The Geometry of Four-Manifolds

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PREFACE

This book grew out of two lecture courses given by the first author in Oxford in 1985 and 1986. These dealt with the applications of Yang–Mills theory to 4-manifold topology, which, beginning in 1982, have grown to occupy an important place in current research. The content of the lectures was governed by two main aims, and although the treatment of the material has been expanded considerably in the intervening years, some of the resulting structure is preserved in the present work. The primary aim is to give a self-contained and comprehensive treatment of these new techniques as they have been applied to the study of 4-manifolds. The second aim is to bring together some of the developments in Yang–Mills theory itself, placed in the framework of contemporary differential and algebraic geometry. Leaving aside the topological applications, ideas from Yang–Mills theory—developed by many mathematicians since the late 1970's—have played a large part in fixing the direction of modern research in geometry. We have tried to present some of these ideas at a level which bridges the gap between general text books and research papers.

These two aims are reflected in the organization of the book. The first provides the main thread of the material and begins in Chapter 1 with the mysteries of 4-manifold topology—problems which have been well-known in that field for a quarter of a century. It finishes in the last chapters, when some of these problems are, in part, resolved. On the way to this goal we make a number of detours, each with the purpose of expounding a particular area of interest. Some are only tangentially related, but none are irrelevant to our principal topic. It may help the reader to signpost here the main digressions.

The first is in Chapter 3, which deals for the most part with the description of instanton solutions on the 4-sphere; some of the facts which emerge are an ingredient in later arguments (in Chapters 7 and 8, for example) and serve as a model for more general results, but their derivation is essentially independent from the rest of the book. Chapter 6 is concerned with the proof of a key theorem which provides a route from differential to algebraic geometry. This result underpins calculations in Chapters 9 and 10, but it could be taken on trust by some readers. In Chapter 7, only the last section is central to the subject matter of the book, and the main topological results can be obtained without the rather lengthy analysis which it contains. The reader who wants only to discover how Yang–Mills theory has been applied to 4-manifold topology might want to read only Chapter 1, the first part of Chapter 2, and Chapters 4, 5, 8, and 9.

The ten chapters are each reasonably self-contained and could, to a large extent, be read as individual articles on different topics. In general we have tried to avoid duplicating material which is readily available elsewhere.

Almost all of the results have appeared in research papers but we have spent some time looking for different, or simplified, proofs and for a streamlined exposition. Where other books already cover a topic in detail, we have tried to keep our treatment brief. While we hope that readers with a wide range of backgrounds will be able to get something useful from the book, we have assumed a familiarity with a definite body of background material, well represented in standard texts, roughly equivalent to first-year graduate courses in topology, differential geometry, algebraic geometry, and global analysis. The pre-requisites in analysis are summarized in the appendix; for the other subjects we hope that the references given will enable the reader to track down what is needed.

There are notes at the end of each chapter which contain a commentary on the material covered. Nearly all the references have been consigned to these notes. We feel that this streamlines the main text, although perhaps at the cost of giving precise references at all points. We have tried to acknowledge the original sources for the ideas and results discussed, and hasten to offer our apologies for any oversights in this regard.

Turning away from the content of the book, we should now say what is missing. First, although the subject of Yang–Mills theory, as an area of mathematical research, is rooted in modern physics, we have not discussed this side of the story except in passing. This is not to deny the importance of concepts from physical theories in the topics we treat. Indeed, throughout the last decade this area in geometry has been continually enriched by new ideas from that direction, and it seems very likely that this will continue. We are not, however, the right authors to provide an account of these aspects. Secondly, we have not given an exhaustive treatment of all the results on 4-manifolds which have been obtained using these techniques, nor have we tried to bring the account up-to-date with all the most recent developments; this area is still very active, and any such attempt would inevitably be overtaken by events. We hope that, by concentrating on some of the central methods and applications, we have written a book which will retain its value. Finally, while we have tried to give a thorough treatment of the theory from its foundations, we feel that there is still considerable scope for improvement in this respect. This holds both for a number of technical points and also, at a more basic level, in the general ethos of the interaction between Yang–Mills theory and 4-manifold topology. The exploratory drive of the early work in this field has not yet been replaced by any more systematic or fundamental understanding. Although the techniques described here have had notable successes, it is at present not at all clear what their full scope is, nor how essential they are to the structure of 4-manifolds. Looking to the future, one might hope that quite new ideas will emerge which will both shed light on these points and also go further in revealing the nature of differential topology in four dimensions. In any case, we hope that this book will help the reader to appreciate the fascination of these fundamental problems in geometry and topology.

It is a pleasure to record our thanks to a number of people and institutions for their help in the writing of this book. We are both indebted to our common doctoral supervisor, Sir Michael Atiyah, who originally suggested the project and has been a great source of encouragement throughout. Together with Nigel Hitchin, he also introduced us to many of the mathematical ideas discussed in the book. We have learnt a tremendous amount from Cliff Taubes and Karen Uhlenbeck, whose work underpins the analytical side of the theory, and from Werner Nahm. We should also like to take this opportunity to record the significant contribution that discussions in 1981 with Mike Hopkins and Brian Steer made in the early development of this subject.

The first author wishes to thank Nora Donaldson and Adriana Ortiz for their encouragement and help with the typing of the manuscript, and The Institute for Advanced Study, All Souls College and The Mathematical Institute, Oxford, for support. The second author is grateful for hospitality and support provided by Balliol College, The Institute for Advanced Study, the Mathematical Sciences Research Institute and the United States' National Science Foundation. Finally, we should like to thank the staff at Oxford University Press for their patience in awaiting the manuscript, and for their efficiency in the production of this book.

Oxford
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S.K.D.
P.B.K.

CONTENTS

1

FOUR-MANIFOLDS

This chapter falls into three parts. In the first we review some standard facts about the geometry and topology of four-manifolds. In the second we discuss a number of results which date back to the 1960's and before; in particular we give an account of a theorem of Wall which accurately portrays the limited success, in four dimensions, of the techniques which were being used to such good effect at that time in the study of high-dimensional manifolds. This discussion sets the scene for the new developments which we will describe in the rest of this book. In the third section we summarize some of the main results on the differential topology of four-manifolds which have sprung from these developments. The proofs of these results are given in Chapters 8, 9, and 10. The intervening chapters work, with many digressions, through the background material required for these proofs.

This first chapter has an introductory nature; the material is presented informally, with many details omitted. For thorough treatments we refer to the sources listed in the notes at the end of the chapter.

1.1 Classical invariants

1.1.1 Homology

In this book our attention will be focused on compact, simply connected, differentiable four-manifolds. The restriction to the simply connected case certainly rules out many interesting examples: indeed it is well known that any finitely presented group can occur as the fundamental group of a four-manifold. Furthermore, the techniques we will develop in the body of the book are, in reality, rather insensitive to the fundamental group, and much of our discussion can easily be generalized. The main issues, however, can be reached more quickly in the simply connected case. We shall see that for many purposes four-manifolds with trivial fundamental group are of beguiling simplicity, but nevertheless the most basic questions about the differential topology of these manifolds lead us into new, uncharted waters where the results described in this book serve, at present, as isolated markers.

After the fundamental group we have the homology and cohomology groups, $H_i(X; \mathbb{Z})$ and $H^i(X; \mathbb{Z})$, of a four-manifold X. For a closed, oriented four-manifold, Poincaré duality gives an isomorphism between homology and cohomology in complementary dimensions, i and $4 - i$. So, when X is simply connected, the first and third homology groups vanish and all the homological information is contained in H_2. The universal coefficient theorem for cohomology implies that, when H_1 is zero, $H^2(X; \mathbb{Z}) = \mathrm{Hom}(H_2(X; \mathbb{Z}), \mathbb{Z})$ is a free abelian group. In turn, by Poincaré duality, the homology group $H_2 = H^2$ is free.

There are three concrete ways in which we can realize two-dimensional homology, or cohomology, classes on a four-manifold, and it is useful to be able to translate easily between them (this is standard practice in algebraic geometry). The first is by *complex line bundles*, complex vector bundles of rank 1. On any space X a line bundle L is determined, up to bundle isomorphism, by its Chern class $c_1(L)$ in $H^2(X; \mathbb{Z})$ and this sets up a bijection between the isomorphism classes of line bundles and H^2. The second realization is by smoothly embedded two-dimensional oriented surfaces Σ in X. Such a surface carries a fundamental homology class $[\Sigma]$ in $H_2(X)$. Given a line bundle L we can choose a general smooth section of the bundle whose zero set is a surface representing the homology class dual to $c_1(L)$. Third, we have the de Rham representation of real cohomology classes by differential forms.

Let X be a compact, oriented, simply connected four-manifold. (The choice of orientation will become extremely important in this book.) The Poincaré duality isomorphism between homology and cohomology is equivalent to a bilinear form:

$$Q\colon H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

This is the *intersection form* of the manifold. It is a unimodular, symmetric form (the first condition is just the assertion that it induces an isomorphism between the groups H_2 and $H^2 = \mathrm{Hom}(H_2, \mathbb{Z})$). We will sometimes write $\alpha . \beta$ for $Q(\alpha, \beta)$, where $\alpha \in H_2$, and also $Q(\alpha)$ or α^2 for $Q(\alpha, \alpha)$. Geometrically, two oriented surfaces Σ_1, Σ_2 in X, placed in general position, will meet in a finite set of points. To each point we associate a sign ± 1 according to the matching of the orientations in the isomorphism,

$$TX = T\Sigma_1 \oplus T\Sigma_2,$$

of the tangent bundles at that point. The intersection number $\Sigma_1 . \Sigma_2$ is given by the total number of points, counted with signs. The pairing passes to homology to yield the form Q. Going over to cohomology, the form translates into the cup product:

$$H^2(X) \times H^2(X) \longrightarrow H^4(X) = \mathbb{Z}.$$

Thus the form is an invariant of the oriented homotopy type of X (and depends on the orientation only up to sign). In terms of de Rham cohomology, if ω_1, ω_2 are closed 2-forms representing classes dual to Σ_1, Σ_2, the intersection number $Q(\Sigma_1, \Sigma_2)$ is given by the integral:

$$\int_X \omega_1 \wedge \omega_2.$$

To see this correspondence between the integration and intersection definitions one chooses forms ω_i supported in small tubular neighbourhoods of the surfaces. Locally, near an intersection point, we can choose coordinates (x, y, z, w) on X so that Σ_1 is given by the equations $x = y = 0$, and Σ_2 by $z = w = 0$. For the dual forms we can take:

$$\omega_1 = \psi(x, y)\mathrm{d}x\mathrm{d}y, \quad \omega_2 = \psi(z, w)\mathrm{d}z\mathrm{d}w,$$

where ψ is a bump function on $\mathbb{R}^2$, supported near (0, 0) and with integral 1. The 4-form $\omega_1 \wedge \omega_2$ is now supported near the intersection points, and for each intersection point we can evaluate the contribution to the total integral in the coordinates above:

$$\int \psi(x, y)\psi(z, w)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}w = \pm 1$$

depending on orientations.

If we choose a basis for the free abelian group H_2, the intersection form is represented by a matrix with integer entries. The matrix is symmetric, and has determinant equal to ± 1 (this is the unimodular condition—a matrix with integer entries has an inverse of the same kind if and only if its determinant is ± 1). As we will explain below, the form on the integral homology contains more information than that on the corresponding real vector space $H_2(X; \mathbb{R})$. The latter is of course classified up to equivalence by its rank—the second Betti number b_2 of the manifold—and *signature*. Following standard notation we write

$$b_2 = b^+ + b^-, \tag{1.1.1}$$

where b^+, b^- are the dimensions of maximal positive and negative subspaces for the form on H_2. (In the familiar way we can identify the bilinear form with the associated quadratic form $Q(x)$.) The *signature* τ of the oriented four-manifold is then defined to be the signature of the form:

$$\tau = b^+ - b^-.$$

1.1.2 Some elementary examples

(i) The four-sphere S^4 has zero second homology group and so all intersection numbers vanish.

(ii) The complex projective plane $\mathbb{CP}^2$ is a simply connected four-manifold whose second homology is $\mathbb{Z}$. The standard generator is furnished by the fundamental class of a projective line $\mathbb{CP}^1 \subset \mathbb{CP}^2$. (The projective line is, of course, diffeomorphic to a two-sphere—the 'Riemann sphere'.) Two lines meet in a point and the conventional orientation is fixed so that this self-intersection number is 1. Thus the intersection form is represented by the 1×1 matrix (1). We write $\overline{\mathbb{CP}^2}$ for the same manifold equipped with the opposite orientation; so this manifold has intersection form (-1). (Note that there is no orientation reversing diffeomorphism of $\mathbb{CP}^2$.)

(iii) In the product manifold $S^2 \times S^2$ standard generators for the homology are represented by the embedded spheres $S^2 \times \{pt\}$ and $\{pt\} \times S^2$. These spheres intersect transversely in one point in the four-manifold and each has zero self-intersection. The intersection matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(iv) We can think of $S^2 \times S^2$ as being obtained from the trivial line bundle $S^2 \times \mathbb{C}$ by compactifying each fibre separately with a 'point at infinity'. More generally we can do the same thing starting with any complex line bundle over S^2. The line bundles are classified by the integers, via their first Chern class, so we get a sequence of four-manifolds M_d, $d \in \mathbb{Z}$. In each case H_2 is two dimensional; we can take generators to be the class of a two-sphere fibre and the zero section of our original bundle. Then the intersection matrix is

$$Q_d = \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}.$$

Now it is easy to see that there are only two diffeomorphism classes realized by these manifolds; M_d is a diffeomorphic to $M_0 = S^2 \times S^2$ if d is even and to M_1 if d is odd. This is because the integer d detects the homotopy class of the transition function for the original line bundle in $\pi_1(S^1) = \mathbb{Z}$, while the manifold M_d, as the total space of a two-sphere bundle, depends only on the image of this in $\pi_1(SO(3)) = \mathbb{Z}/2$. It follows of course that the quadratic forms above depend, up to isomorphism, only on the parity of d, which one can readily verify by a suitable change of basis. All the forms have $b^+ = b^- = 1$; however the forms for d odd and d even are *not* equivalent over the integers, so M_1 is not diffeomorphic to $S^2 \times S^2$. The two non-equivalent standard models are:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.1.2}$$

We say an integer quadratic form Q is of *even type* if $Q(x)$ is even for all x in the lattice, and that the form is of *odd type* if it is not of even type. Then we see that the form Q_d is even if and only if d is even.

(v) For any two four-manifolds X_1, X_2 we can make the *connected sum* $X_1 \# X_2$. If X_1, X_2 are simply connected, so is the connected sum; $H_2(X_1 \# X_2)$ is the direct sum of the $H_2(X_i)$ and the intersection form is the obvious direct sum. Starting with the basic building blocks above, we can make many more four-manifolds: for example by taking sums of copies of $\mathbb{CP}^2$ with appropriate orientations we get manifolds $l\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2$ with forms:

$$\mathrm{diag}(\underbrace{1, \ldots, 1}_{l}, \underbrace{-1, \ldots, -1}_{m}) = l(1) \oplus m(-1).$$

In fact the manifold $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is diffeomorphic to M_1 of (iv). One can see this by thinking of the Hopf fibration $S^1 \to S^3 \to S^2$. The complement of a small ball in $\mathbb{CP}^2$ can be identified with the disc bundle over S^2 (a line in $\mathbb{CP}^2$) associated with this circle bundle. When we make the connected sum we glue two of these disc bundles, with opposite orientations, along their boundary spheres to get the S^2 bundle considered in (iv).

1.1.3 Unimodular forms

How far do these examples go to cover the possible unimodular forms? It turns out that the algebraic classification of unimodular *indefinite* forms is rather simple. Any odd indefinite form is equivalent over the integers to one of the $l(1) \oplus m(-1)$ and any even indefinite form to one of the family $l\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus m E_8$, where E_8 is a certain positive definite, even form of rank 8 given by the matrix:

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & & & & & \\ 0 & 2 & 0 & -1 & & & & \\ -1 & 0 & 2 & -1 & & & & \\ & -1 & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}. \qquad (1.1.3)$$

In other words, indefinite, unimodular forms are classified by their rank, signature and type. (This is the *Hasse–Minkowski* classification of indefinite forms.) Thus we have found, so far, four-manifolds corresponding to all the odd indefinite forms but only the forms $l\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the even family.

The situation for definite forms is quite different. For each fixed rank there are a finite number of isomorphism classes, but this number grows quite rapidly with the rank—there are many exotic forms, E_8 being the prototype, not equivalent to the standard diagonal form. In fact, up to isomorphism, there is just one even positive-definite form of rank 8, two of rank 16, namely $E_8 \oplus E_8$ and E_{16}, and five of rank 24, including $3E_8$, $E_8 \oplus E_{16}$ and the Leech lattice.

Notice that we only consider above those definite forms whose rank is a multiple of eight: this is due to the following algebraic fact. For any unimodular form Q, an element c of the lattice is called *characteristic* if

$$Q(c, x) = Q(x, x) \bmod 2$$

for all x in the lattice; then if c is characteristic we have

$$Q(c, c) = \text{signature}(Q) \mod 8. \tag{1.1.4}$$

If Q is even the element 0 is characteristic, and we find that the signature must be divisible by 8. (Note that characteristic elements can always be found, for any form.)

1.1.4 The tangent bundle: characteristic classes and spin structures

In general one obtains invariants of smooth manifolds, beyond the homology groups themselves, as characteristic classes of the tangent bundle. For an oriented four-manifold X the characteristic classes available comprise the Stiefel–Whitney classes $w_i(TX) \in H^i(X; \mathbb{Z}/2)$ and the Euler and Pontryagin classes $e(X)$, $p_1(TX) \in H^4(X; \mathbb{Z}) = \mathbb{Z}$. The second Stiefel–Whitney class w_2 can be obtained from the mod 2 reduction of the intersection form by the Wu formula:

$$Q(w_2(TX), \alpha) = Q(\alpha, \alpha) \mod 2, \tag{1.1.5}$$

for all $\alpha \in H^2(X; \mathbb{Z}/2)$. This is especially easy to see when X is simply connected. Then any mod 2 class is the reduction of an integral class and so can be represented by an oriented embedded surface Σ. We have:

$$\langle w_2(TX), [\Sigma] \rangle = \langle w_2(T\Sigma \oplus \nu_\Sigma), [\Sigma] \rangle = \langle w_2(T\Sigma) + w_2(\nu_\Sigma), [\Sigma] \rangle,$$

where ν_Σ is the normal bundle. The Wu formula follows for, on the oriented two-plane bundles $T\Sigma$ and ν_Σ, the class w_2 is the mod 2 reduction of the Euler class; $e(\nu_\Sigma)$ is the self-intersection number $\Sigma . \Sigma$ of Σ, and $e(T\Sigma)$ is the Euler characteristic $2 - 2 \cdot \text{genus}(\Sigma)$, which is even. It is in fact the case that for any oriented four-manifold w_1 and w_3 are both zero. This is trivial for simply connected manifolds and we see that in this case the Stiefel–Whitney classes give no extra information beyond the integral intersection form.

The Euler and Pontryagin classes of a four-manifold can both be obtained from the rational cohomology ring. For the Euler class we have the elementary formula

$$e(TX) = \Sigma(-1)^i b_i,$$

the alternating sum of the Betti numbers b_i. The Pontryagin class is given by a deeper formula, the Hirzebruch Signature Theorem in dimension 4,

$$p_1(TX) = 3\tau(X) = 3(b^+ - b^-). \tag{1.1.6}$$

So in sum we see that all the characteristic class data for a simply connected four-manifold is determined by the intersection form on H_2.

In any dimension $n > 1$ the special orthogonal group $SO(n)$ has a connected double cover $\text{Spin}(n)$. If V is a smooth oriented n-manifold with a Riemannian metric, the tangent bundle TV has structure group $SO(n)$. The Stiefel–Whitney class w_2 represents the obstruction to lifting the structure group of TV to Spin(n). Such a lift is called a 'spin structure' on V. If $w_2 = 0$ a spin structure exists and, if also $H^1(X; \mathbb{Z}/2) = 0$, it is unique. In particular a simply connected four-manifold has a spin structure if and only if its intersection form is even, and this spin structure is unique.

A special feature, which permeates four-dimensional geometry, is the fact that Spin(4) splits into a product of two groups: Spin(4) = $SU(2) \times SU(2)$. One way to understand this runs as follows. Distinguish two copies of $SU(2)$ by $SU(2)^+$, $SU(2)^-$ and let S^+, S^- be their fundamental two-dimensional complex representation spaces. Then $S^+ \otimes_{\mathbb{C}} S^-$ has a natural Hermitian metric and also a complex symmetric form (the tensor product of the skew forms on S^+, S^-). Together these define a real subspace $(S^+ \otimes S^-)_{\mathbb{R}}$, the space on which the symmetric form is equal to the metric. The symmetry group $SU(2)^+ \times SU(2)^-$ acts on $S^+ \otimes S^-$, preserving the real subspace, and this defines a map from $SU(2)^+ \times SU(2)^-$ to $SO(4)$ which one can verify to be a double cover. In the same way a spin structure on a four-manifold can be viewed as a pair of complex vector bundles S^+, S^-—the spin bundles—each with structure group $SU(2)$, and an isomorphism $S^+ \otimes S^- = TX \otimes \mathbb{C}$, compatible with the real structures. (We will come back to spin structures in Chapter 3.)

1.1.5 Self-duality and special isomorphisms

The splitting of Spin(4) is related to the decomposition of the 2-forms on a four-manifold which will occupy a central position throughout this book. On an oriented Riemannian manifold X the $*$ operator interchanges forms of complementary degrees. It is defined by comparing the natural metric on the forms with the wedge product:

$$\alpha \wedge *\beta = (\alpha, \beta)\,\mathrm{d}\mu \tag{1.1.7}$$

where $\mathrm{d}\mu$ is the Riemannian volume element. So, on a four-manifold, the $*$ operator takes 2-forms to 2-forms and we have $** = 1_{\Lambda^2}$. The *self-dual* and *anti-self-dual* forms, denoted Ω_X^+, Ω_X^- respectively, are defined to be the ± 1 eigenspaces of $*$, they are sections of rank-3 bundles Λ^+, Λ^-:

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-, \quad \alpha \wedge \alpha = \pm|\alpha|^2\,\mathrm{d}\mu, \quad \text{for } \alpha \in \Lambda^\pm. \tag{1.1.8}$$

Reverting to the point of view of representations, the splitting of Λ^2 corresponds to a homomorphism $SO(4) \to SO(3)^+ \times SO(3)^-$. But $SU(2)$ can be identified with Spin(3) and the whole picture can be expressed by

$$\begin{array}{ccccc} \mathrm{Spin}(4) = & SU(2)^+ \times SU(2)^- & = \mathrm{Spin}(3)^+ \times \mathrm{Spin}(3)^- \\ \downarrow & \downarrow & \\ SO(4) \longrightarrow & SO(3)^+ \times SO(3)^-. & \end{array} \tag{1.1.9}$$

Over a four-manifold the $*$-operator on two-forms, and hence the self-dual and anti-self-dual subspaces, depend only on the conformal class of the

Riemannian metric. It is possible to turn this around, and regard a conformal structure as being defined by these subspaces. This is a point of view we will adopt at a number of points in this book. Consider first the intrinsic structure on the six-dimensional space $\Lambda^2(U)$ associated with an oriented four-dimensional vector space U. The wedge product gives a natural indefinite quadratic form q on U, with values in the line Λ^4. A choice of volume element makes this into a real-valued form. Plainly this form has signature 0; a choice of conformal structure on U singles out maximal positive and negative subspaces Λ^+, Λ^- for q. Note in passing that the null cone of the form q on Λ^2 has a simple geometric meaning—the rays in the null cone are naturally identified with the oriented two-planes in U. On the other hand, given a metric, this set of rays can be identified with the set of pairs $(\omega^+, \omega^-) \in \Lambda^+ \times \Lambda^-$ such that $|\omega^+| = |\omega^-| = 1$. So we see that the Grassmannian of oriented two-planes in a Euclidean four-space can be identified with $S^2 \times S^2$.

Now, in the presence of the intrinsic form q one of the subspaces, say Λ^-, determines the other; it is the annihilator with respect to q. The algebraic fact we wish to point out is that for any three-dimensional negative subspace $\Lambda^- \subset \Lambda^2$ there is a unique conformal structure on U for which this is the anti-self-dual subspace. (Note that the discussion depends on the volume element in Λ^4 only through the orientation; switching orientation just switches Λ^+ and Λ^-.) This is a simple algebraic exercise: it is equivalent to the assertion that the representation on Λ^2 exhibits $SL(U) = SL(4, \mathbb{R})$ as a double cover of the identity component of $SO(\Lambda^2, q) = SO(3, 3)$. This is another of the special isomorphisms between matrix groups. (The double cover $SO(4) \to SO(3) \times SO(3)$ considered before can be derived from this by taking maximal compact subgroups.)

For purposes of calculation we can exploit this representation of conformal structures as follows. Fix a reference metric on U and let Λ_0^+, Λ_0^- be the corresponding subspaces. Any other negative subspace Λ^- can be represented as the graph of a unique linear map,

$$m: \Lambda_0^- \longrightarrow \Lambda_0^+, \tag{1.1.10}$$

such that $|m(\omega)| < |\omega|$ for all non-zero ω in Λ_0^- (see Fig. 1). Thus there is a bijection between conformal structures on U and maps m from Λ_0^- to Λ_0^+ of operator norm less than 1. We can identify the new subspace Λ^- with Λ_0^-, using 'vertical' projection, and similarly for Λ^+. Then if α is a form in Λ^2, with components (α^+, α^-) in the old decomposition, the self-dual part with respect to the new structure is represented by:

$$(1 + mm^*)^{-1} (\alpha^+ + m\alpha^-). \tag{1.1.11}$$

This discussion goes over immediately to an oriented four-manifold X; given a fixed reference metric, we can identify the conformal classes with bundle maps $m: \Lambda^- \to \Lambda^+$ with operator norm everywhere less than 1.

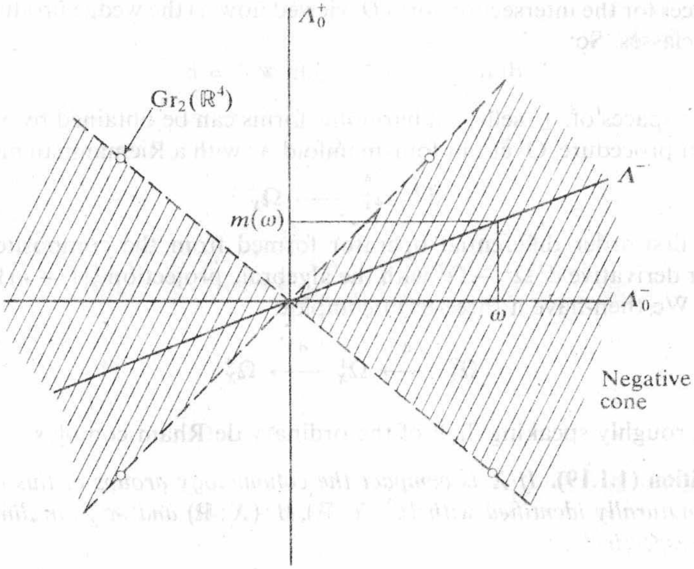

Fig. 1

1.1.6 *Self-duality and Hodge theory*

On any compact Riemannian manifold the Hodge Theory gives preferred representatives for cohomology classes by harmonic differential forms. Recall that one introduces the formal adjoint operator,

$$d^* : \Omega^p \longrightarrow \Omega^{p+1}, \tag{1.1.12}$$

associated with the intrinsic exterior derivative by the metric, so that

$$\int (\mathrm{d}\alpha, \beta) = \int (\alpha, \mathrm{d}^*\beta); \tag{1.1.13}$$

in the oriented case $\mathrm{d}^* = \pm * \mathrm{d} *$. The Hodge theorem asserts that a real cohomology class has a unique representative α with:

$$\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0. \tag{1.1.14}$$

For a compact, oriented four-manifold there is an interaction between the splitting of Λ^2 and the Hodge theory, which will be central to much of the material in this book. First, the harmonic two-forms are preserved by the $*$ operator (which interchanges $\ker \mathrm{d}$ and $\ker \mathrm{d}^*$), so given a metric we get a decomposition,

$$H^2(X; \mathbb{R}) = \mathscr{H}^+ \oplus \mathscr{H}^-, \tag{1.1.15}$$

into the self-dual and anti-self-dual (ASD) harmonic 2-forms. It follows immediately from the definition that these are maximal positive and negative