.3 The Divergence and the Divergence Theorem	59
2.4 Stokes' Theorem and Curl	64
2.5 Green's Identities	68
2.6 Orthogonal Curvilinear Coordinates	70

2.7	Evaluation of Line and Surface Integrals	74
2.8	Suffix Notation	79
3.	COMPLEX VARIABLES	83
3.1	Complex Numbers	83
3.2	Functions of a Complex Variable	87
3.3	Integration in the Complex Plane	92
3.4	Cauchy's Theorem	97
3.5	Cauchy's Integral Formula	101
3.6	Taylor's Theorem	103
3.7	Laurent's Expansion	104
3.8	Poles and Essential Singularities	106
3.9	Cauchy's Residue Theorem	107
3.10	Applications of the Residue Theorem to Evaluate Real Integrals	111
3.11	Contour Integration Applied to the Summation of Series	121
3.12	Conformal Representation	123
3.13	Laplace's Equation in Two Dimensions	128
3.14	Applications to Hydrodynamics	131
4.	PARTIAL DIFFERENTIAL EQUATIONS	139
4.1	Classification of Second-Order Equations	139
4.2	Boundary Conditions for Well-Posed Problems	144
4.3	Method of Separation of Variables	147
4.4	Problems with Cylindrical Boundaries	157
4.5	Application of Green's Second Identity: Green's Function	163
4.6	The Dirac Delta Function in Three Dimensions	166
4.7	The Method of Images	168
4.8	Green's Function for the Wave Equation	172
4.9	Fourier Transforms	179
4.10	Application of Fourier Transforms to the Solution of Partial Differential Equations	187

5. SPECIAL FUNCTIONS	205
5.1 The Gamma Function $\Gamma(x)$	205
5.2 The Beta Function	210
5.3 Legendre Polynomials	213
5.4 The Error Function $\operatorname{erf}(x)$	224
6. MATRIX ALGEBRA AND LINEAR EQUATIONS	227
6.1 Definitions	227
6.2 Algebra of Matrices	228
6.3 Linear Equations	230
6.4 Further Discussion of Compatibility	237
6.5 Determinants	239
6.6 Inverse of a Square Matrix	242
6.7 Cramer's Rule	246
6.8 Eigenvalue Problems	253
6.9 Real Symmetric Matrices	258
6.10 The Cayley–Hamilton Equation	267
7. VARIATIONAL CALCULUS	271
7.1 Taylor's Theorem for Several Variables	271
7.2 Maxima and Minima	274
7.3 Constrained Maxima and Minima: Lagrange Multipliers	280
7.4 Stationary Definite Integrals	287
7.5 Isoperimetric Problems	294
USEFUL FORMULAE	299
BIBLIOGRAPHY	307
INDEX	311

1

LINEAR ORDINARY DIFFERENTIAL EQUATIONS

1.1 First-Order Equations

General form:

$$\frac{dy}{dx} + p(x)y = r(x), \quad \text{or} \quad y' + p(x)y = r(x), \quad \text{where } y' = \frac{dy}{dx}.$$

Homogeneous equation $y' + p(x)y = 0$.

Solve by separating the variables:

$$\begin{aligned} \int \frac{dy}{y} &= - \int p(x)dx + C_1, \quad C_1 = \text{constant} \\ \therefore \ln y &= - \int p(x)dx + C_1 \end{aligned}$$

\therefore The general solution is $y = Ce^{-\int p(x)dx}$,

$$C = e^{C_1} = \text{arbitrary constant}$$

This solution may also be derived by means of an integrating factor, as described below for the inhomogeneous equation.

Example Find the general solution of $y' + x^2y = 0$.

$$\begin{aligned} \int \frac{dy}{y} &= - \int x^2dx + C_1, \\ \therefore \ln y &= -\frac{1}{3}x^3 + C_1 \\ \therefore y &= Ce^{-\frac{x^3}{3}}. \end{aligned}$$

If $y = 2$ when $x = 0$, then $C = 2$ and $y = 2e^{-\frac{x^3}{3}}$.

Inhomogeneous equation $y' + p(x)y = r(x)$.

This is solved by multiplying by the integrating factor $f(x) \equiv e^{\int p(x)dx}$:

$$fy' + fpy \equiv \frac{d}{dx} \left(y(x)e^{\int p(x)dx} \right) = r(x)e^{\int p(x)dx}$$

$$\therefore y(x)e^{\int p(x)dx} = \int r(x)e^{\int p(x)dx} dx + C$$

$$\therefore y = e^{-\int p(x)dx} \int r(x)e^{\int p(x)dx} dx + Ce^{-\int p(x)dx}$$

= particular integral

+ solution of the homogeneous equation

Example Find the general solution of $y' + x^2y = x^2$.

$$\text{Integrating factor} = e^{\int x^2 dx} = e^{\frac{x^3}{3}}$$

$$\therefore \frac{d}{dx} \left(y(x)e^{\frac{x^3}{3}} \right) = x^2 e^{\frac{x^3}{3}}$$

$$\therefore y(x)e^{\frac{x^3}{3}} = \int x^2 e^{\frac{x^3}{3}} dx + C$$

$$\therefore y = 1 + Ce^{-\frac{x^3}{3}}$$

If $y = 2$ when $x = 0$, then $C = 1$ and $y = 1 + e^{-\frac{x^3}{3}}$.

Problems 1A

Find the general solution of:

1. $y' - 4y = 2x - 4x^2$. $[y = x^2 + Ce^{4x}]$
2. $xy' + 2y = 4e^{x^2}$. $[y = (C + 2e^{x^2})/x^2]$
3. $y' + 2y \tan x = \sin^2 x$. $[y = C \cos^2 x + \cos^2 x(\tan x - x)]$
4. $y' + y \cot x = \sin 2x$. $[y = \frac{2}{3} \sin^2 x + C \operatorname{cosec} x]$
5. $\sin xy' - y \cos x = \sin 2x$. $[y = 2 \sin x \ln(\sin x) + C \sin x]$
6. $x \ln xy' + y = 2 \ln x$. $[y = \ln x + C/\ln x]$
7. $y' + \frac{2y}{x} = e^x$. $[y = C/x^2 + (1 - 2/x + 2/x^2)e^x]$
8. $(x - 1)y' + 3y = x^2$. $[(x - 1)^3 y = C + x^5/5 - x^4/2 + x^3/3]$
9. $(x + 1)y' + (2x - 1)y = e^{-2x}$. $[e^{2x}y = C(x + 1)^3 - \frac{1}{3}]$

10. $y' + \frac{y}{x} = \frac{1}{2} \sin\left(\frac{x}{2}\right)$. $[y = -\cos\frac{x}{2} + \frac{2}{x} \sin\frac{x}{2} + \frac{C}{x}]$
11. $(1-x^2)y' + x(y-a) = 0$. $[y = a + C(1-x^2)^{\frac{1}{2}}]$
12. $y' - (1+\cot x)y = 0$. $[y = Ce^x \sin x]$
13. $(1+x^2)y' + xy = 3x + 3x^3$. $[y = 1 + x^2 + C(1+x^2)^{-\frac{1}{2}}]$
14. $\sin x \cos xy' + y = \cot x$. $[y = (C + \ln \tan x)/\tan x]$

Solve:

15. $y' + 2xy = 4x$, $y(0) = 3$. $[y = 2 + e^{-x^2}]$
16. $y' \coth 2x = 2y - 2$, $y(0) = 0$. $[y = 1 - \cosh 2x]$
17. $y' + ky = e^{-kx}$, $y(0) = 1$. $[y = (1+x)e^{-kx}]$
18. $y' = a(y-g)$, $y(0) = b$. $[y = g + (b-g)e^{ax}]$
19. $yy' = 2a$, $y(0) = 0$. $[y^2 = 4ax]$
20. $yy' + x = 0$, $y(0) = a$. $[x^2 + y^2 = a^2]$
21. $yy' + \frac{b^2x}{a^2} = 0$, $y(0) = b$. $\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right]$
22. $(x+1)y' = y - 3$, $y(0) = 8$. $[y = 5x + 8]$
23. $2xy' + y = 0$, $y(1) = 1$. $[xy^2 = 1]$
24. $(1+x^2)y' = \sqrt{y}$, $y(0) = 0$. $[y = \frac{1}{4}(\tan^{-1} x)^2]$
25. $\frac{di}{dt} + 3i = \sin 2t$, $i = 0$ when $t = 0$. $[i = \{\sin(2t - \alpha) + e^{-3t} \sin \alpha\}/\sqrt{13}]$,
where $\tan \alpha = \frac{2}{3}$
26. Water runs out through a hole in the base of a circular cylindrical tank at speed $\sqrt{2gh}$ ft/s, where $g = 32$ ft/s² and h is the water depth. If the tank is 2 ft in height, 1 ft in diameter and is full at time $t = 0$, calculate the time at which half the water has run out when the effective area of the hole is 0.25 in². [47 s]
27. The current i in a circuit satisfies $L di/dt + Ri = E$, where L , R , E are constants. Show that when t is large the current is approximately equal to E/R .

If, instead, $E = E_0 \cos \omega t$, where E_0 , ω are constants, show that when t is large

$$i \approx \frac{E_0 \cos(\omega t - \epsilon)}{\sqrt{R^2 + \omega^2 L^2}}, \quad \text{where } \tan \epsilon = \frac{\omega L}{R}.$$

1.2 Second-Order Equations with Constant Coefficients

Homogeneous equation $y'' + ay' + by = 0$, $a, b = \text{constants}$.

Inhomogeneous equation $y'' + ay' + by = r(x)$.

General solution:

$$y = Ay_1(x) + By_2(x) + y_p(x), \quad A, B = \text{constant}$$

where y_1 , y_2 are any two linearly independent solutions of the homogeneous equation, called *basis functions* or *complementary functions*, and y_p is a *particular integral* that yields $r(x)$ when substituted into the equation.

Solution of the homogeneous equation Because $d(e^{\lambda x})/dx = \lambda e^{\lambda x}$, $y = e^{\lambda x}$ will be a solution of the homogeneous equation if λ is a solution of the *characteristic equation*

$$\lambda^2 + a\lambda + b = 0, \quad \text{i.e. for } \lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \lambda_1, \lambda_2. \quad (1.2.1)$$

Case 1 $\lambda_1 \neq \lambda_2$:

$y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are linearly independent and the general solution is therefore

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}. \quad (1.2.2)$$

The values of the constants A , B are fixed by the *boundary conditions*.

Example Solve $y'' + 2y' - 8y = 0$, $y(0) = 1$, $y'(0) = 0$.

Characteristic equation : $\lambda^2 + 2\lambda - 8 = 0$

$$\therefore \lambda = -4, 2$$

$$\therefore y(x) = Ae^{-4x} + Be^{2x}.$$

At $x = 0$: $y = 1$, and $y' = 0$

$$\therefore A + B = 1$$

$$\text{and } -4A + 2B = 0.$$

$$\therefore y = \frac{e^{-4x} + 2e^{2x}}{3}.$$

Case 2 $\lambda_1 = \lambda_2 \equiv \lambda$:

The two solutions in (1.2.2) are not independent. The differential equation can now be written in the factored form

$$y'' + ay' + b \equiv \left(\frac{d}{dx} - \lambda \right) \left(\frac{d}{dx} - \lambda \right) y = 0.$$

If $z = \frac{dy}{dx} - \lambda y$, then $z' - \lambda z = 0$, i.e. $z = Be^{\lambda x}$, $B = \text{constant}$,

$$\therefore y' - \lambda y = Be^{\lambda x}.$$