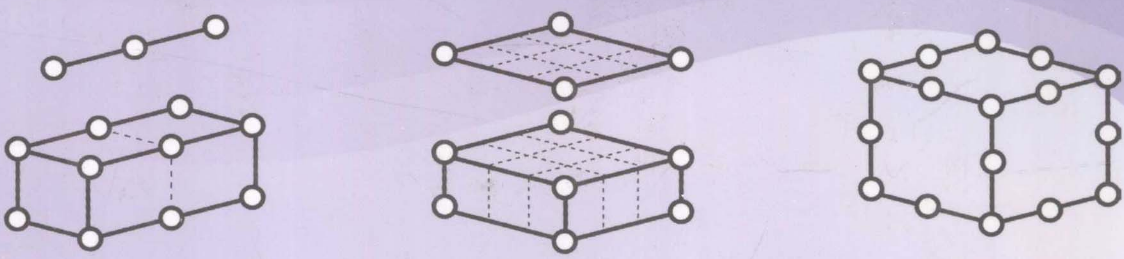
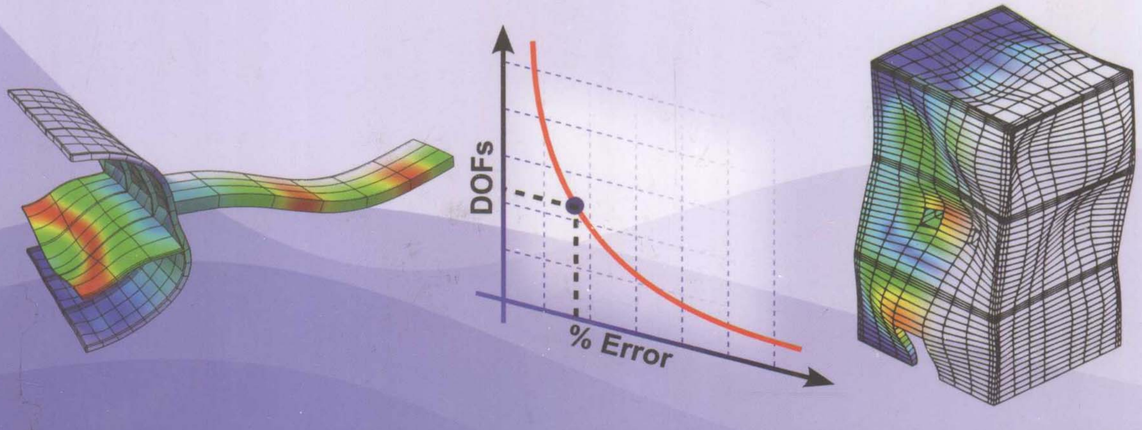
Finite Element Analysis of Structures through Unified Formulation

Erasmo Carrera, Maria Cinefra, Marco Petrolo and Enrico Zappino



$$k_{xx} = -(\lambda+2G)\, \partial_{xx} - G\, \partial_{zz} - G\, \partial_{yy}$$
$$k_{xy} = -G\, \partial_{yx} - \lambda\, \partial_{xy}$$



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FINITE ELEMENT ANALYSIS OF STRUCTURES THROUGH UNIFIED FORMULATION

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About the Authors

Erasmo Carrera

Erasmo Carrera graduated in Aeronautics in 1986 and in Space Engineering in 1988 from the Politecnico di Torino. He obtained a PhD in Aerospace Engineering in 1991 within the framework of a joint PhD programme between the Politecnico di Milano, the Politecnico di Torino and the Università di Pisa. He became assistant professor in 1992. He has continuously held courses at Bachelor, Master and PhD levels on Fundamentals of Theory of Structures, Aerospace Structures, Nonlinear Problems, Plates and Shells, Thermal Stress, Composite Materials, Multifield Problems and Computational Aeroelasticity. Currently he is a full professor in the Department of Mechanical and Aerospace Engineering. He has also been a visiting professor at the University of Stuttgart, Virginia Tech, Supmeca and the Centre of Research Public Henri Tudor.

His research topics cover: composite materials, nonlinear problems and the stability of structures, contact mechanics, multibody dynamics, finite elements, path-following methods in nonlinear finite element (FE) analysis, meshless methods, unconventional lifting systems, smart structures, thermal stress for coupled and uncoupled problems, multifield interaction, aeroelasticity, panel flutter, wind blades, explosion effects on flying aircraft, advanced theories for beams, plates and shells, mixed variational methods; zigzag, mixed and layer-wise modellings for multilayered beams, plates and shells; local–global methods and the Arlequin-type approach; advanced structural models for wings, fuselage and complete aircraft/spacecraft through the introduction of the so-called component-wise approach; failure and progressive failure analysis of laminated structures; inflatable structures for manned and unmanned space applications; and the design and analysis of full composite aircraft, including trikes and unmanned aerial vehicles (UAVs).

Professor Carrera developed the Reissner mixed variational theorem (RMVT) as a natural extension of the principle of virtual displacements to layered structure analysis. He introduced the unified formulation, or CUF (Carrera Unified Formulation), as a tool to establish a new framework in which beam, plate and shell theories can be developed for metallic and composite multilayered structures under mechanical, thermal, electrical and magnetic loadings. The CUF has been applied extensively to both strong and weak forms (FE and meshless solutions). The main feature of the CUF is that it permits any expansion of the unknown variables over the thickness/cross-section domain to be handled in a compact manner. Governing equations are in fact obtained in terms of a few fundamental nuclei whose forms do not depend on either the order of the expansion or the base functions used. As a result, the CUF allows the so-called best theory diagram (BTD) (which shows the minimum number of unknown variables vs the error on an assigned parameter) to be computed for a given problem. The BTD is a way of enhancing axiomatic and asymptotic approaches in the theory of structures.

Professor Carrera is the author and coauthor of about 500 papers on the above topics, most of which have been published in primary international journals, as well as of two recent books published by John Wiley & Sons, Ltd. His papers have received about 5000 citations with an h-index=39 (data from Scopus). He has held invited seminars in various European and North American universities, as well as plenary talks at international conferences. Professor Carrera serves as the Associate Editor for *Composite Structures*, *Journal of Thermal Stress*, *Mechanics of Advanced Structures*, *Computer and Structures* and the *International Journal of Aeronautical and Space Sciences*. He is founder and Editor-in-Chief of *Advances in Aircraft and Spacecraft Science*; acts as a reviewer for about 80 journals; and is on the Editorial Board of many international conferences. He is also in charge of the chapter on 'Shells' in the *Encyclopedia of Thermal Stress*, published by Springer. Professor Carrera is the founder of the non-profit international conference DeMEASS and the main organizer of ICMNMMCS (Turin, June 2012, co-chaired by Professor A. Ferreira), the ECCOMASS SMART 13 conference (Turin, June 2013) and ISVCS IX (Courmayeur, July 2013). He is member of the Distinguished Professor Board at King Abdulaziz University (Saudi Arabia). He has been a member of PhD and Habilitation committees in Germany, France, the Netherlands and Portugal. He is president of the Piedmont Section of AIDAA (Associazione Italiana di Aeronautica ed Astronautica).

Professor Carrera has been responsible for various research contracts granted by public and private national (including regional) and international institutions such as IVECO, the Italian Ministry of Education, the European Community, the European Space Agency, Alenia Spazio, Thales Alenia Space and Regione Piemonte. Among other projects, he has been responsible for the structural design and analysis of a full composite aircraft, named Sky-Y, by Alenia Aeronautica Torino, the first fully composite UAV made in Europe.

Professor Carrera is founder and leader of the MUL2 Group at the Politecnico di Torino. This group is considered one of the most active research teams in the Politecnico; it has acquired a significant international reputation in the field of *mul*tilayered structures subjected to *mul*tifield loadings; see also www.mul2.com. He is one of the Highly Cited Researchers by Thomson Reuters in both the Engineering and Materials Sections.

Maria Cinefra

Maria Cinefra is a research assistant at the Politecnico di Torino. She gained a BSc in Aerospace Engineering at the Politecnico di Torino in March 2007 with a thesis on the finite element method (FEM) in elliptic differential equations. Afterwards, she undertook an MSc in Aerospace Engineering at the Politecnico di Torino and gained her Master's degree, summa cum laude, in December 2008 from her work on the thermomechanical analysis of functionally graded material (FGM) shells. She began her PhD in January 2009, under the supervision of Professor Erasmo Carrera, on a research project related to the thermomechanical design of multilayered plates and shells embedding FGM layers. She was enrolled in a PhD with a foreign co-advisor, Professor Olivier Polit, at the University of Paris Ouest Nanterre. Her research project was funded by the Fonds National de la Recherche of Luxembourg and was performed in collaboration with the CRP Henri Tudor of Esch (Luxembourg). She was given the award for the best PhD paper (Ian Marshall's Award) at the 16th International Conference on Composite Structures (28–30 June 2011, Porto, Portugal). In January 2012, she was admitted to the final exam of her PhD and presented the defence of her thesis in April

2012. Since 2010, she has worked as a teaching assistant at the Politecnico di Torino on the courses Nonlinear Analysis of Structures, Structures for Space Vehicles and Fundamentals of Structural Mechanics. She is currently collaborating with the Department of Mathematics at Pavia University in order to develop a mixed shell FE based on the CUF for analysing composite structures. She has collaborated with Professor Ferreira, Editor of the *Composite Structures Journal*, on the radial basis functions method combined with the CUF. Dr Cinefra works as a reviewer for international journals such as *Composite Structures* and *Mechanics of Advanced Materials and Structures*. She is currently working on the STEPS regional project, in collaboration with Thales Alenia Space, and is also working on an extension of the shell FE, based on the CUF, to the analysis of multifield problems.

Marco Petrolo

Marco Petrolo is a Research Fellow at the School of Aerospace, Mechanical and Manufacturing Engineering, RMIT University, Melbourne, Australia. He was Post-Doc fellow at the Politecnico di Torino, Italy. He works in Professor Carrera's research group on various topics related to the development of refined structural models of composite structures. His research activity is connected with the structural analysis of composite lifting surfaces; refined beam, plate and shell models; component-wise approaches; and axiomatic/asymptotic analyses. He is the author and coauthor of some 50 publications, including 2 books and 25 articles that have been published in peer-reviewed journals.

Dr Petrolo gained his PhD in Aerospace Engineering at the Politecnico di Torino in April 2012, presenting a thesis on advanced aeroelastic models for the analysis of lifting surfaces made of composite materials. He also has an MSc in Aerospace Engineering from the Politecnico di Torino, an MSc in Aerospace Engineering from TU Delft (the Netherlands) and a BSc in Aerospace Engineering from the Politecnico di Torino. He has worked as an intern at EADS (Germany) and, as a Fulbright scholar, spent research periods at San Diego State University and the University of Michigan (USA). Dr Petrolo was appointed Adjunct Professor in Fundamentals of Strength of Materials (part of the BSc in Mechanical Engineering at the Turin Polytechnic University in Tashkent, Uzbekistan).

Enrico Zappino

Enrico Zappino is a post-doctoral fellow at the Politecnico di Torino. He has been in Professor Carrera's research group since 2010. His research activities concern structural analysis using classical and advanced models, multi-field analysis, and composite materials analysis. He is the coauthor of many works published in several international peer-reviewed journals. He obtained his PhD in April 2014, presenting a thesis on variable kinematic 1D, 2D, and 3D models for the analysis of aerospace structures. He also gained his BSc in Aerospace Engineering at the Politecnico di Torino in October 2007, presenting a thesis on advanced wing structures. He then obtained an MSc from the same university in July 2010, with a thesis on higher-order one-dimensional structural models applied to static, dynamic, and aeroelastic analysis. He was involved in many research programs supported by the European Space Agency and the European Union in cooperation with many European industrial and academic partners. From 2011, Dr. Zappino has worked as a teaching assistant at the Politecnico di Torino on the course of Aeroelasticity. In 2014, he was appointed as Adjunct Professor in Fundamentals of Strength of Materials at the Turin Polytechnic University in Tashkent, Uzbekistan.

Preface

This book deals with the finite element method (FEM) used for analysing the mechanics of structures in the case of linear elasticity. The novelty of this book is that the finite elements (FEs) are formulated on the basis of a class of theories of structures known as the Carrera Unified Formulation (CUF).

The CUF provides one-dimensional (beam) and two-dimensional (plate and shell) theories that go beyond classical theories (those of Euler, Kirchhoff, Reissner, Mindlin, Love) by exploiting a condensed notation and by expressing the displacement fields over the cross-section (the beam case) and along the thickness (plate and shell cases) in terms of base functions whose forms and orders are arbitrary. The condensed notation leads to the so-called *fundamental nucleus* (FN) of all the FEM matrices and vectors involved. The fundamental nuclei (FNs) and the related assembly technique are schematically shown in Table 1. The FNs consist of a few mathematical statements whose forms are independent of the theory of structures (TOS) employed. The FNs stem from the 3D elasticity equations via the principle of virtual displacements (PVD) and can be easily obtained for the 3D, 2D and 1D cases. This table will be reintroduced at the beginning of each chapter of this book that deals with 3D, 2D and 1D models to highlight the relevant fundamental nucleus.

The 1D and 2D FEs that stem from the CUF have enhanced capabilities since they can obtain results that are usually only provided by 3D elements with much lower computational costs. The 1D elements are particularly advantageous since they can deal with 2D and 3D problems in a proper manner.

The 1D and 2D CUF models are described in various chapters of this book. Particular attention has been paid to 1D and 2D FEs with only pure displacement degrees of freedom. The displacement unknowns of such FEs are defined over the physical surfaces of the real 3D body; this means that the definitions of mathematical reference axes (for beams) or reference surfaces (for plates and shells) are not needed. This capability is extremely important in an FEM/CAD coupling scenario. The modifications carried out in an FEM model can, in fact, be implemented directly in a CAD model (and vice versa) since physical surfaces are taken into account.

The concluding chapters of the book offer an overview of some of the most important features of the CUF models. In particular, the following topics are emphasized: multifield loads can be easily implemented; layered structures can be analysed; 1D, 2D and 3D models can be combined straightforwardly; and the CUF can lead to a definition of the BTD to evaluate the effectiveness of any structural theory. Numerical examples appear throughout the book on classical and non-classical TOS problems.

Table 1 A schematic description of the CUF and the related fundamental nucleus of the stiffness matrix for 3D, 2D and 1D models

Equilibrium equations in Strong Form $\longrightarrow$ $\delta L_i = \int_V \delta \boldsymbol{u} \boldsymbol{k} \boldsymbol{u} dV + \int_S \ldots dS$

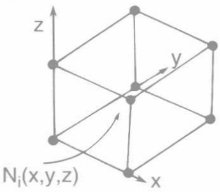
$$\underbrace{\begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix}}_{\boldsymbol{k}} \underbrace{\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}}_{\boldsymbol{u}} = \underbrace{\begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}}_{\boldsymbol{p}}$$

$k_{xx} = -(\lambda + 2G)\, \partial_{xx} - G\, \partial_{zz} - G\, \partial_{yy};$
$k_{xy} = -\lambda\, \partial_{xy} - G\, \partial_{yx};$
$k_{xz} = \ldots$

$\lambda = (E\nu)/[(1+\nu)(1-2\nu)]; \quad G = E/[2(1+\nu)]$

$\boldsymbol{u} = \boldsymbol{u}(x, y, z)$
$\delta \boldsymbol{u} = \delta \boldsymbol{u}(x, y, z)$

The diagonal (e.g. k_{xx}) and the non-diagonal (e.g. k_{xy}) terms can be obtained through proper index permutations.

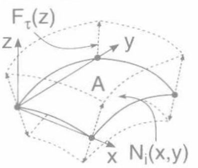


$\boldsymbol{u} = N_i(x, y, z)\boldsymbol{u}_i$
$\delta \boldsymbol{u} = N_j(x, y, z)\delta \boldsymbol{u}_j$

3D FEM Formulation $\longrightarrow$ $\delta L_i = \delta \boldsymbol{u}_j \boldsymbol{k}^{ij} \boldsymbol{u}_i$

$$k_{xx}^{ij} = (\lambda + 2G) \int_V N_{j,x} N_{i,x} dV + G \int_V N_{j,z} N_{i,z} dV + G \int_V N_{j,y} N_{i,y} dV;$$

$$k_{xy}^{ij} = \lambda \int_V N_{j,y} N_{i,x} dV + G \int_V N_{j,x} N_{i,y} dV$$

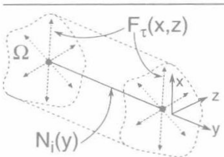


$\boldsymbol{u} = N_i(x, y) F_\tau(z) \boldsymbol{u}_{\tau i}$
$\delta \boldsymbol{u} = N_j(x, y) F_s(z) \delta \boldsymbol{u}_{sj}$

2D FEM Formulation $\longrightarrow$ $\delta L_i = \delta \boldsymbol{u}_{sj} \boldsymbol{k}^{\tau sij} \boldsymbol{u}_{\tau i}$

$$k_{xx}^{\tau sij} = (\lambda + 2G) \int_\Omega N_{i,x} N_{j,x} d\Omega \int_h F_\tau F_s dz + G \int_\Omega N_i N_j d\Omega \int_h F_{\tau,z} F_{s,z} dz + G \int_V N_{i,y} N_{j,y} d\Omega \int_h F_\tau F_s dz;$$

$$k_{xy}^{\tau sij} = \lambda \int_\Omega N_{i,y} N_{j,y} d\Omega \int_h F_\tau F_s dz + G \int_\Omega N_{i,x} N_{j,y} d\Omega \int_h F_\tau F_s dz$$



$\boldsymbol{u} = N_i(y) F_\tau(x, z) \boldsymbol{u}_{\tau i}$
$\delta \boldsymbol{u} = N_j(y) F_s(x, z) \delta \boldsymbol{u}_{sj}$

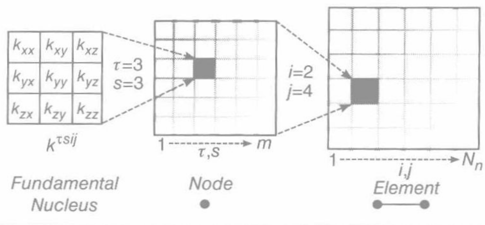
1D FEM Formulation $\longrightarrow$ $\delta L_i = \delta \boldsymbol{u}_{sj} \boldsymbol{k}^{\tau sij} \boldsymbol{u}_{\tau i}$

$$k_{xx}^{\tau sij} = (\lambda + 2G) \int_l N_i N_j dy \int_A F_{\tau,x} F_{s,x} dA + G \int_l N_i N_j dy \int_A F_{\tau,z} F_{s,z} dA + G \int_l N_{i,y} N_{j,y} dy \int_A F_\tau F_s dA;$$

$$k_{xy}^{\tau sij} = \lambda \int_l N_{i,y} N_j dy \int_A F_\tau F_{s,x} dA + G \int_l N_i N_{j,y} dy \int_A F_{\tau,x} F_s dA$$

CUF leads to the automatic implementation of any theory of structures through 4 loops (i.e. 4 indexes):

- $\boldsymbol{\tau}$ and $\boldsymbol{s}$ deal with the functions that approximate the displacement field and its virtual variation along the plate/shell thickness ($F_\tau(z), F_s(z)$) or over the beam cross-section ($F_\tau(x, z), F_s(x, z)$);
- $\boldsymbol{i}$ and $\boldsymbol{j}$ deal with the shape functions of the FE model, (3D:$N_i(x, y, z), N_j(x, y, z)$; 2D:$N_i(x, y), N_j(x, y)$; 1D:$N_i(y), N_j(y)$).



This table shows the essential features of the CUF. The strong form of the equilibrium equations allows one to derive a compact formulation for the fundamental nucleus. The nine elements of the FN can be written using only 2 terms. In this table, k_{xx} and k_{xy} are reported. All the remaining terms can be derived by a permutation of the indexes. This compact formulation is used to derive the 3D, 2D and 1D models in weak form.

This book follows on from two recent books where the CUF was applied to shell, plate and beam models: *Plates and Shells for Smart Structures: Classical and Advanced Theories for Modeling and Analysis* (E. Carrera, S. Brischetto and P. Nali, John Wiley & Sons, Ltd, 2011) deals with refined shell and plate models for smart structures; and *Beam Structures: Classical and Advanced Theories* (E. Carrera, G. Giunta and M. Petrolo, John Wiley & Sons, Ltd, 2011) deals with refined beam models. Analytical and FE formulations were introduced in both these books.

Nomenclature and Acronyms

The main symbols and acronyms that are defined in the book are listed below. Unless otherwise stated, the following definitions will be valid throughout the entire book.

Symbols

$\boldsymbol{B}, b$ Differential operator of the strain–displacements relations
B2, B3, B4 Beam elements with two, three and four nodes
$\boldsymbol{C}$ Hooke's law stiffness matrix
$C_{11}, C_{12}, C_{21}, C_{13}, C_{23}, C_{44}$ Hooke's law stiffness coefficients
E Young's modulus
F_τ, F_s Expansion functions
G Shear modulus
$\boldsymbol{g}$ Body forces per unit volume vector
g_x, g_y, g_z Body forces per unit volume components
H Metric factor
i, j Shape function indexes
k Layer index
$\boldsymbol{K}$ Stiffness matrix
$\mathbf{k}^{\tau sij}$ Fundamental nucleus of the stiffness matrix
$k_{xx}^{\tau sij}, k_{xy}^{\tau sij}, \ldots, k_{zz}^{\tau sij}$ Components of the stiffness matrix fundamental nucleus
L3, L4, L6, L9 Lagrange cross-section elements with three, four, six and nine nodes
L_{ext} Work of the external forces
L_{ine} Work of the inertial forces
L_{int} Work of internal forces
M Number of terms in an expansion
$\boldsymbol{M}$ Mass matrix
$\mathbf{m}^{\tau sij}$ Mass matrix fundamental nucleus
$m_{xx}^{\tau sij}, m_{xy}^{\tau sij}, \ldots, m_{zz}^{\tau sij}$ Components of the mass matrix fundamental nucleus
N Expansion order of F_τ, F_s
$\boldsymbol{n}$ Normal unit vector
N_a, N_b, N_c, N_d MITC4 interpolating shape functions for shear stresses
N_i, N_j Shape functions
N_m MITC9 interpolating shape functions for membrane stresses

$\boldsymbol{P}, \boldsymbol{p}$ Load vector
P_x, P_y, P_z Point load components
p_x, p_y, p_z Surface load components
q_x, q_y, q_z Line load components
R Radius of curvature
$\boldsymbol{U}, \boldsymbol{u}$ Displacement vector
u_x, u_y, u_z Displacement components
$\mathbf{u}_\tau$ Generalized displacement vector
$\mathbf{u}_{\tau i}$ Nodal displacement vector
$\ddot{\boldsymbol{U}}, \ddot{\boldsymbol{u}}$ Acceleration vector
V Volume
x, y, z Orthogonal Cartesian reference system
α, β, z Curvilinear coordinates
δ Virtual variation
$\boldsymbol{\epsilon}$ Strain vector
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$ Axial strain components
$\varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{xz}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}$ Shear strain components
κ Shear correction factor
ν Poisson's ratio
ρ Material density
$\boldsymbol{\sigma}$ Stress vector
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ Axial stress components
$\sigma_{xy}, \sigma_{yz}, \sigma_{xz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$ Shear stress components
τ, s Expansion function indexes
ϕ Rotation

Acronyms

1D/2D/3D One-/Two-/Three-Dimensional
BC Boundary Condition
BS Beam Semimonocoque
BTD Best Theory Diagram
CAD Computer-Aided Design
CLT Classical Lamination Theory
CNT Carbon Nanotube
CPT Classical Plate Theory
CST Classical Shell Theory
CUF Carrera Unified Formulation
CW Component-Wise
DOF Degree of Freedom
EBBT Euler–Bernoulli Beam Theory
ESL Equivalent Single Layer
ESLM ESL Model
FE Finite Element
FEA Finite Element Analysis

FEM Finite Element Method
FGM Functionally Graded Material
FN/FNs Fundamental Nucleus/Nuclei
FSDT First-order Shear Deformation Theory
HOT Higher-Order Theory
IC Interlaminar Continuity
LE Lagrange Expansion
LFAT Love First Approximation Theory
LM Lagrange Multiplier
LSAT Love Second Approximation Theory
LW Layer-Wise
LWM LW Model
MAAA Mixed Axiomatic–Asymptotic Approach
MAC Modal Assurance Criterion
MCS Multi-Component Structures
MFP Multifield Problem
MITC Mixed Interpolation of Tensorial Components
MLS Multilayered Structure
NRP Nanotube Reinforced Polymer
ODE/PDE Ordinary/Partial Differential Equations
PL Poisson Locking
PS Pure Semimonocoque
PVD Principle of Virtual Displacements
PVW Principle of Virtual Work
RMVT Reissner Mixed Variational Theorem
SDT Shear Deformation Theory
TBT Timoshenko Beam Theory
TE Taylor Expansion
TL Thickness Locking
TOS Theory of Structures
WRM Weight Residual Method
ZZ Zigzag

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