MATHEMATICAL MODELS IN THE SOCIAL AND BEHAVIORAL SCIENCES

ANATOL RAPOPORT

Institute for Advanced Studies
Vienna, Austria

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Mathematical Models in the Social and Behavioral Sciences

Preface

This book, first published in German (*Mathematische Methoden in den Sozialwissenschaften*, Würzburg: Physica-Verlag, 1980), grew out of courses I gave as guest professor at the Institute for Advanced Studies in Vienna. Quantitative and mathematical approaches to the social sciences are incorporated both in the teaching curriculum and in the research approaches of that institution. However, the very fact that these approaches are specifically mentioned in the description of the Institute's program attests to the circumstance that these methods are still not generally regarded as indispensable in the development of the social sciences as they obviously are regarded in the development of the physical sciences. For this reason, a deliberate orientation of the social sciences toward quantification and mathematization still requires justification.

The most widely accepted justifications of any approach are those based on pragmatic expectations. So it has been with the natural sciences, which enjoy social support in proportion to their contributions to control over the natural environment, affluence, power, and so on. Can the social sciences aspire to a similar status on similar grounds? If so, can this status be achieved by transplanting the spectacularly successful mathematical methods of theory construction from their natural habitat in the physical sciences to the soil of social science? If, on the other hand, the triumphal unfolding of mathematized theory cannot be duplicated in the social sciences, is there some other justification for infusing mathematical ways of thinking into the social sciences?

While teaching courses with the explicit goal of inculcating mathematical concepts and reasoning into thinking about social phenomena, I was constantly confronted with such questions. Quality and effectiveness of teaching depend in large measure on the motivation of the students. The courses on which this book is based were given in the late 1960s and in the 1970s, mostly to young Austrians seeking postgraduate training in sociology or political science and to students already equipped with some mathematical skills and seeking to apply them outside the conventional fields such as the physical sciences or technology. Of these students, only about 15% continued their careers in academic pursuits, where they could apply what they learned by simply teaching it to *their* students. The majority sought (and usually found) positions in public or private sectors of the Austrian economy or as "professional" sociologists, political scientists, or statisticians in public administration bodies. For this reason, whatever purely intellectual interest these young people had in the input or impact of mathematics in the social sciences stemmed from already existing personal predilections. The maintenance of the interest depended on the strength of these predilections, not on reinforcements of specific career requirements. Such leanings toward and appreciation of mathematical ways of thinking were frequently manifested in students in the Mathematics and Computer Science Department of the Institute but seldom by those in the two social science departments (sociology and political science). For most of these young people, the raison d'être of the social sciences was the betterment of society, and "relevance" was an instantaneous litmus test for taking an idea seriously or dismissing it. A strong "antipositivist" bias was coupled with this attitude. To the extent that it found articulate expression, it was embodied in a conviction that the "methods of natural science," in particular, the primacy of "objective" data, quantification of observations, mathematical deduction, and so on, are in principle inapplicable in the social sciences, and attempts to force the latter into the mold of the natural sciences would serve to sterilize them or, worse, put them at the service of ossified privilege or antihuman forces. Thus since acquaintance with quantitative and mathematical methods in the social sciences was required of all social science students at the Institute, some answers to these misgivings had to be incorporated in any purposeful teaching of these approaches.

In America, the problem of justification is different, almost an opposite one, as it were. There the "positivistic" approaches to the behavioral sciences proliferated luxuriantly and found wide acceptance, and with them mathematical modeling came into its own. Questions of "relevance" are raised more often in the technocratic vein. Will mathematization of the social sciences, following the path blazed by the physical sciences, render social events more controllable and so help close the gap between our mastery of the physical environment and our helplessness in the face of social forces? Since this English version of the book is addressed also to American readers, some answers to questions of this sort have to be given.

In deciding to face these questions (as has been said, for pedagogical reasons), I clearly did not intend to "sell" mathematization of the behavioral sciences to either ideologically or technocratically oriented students on their own terms. For that matter, no convincing case can be made on the basis of contemporary contributions of mathematical approaches to the social sciences that anything resembling the magnificent edifice of the physical sciences is being constructed. The emancipation of humans from the blind forces of history is not in the making. From the numerous examples in this book (a fairly representative sample reflecting the state of the art), it should be clear that mathematical modeling in the social sciences is a highly opportunistic venture. Not what seems especially important for the solution of pressing social problems but rather what seems to fit into a mathematical model is usually chosen for "applying" mathematical methods worked out previously in an abstract context.

This book is offered as an attempt to demonstrate the *integrative* function of the mathematical mode of cognition. It was my hope that once this integrative function is appreciated, the questions posed by ideological or pragmatic demands will be seen in a different light. Thus the goal pursued in this book is that of restructuring habits of thinking about social phenomena. The advantages of this restructuring cannot be seen clearly in advance; they can be appreciated only after the restructuring has taken place. Here I can state only the grounds for my own faith in the beneficial effects of mathematical thinking. Mathematics is the lingua franca of all science because it is contentless. Whenever a scientific theory can be represented and developed in mathematical language, it is thereby related to all other theories formulated and developed in that language. If the ideal of the "unity of science" bridging both diverse contents and cultural differences can be achieved at all, this will be done via mathematization.

This book is structured to emphasize this integrative function of mathematics. The principal theme of each part is a class of mathematical tools (models or methods) rather than a content area. In this way, the different areas of social science encompassed by mathematical analogies come into unified focus. Models based on differential equations deal with changes in the sex composition of human populations (Chapter 1), with contagion processes (Chapter 2), or with arms races (Chapter 3). Essentially the same stochastic model represents certain aspects of social mobility (Chapter 10), certain demographic dynamics (Chapter 11), and suggests forecasts of the recruitment needs of an organization (Chapter 9). Set theoretic language provides the conceptual framework for a rigorous analysis of certain types of conflict resolution—for example, those modeled by cooperative games (Chapter 19)—and also for the principles underlying democratic collective decisions (Chapter 18).

The content areas represented here range over psychology, sociology, political science, and anthropology. Mathematical economics is not represented (except tangentially through utility theory and the theory of games), first because this area was not included in the courses on which the book is based; second, because there is no dearth of textbooks and treatises on that subject.

Restrictions of space have limited presentations of applications to a few chosen examples to illustrate each method. Topics of importance regretfully omitted from the final version include cluster analysis (see Tryon and Bailey, 1970), latent structure analysis (see Lazarsfeld and Henry, 1968), and spectral analysis (see Mayer and Arney, 1973–1974). An extensive treatment of the mathematical methods discussed in Parts II and III can be found in Kemeny and Snell (1962) and in Fararo (1973). Throughout the book, the interested reader is repeatedly referred to more detailed treatments of topics discussed. "Classical" methods (e.g., differential equations) are, of course, explained in textbooks available everywhere.

One further word is in order about the selection of examples of mathematical models. Among these, not only apparently interesting and fruitful ones will be found, but also questionable and even clearly sterile ones. These were included to show the development of mathematical models in the social sciences in historical perspective and at times to provide leverage for instructive criticism. Learning from the mistakes of others is less painful than learning from one's own.

I am grateful to John Wiley & Sons for providing the opportunity to write another version of this book and so to incorporate some revisions, which I felt were needed. Some chapters of the German version were combined, others separated, some substantially rewritten or expanded to accommodate recently published material. Chapter 6 of the present version has been added.

I take further pleasure in expressing heart-felt thanks to Ilse Reichel and Ingrid Schmidbauer, who typed the manuscript in a language foreign to them, and to Regina-Cernavskis and Gerda Suppanz for preparing the figures.

ANATOL RAPOPORT
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A Note on Notation

Letters

Sets are usually designated by capital italic letters, elements of sets by lower case letters. Random variables are denoted by capital letters, specific values assumed by them by lower case letters. For example, $\Pr[X \leqslant x]$ designates the probability that a random variable X assumes a value no larger than x.

Vectors are denoted by lower case letters; for example, $x = (x_1, x_2, \ldots, x_n)$. Matrices are denoted by capital letters, their scalar elements by lower case letters; for example, $M = (m_{ij})$ designates a matrix with elements m_{ij} in ith row, jth column.

Parentheses

Sets represented by their elements are usually indicated by braces; for example, $\{x, y, z\}$. Accordingly, $U(\{x, y, z\})$ designates the value of a function U whose argument is the set $\{x, y, z\}$ (see Chapter 12). Sets consisting of single elements are also designated in order to adhere to the logic of operations on sets. For example, $Y - \{x\}$ designates the set resulting when the element x is deleted from the set Y (since "subtraction" is a binary operation on sets, not on sets and elements). Where there is no danger of confusion, braces are omitted in functions on sets. For instance, $v(abc)$ instead of $v(\{a, b, c\})$ denotes the value of an n-person cooperative game to a set of players $\{a, b, c\}$ acting as a coalition (see Chapter 19).

Round parentheses are used for *ordered* n-tuples. For instance, $(abcd)$ may indicate that objects a, b, c, d are preferred in that order (see Chapter 12).

Open intervals are set in parentheses, closed intervals in brackets. For example, (a, b) denotes the sets of all real numbers larger than a and smaller than b; $[a, b]$ the set of all real numbers equal to or larger than a and equal to or smaller than b. The meaning of a half-open (half-closed) interval, $[a, b)$ or $(a, b]$, is clear.

Angular parentheses are used to enclose the elements defining mathematical objects. For example, $G = \langle N, V \rangle$ designates a graph G defined by a set of vertices N and a binary relation V (see Chapter 14). $\langle N, v \rangle$ designates a cooperative game defined by a set of players N and a characteristic function v (see Chapter 19).

Summations

In addition to the usual symbol of summation over consecutively indexed quantities—for example, $\sum_{i=1}^{n}$—the following special summation symbols are used. $\sum_y'$ designates summation over all y distinct from a specific x. $\sum_{i \in S} x_i$ designates summation over x_i, whose indices i are elements of a given set S. When the numbers x belong to a set fixed in a given context, summation over these numbers is denoted by $\sum_x$.

Some Special Symbols Used in This Book

$x \succ_S y$:	Imputation x dominates imputation y via a given set of players S (see Chapter 19).
$x \succ y$:	Imputation x dominates imputation y via unspecified set or sets of players.
$x \succ y$:	x is preferred to y (see Chapter 12).
$x \precsim y$ or $y \succsim x$:	x is not preferred to y.
$x \sim y$:	x is not preferred to y, and y is not preferred to x (also, "x is indifferent to y").
$x \cong y$:	x is approximately equal to y.
$x \sim y$:	x is proportional to y.
$\Rightarrow$:	implies.
$\Leftrightarrow$:	implies and is implied by.
$\exists$:	there exists. For instance, "$y \notin I_0 \Rightarrow \exists x \in I_0,\ x \succ y$" is read, "for every imputation y not an element of the set I_0, there exists an imputation x in I_0 which dominates y."

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Introduction: Goals and Means

Scientific knowledge, as distinguished from other modes of cognition (poetic, introspective, religious, tradition-preserving), is characterized by objectivity, generality, and precision. Of these criteria, generality and precision are matters of degree. Objectivity is more nearly an all-or-none criterion: it enters the definition of a *fact*, a state of affairs reported in equivalent ways by *independent* observers.

That generality of scientific knowledge is a matter of degree is reflected in the paradigm of a scientific assertion: "If . . . , then . . . ." The "if" part specifies conditions under which observations are to be made, the "then" part specifies what will be observed. The range of conditions under which a given state of affairs is expected to be observed serves as a measure of generality of a scientific assertion.

All scientific knowledge is shared by means of language. Linguistic utterances carry both denotative and connotative meanings. The former relate words and combinations of words to *referents* (things, events, and so on), which are assumed to constitute knowable reality. The latter evoke associations, which may or may not be relevant to the denotative meanings. Scientific knowledge is coded and communicated in ways that tend to reduce the connotative components of meaning. Words purged of connotative meaning are called *terms*. Terms are more precise than words of common usage, because their ranges of meaning are narrower. The Latin names of plants and animals are common examples. They denote species, which are usually narrower and hence more precise designations than common names of plants and creatures. The meanings of terms used in physics (for example, force, energy, power, and so on) are further examples of terms, that is, words from which all the usual connotations (related to, say, traits of human character) have been removed.

Mathematics is a language in which the ideals of objectivity, generality, and precision are realized in the highest degree. Typically (though not exclusively), assertions in this language refer to quantities. Terms referring to quantities are obviously both more objective and more precise than common usage words with quantitative denotations. The statement "This table is large" leaves much to be desired with regard to objectivity and precision; it may reflect the speaker's experience with tables or the intended use for the table. "This table is 250 cm long and 120 cm wide, hence 3 m^2 in area" is more objective and more precise. The connotations of "large" (which may depend on both the speaker's and the listener's experiences and attitudes) have been eliminated from the original statement, and its meaning has been thereby substantially narrowed.

Furthermore, the statement "The area of a rectangular table equals the product of length and width" *generalizes* the idea conveyed. Stated as a *formula*, A (area) $= L$ (length) $\times W$ (width), the assertion becomes a compact representation of a potential *infinity* of assertions, since any numbers can be substituted for L and for W, whereupon A becomes known. Here the "If..., then..." format is realized: If the length of a table is L and its width W, then its area is A.

In attempting to systematize modalities of cognition, philosophers (and, in fact, most people) distinguish between *quantities* and *qualities*. Unlike the former, the latter appear to be impervious to counting or measurement. However, advances in the physical sciences have revealed quantitative aspects of many categories perceived by our senses as "qualities." Color, pitch, and tone timbre are notable examples. The pitch of a pure tone is completely determined by the frequency of air waves it produces. Analogously, the colors of the rainbow are manifestations of frequencies of electromagnetic radiation. The timbre of a tone is determined by the relative amplitudes of its overtones. In view of the objectivity and greater precision of quantitative descriptions and because these descriptions lend themselves more easily to generalizing summarization, it is easy to see why mathematics was incorporated into the language of science in the late Renaissance, from the very inception of what we now call "modern" science.

The cognitive fruits of mathematical language became fully apparent when the power of mathematical *deduction* was tapped. Scientific knowledge is accumulated by two modes of reasoning, induction and deduction. Induction proceeds from particular observations to generalized inferences. Thus by observing that every human eventually dies, we are moved to infer that all humans are mortal—an inductive inference. Deduction proceeds from general principles to conclusions about particular cases. If all humans are mortal and Socrates is a human, we conclude that Socrates is mortal—a deductive conclusion. There is a crucial destinction between inductive and deductive modes of cognition. Induction depends on observations; deduction does not. Deduction amounts only to manipulations of symbols; in its most elementary form, of words. For instance, from the assertion "John is the husband of Mary" we can deduce the assertion "Mary is the wife of John." We need not observe John and Mary to establish that the deduction is valid. Its validity depends solely on the rules of English, according to which both assertions say the same thing.

Every deductive conclusion is, in the last analysis, a tautology. It says the same thing as the premises from which it was deduced. "Mary is the wife of

John" is deduced from the premis "John is the husband of Mary" and asserts the same thing. "Socrates is mortal" is deduced from two premises, "All men are mortal" and "Socrates is a man," and says the same thing as these premises. If the premises are "true" (in the sense to be discussed below), the conclusion *must* be "true," provided the deduction was made in accordance with certain specified rules (also to be discussed below). But the *validity* of the deduction does not depend on the "truth" of the premises. Deduction fits the "If..., then..." paradigm. The "if" part refers to the premises; the "then" part to the conclusion.

Premises from which mathematical deductions are made are called *axioms* or *postulates*. A set of such axioms or postulates serves as the foundation of a mathematical system or discipline. For example, Euclidean geometry rests on a number of axioms referring to quantities (for example, "quantities equal to the same quantity are equal to each other") and on a number of postulates referring to idealized spatial concepts (for example, "a straight line is the shortest distance between two points"). Conclusions derived from these axioms and postulates are called *theorems*. For example, the statement "In a right triangle, the square of the hypotenuse has the same area as the sum of the areas of the squares of the other two sides" is a theorem. Ordinary arithmetic rests on axioms that establish rules of addition and multiplication; for example, the distributive rule: $a(b + c) = ab + ac$. That the product of the sum and the difference of two numbers equals the difference of their squares is a theorem derived from these axioms.

The ancient Greeks, who laid the foundations of deductive geometry, regarded the axioms and postulates as "self-evident truths." The theorems derived from them may be anything but self-evident. However, since conclusions arrived at by deduction seem to be compelling, it appeared to some philosophers that absolutely certain truths about the world could be attained by this method. Plato was most insistent in asserting that observations, necessarily perturbed by error and limited to specific instances, were far inferior to instrospective reasoning as a method of discovering truth. There is a scene in *Meno* where Socrates leads a young slave through a proof of a special case of the Pythagorean Theorem in order to demonstrate that truth arrived at by reasoning can be grasped even by a lowly untrained mind.

In modern philosophy of mathematics, the axioms and postulates underlying a mathematical discipline are regarded as no more than conventions, "rules of the game" as it were, rather than "self-evident truths." This view reflects a concept of mathematics as a purely deductive procedure—manipulation of symbols according to prescribed rules. Whatever relations can or ought to be established between the symbols and assertions on the one hand, and things or events in the observable world on the other, raises problems of crucial importance in science but is entirely outside the scope of mathematics as such.

A major impetus to the abandonment of the notion of self-evident truth, as it applies to the postulates of mathematics, was given by the resolution of a problem with which mathematicians wrestled in the eighteenth and early nineteenth centuries. Of the postulates of Euclidean geometry, one in particular made some mathematicians uneasy, namely, the so-called parallel postulate. The postulate asserts that through a point external to a given line, one and only one line can be drawn in the same plane which fails to meet the given line however far it is produced. Somehow the postulate did not seem as self-evident as the others; for example, that between two distinct points one and only one straight line can be drawn. Accordingly, attempts were made to prove this postulate, that is, to derive it as a theorem from the other more compellingly acceptable postulates. All such attempts failed. The resolution of the difficulty came when J. Bolyai in Hungary and N.I. Lobachevsky in Russia each proved independently that the parallel postulate was independent of the others. That is to say, it could be replaced by another postulate incompatible with it without introducing a contradiction into geometry. In consequence, many *non-Euclidean* geometries were constructed. In so-called hyperbolic geometries, an infinity of lines can be passed through a point external to a line in the same plane which fail to meet it ever. In so-called elliptic geometries, there are no parallel lines at all. Consequences of postulates contradicting Euclid's parallel postulate introduce no internal contradictions into the resulting geometries. However, some of these consequences are, of course, incompatible with some theorems of Euclidean geometry. In a hyperbolic geometry, the sum of the angles of a triangle is less than 180°; in an elliptic geometry, greater. In both, the sum of the angles depends on the area of the triangle.

If one adheres to the view that geometry is a mathematical description of "actual" space, one must face the question as to which geometry is the "true" geometry. Since none of them can be invalidated by an internal contradiction, it follows that the answer to this question cannot be found by pure deduction. If the question can be decided at all, the answer can come only through observation, that is, by comparing the conclusions of the different geometries about spatial relations with measurements. In principle, one could measure the sums of angles of triangles to see whether they are equal to, less than, or greater than 180°. Such procedures, however, are beset with difficulties. The excess or deficiency in the sum of the angles of a non-Euclidean triangle becomes very small, hence undetectable by physical measurements, as the area of the triangle becomes small. Triangles accessible to our measurements are "small" in terms of cosmic dimensions, so that "locally" the geometry of physical space may appear Euclidean, even though space "as a whole" may not be. Moreover, a triangle is supposed to be bounded by "straight lines," and the question arises as to what should be taken as a *physical* "straight line." We may identify the path taken by a beam of light with a "straight line," but since 1919 it is known that light beams are "bent" in the neighborhood of large masses.

However we approach the problem of determining the nature of physical space, we see that this involves more than pure deduction. For the sake of clarity, therefore, it seems advisable to define mathematics (and geometry in particular) as a purely deductive discipline, concerned only with the *validity* of propositions, not with their empirically established *truth*. This view severs the link between mathematical concepts and observable referents. Mathematicians become free to construct a rich variety of mathematical systems, each restricted only by the requirement of internal consistency.

Some elementary systems of this sort are familiar to the layperson. Consider the "arithmetic" associated with a 24-hour clock. Six hours after 19:00 on this clock the time is 01:00. We can express this by writing $19+6=1$, which of course is false in "ordinary" arithmetic. Nevertheless, cyclic addition, based on identifying 24 with zero, works very well in telling time and in computing the time any number of hours ahead or ago.

A somewhat more complex example is an "arithmetic" that reflects the results of rotating a cube through multiples of 90° around any of its three axes. The cube can be in any of 24 positions. Starting with an arbitrarily selected position labeled zero, we can reach any other position by a combination of rotations, in fact, in several different ways. Let the elements (analogous to numbers) of our rotation arithmetic be combinations of rotations, each corresponding to the resulting position of the cube. Our arithmetic will contain 24 distinct elements. An "addition" of two such elements will be interpreted as the element that corresponds to the resulting position of the cube starting from zero when the rotations are performed consecutively. Thus the "sum" of two rotations is again a rotation, just as in ordinary arithmetic the sum of two numbers is again a number. Moreover, it turns out that this addition is associative: $a+(b+c)=(a+b)+c$. But this addition is not in general commutative; $a+b$ does not always result in the same position of the cube as $b+a$. (The reader is invited to try it with a die.) Thus rotation arithmetic is different from clock arithmetic. Both have 24 elements and associative additions, but addition in clock arithmetic is commutative, whereas addition in rotation arithmetic is not.

The readings of the clock are *representations* of the clock arithmetic; the positions of the cube are a representation of rotation arithmetic. These representations serve a *heuristic* purpose. They help us visualize the operations. In principle, however, a mathematical system is independent of its possible representations. As pointed out, its validity depends only on its internal consistency. Conversely, the 24-element arithmetic with commutative addition is a *model* of the 24-hour clock, and a particular 24-element arithmetic with non-commutative addition (there are several such arithmetics) is a model of the system of 90° rotations of a cube. The results of operations in these mathematical models can be regarded as predictions of clock readings and of positions of the cube, respectively. A mathematical model *fits* its referent system to the extent that the predictions deduced from its postulates are verified by observations.

Now in the case of the two models just described, we expect the correspondence between model and referent to be perfect, because the physical events involved (namely, the passage of time and the rotations of the cube) are *defined* by the readings of the clock and the resulting positions of the cube, respectively. No events external to those embodied in the models are allowed to interfere. The situation is different when physical events independent of our operations are involved. For instance, if we used another clock to measure the passage of time, our predictions about the readings on our clock might not be confirmed. Models